Blind Deconvolution and Polynomial Factorization

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Motivation and Background

Blind deconvolution is within reach for applications when using cyclic extension, lifting and convex programming [**Ahmed:2012**] or even with gradient-based methods [**Li2016**] - when the data is in lower-dimensional random subspaces. The recent demands for ***sporadic and short messages*** in next generation wireless networks have put blind strategies back into the focus in communication engineering.

prototypical problem: infer on (x, h) given noisy observation of:

$$y_k = (h * x)_k = \sum_l h_l x_{k-l}$$

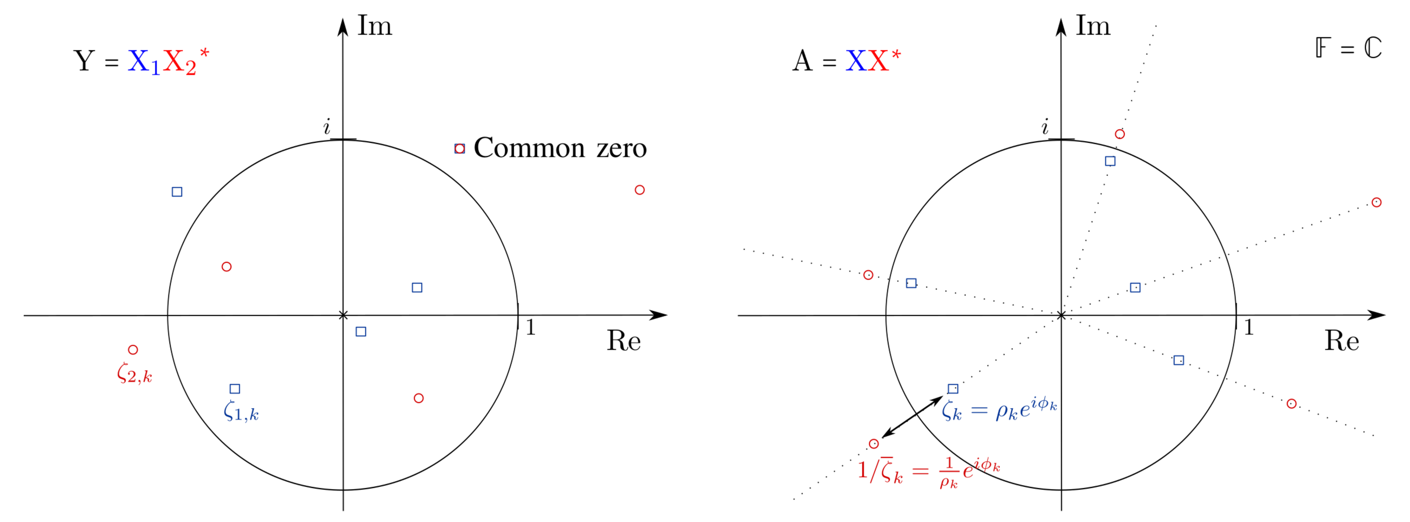
assumptions & open questions:

- single-channel baseband: x and h are finite complex sequences
- sporadic and short messages: length of x and h are similar
- derandomized and even deterministic constructions

z-transform / conjugate-symmetry / factorization:

- $X(z)$ of x and its reciprocal $X^*(z)$ are the polynomials (in $z^{-1} \in \mathbb{C}$):
$$X(z) = \sum_{k=0}^{L-1} x_k z^{-k} \quad \text{and} \quad X^*(z) := z^{1-L}\overline{X(1/\overline{z})} = \sum_{k=0}^{L-1} \underbrace{\overline{x}_{L-1-k}}_{(\overline{x^-})_k} z^{-k}$$
and for convolution $y = h * x$ it holds $Y(z) = H(z)X(z)$.
- $x = \overline{x^-}$ (conjugate-symmetric) $\Leftrightarrow$ self–reciprocal polynomials $X = X^*$, those zeros $\xi \notin \mathbb{T}$ will come in conjugated pairs $(\xi, 1/\overline{\xi}) \in (\mathbb{D}, \mathbb{D}^c)$.
- Wiener-Hopf (WH) factorization $PQ \Rightarrow (P, Q)$ with $\xi_P \in \mathbb{D}$ and $\xi_Q \in \mathbb{D}^c$ (fast Newton method [**Boettcher2013:WienerHopf:Real**] gives directly $p * q \Rightarrow (p, q)$)
- for autocorrelation $a = x * \overline{x^-}$ therefore $A = XX^*$ is self-reciprocal polynomial with even zeros on $\mathbb{T}$, $A(\mathbb{T}) \geq 0$

idea / principle: avoid ambiguities (zeros separated in pre-defined regions)



Using the Wiener-Hopf Factorization

The following ideas are applicable if h is ***minimum phase***, i.e., $\forall \xi \in \mathbb{D}$

- a sufficient condition: $\forall \omega : |\sum_{k=1}^{K-1} h_k e^{i\omega k}| \leq |h_0|$ (LOS channels)

method (a) - maximum-phase signaling:

Tx data signal x is maximum-phase (X^* is minimum-phase)

Rx WH factorization of $Y = HX$ provides (H, X) up scaling

method (b) - autocorrelation signaling:

Tx encode information *directly* into $a = x * \overline{x^-}$, i.e., $A = XX^*$

Rx receive noisy observations of $h * a = h * (x * \overline{x^-})$

... WH factorization for $Y = HA = HXX^*$ giving (HX, X^*).

... compute $A = XX^*$ and decode A

- ☺ methods are straightforward, fast WH implementation can be used
- ☹ empirically, maximum-phase constructions have undesired PAPR

$\text{PAPR}(x) = L\|X\|_{\ell_\infty}^2/\|x\|_{\ell_2}^2 = \mathcal{O}(L)$

Using Autocorrelations

If h is not minimum-phase but H is co-prime from X then a factorization can be obtained by an ***semidefinite program*** [**JH16**]. Stack $u = [x, h]$ and:

$$\mathcal{A}(uu^*) = \mathcal{A}\left(\begin{pmatrix} xx^* & xh^* \\ hx^* & hh^* \end{pmatrix}\right) = \begin{pmatrix} x * \overline{x^-} \\ x * \overline{h^-} \\ h * \overline{x^-} \\ h * \overline{h^-} \end{pmatrix} = b \in \mathbb{C}^{4N-4}$$

Theorem: *([**JH16**] and deterministic case [**WJPH17**]) With measurements b as above: For co-prime polynomials X and H the convex program (SDP) gives uniquely:*

$$U = \arg\min\{\|\mathcal{A}(V) - b\|_{\ell_2} \,|\, 0 \leq V \in \mathbb{C}^{N \times N}\}$$

where $U = uu^*$ and $u = [x, h] \in \mathbb{C}^N$.

method (c) - signaling with $A(\mathbb{T}) \approx$ const > 0: (see [**WJH17**])

- From Wiener-Lee relation for $d := y * \overline{y^-}$:
$$d = (h * \overline{h^-}) * (x * \overline{x^-}) \;\&\; YY^* = (HH^*)(XX^*) = (HH^*)A$$
allowing to compute $(h * \overline{h^-})$ from $y * \overline{y^-}$ if $A(\mathbb{T}) \approx$ const
- ***Huffman sequences*** [**Huf62**] have autocorrelation $a = x * \overline{x^-}$
$$a = [-1, 0 \ldots, E, 0 \ldots, -1] \;\&\; |A(e^{iw})| = |E - 2\cos(w(L-1))|$$
For $K \leq \frac{L}{2}$ it follows $d = [-f, 0 \ldots, Ef, 0 \ldots, -f]$ with $f = h * \overline{h^-}$
- zeros of $A(z)$ are $k = 1 \ldots L-1$ conjugated pairs:
$$\xi_k^{\pm} = R^{\pm 1}\exp(i2\pi\frac{k-1}{L-1}) \text{ with } R^{L-1} = (E + \sqrt{E^2 - 4})/2$$

Tx choose $E > 2$, encode x from $L-1$ bits by choosing from pairs $\xi_k^{\pm}$

Rx estimate $f = h * \overline{h^-}$ and E from $d = y * \overline{y^-}$, solve SDP

... perform rank-one projection of U (SVD) to get $u = [h, x]$

... decode x for the estimated E, e.g., using zeros of X

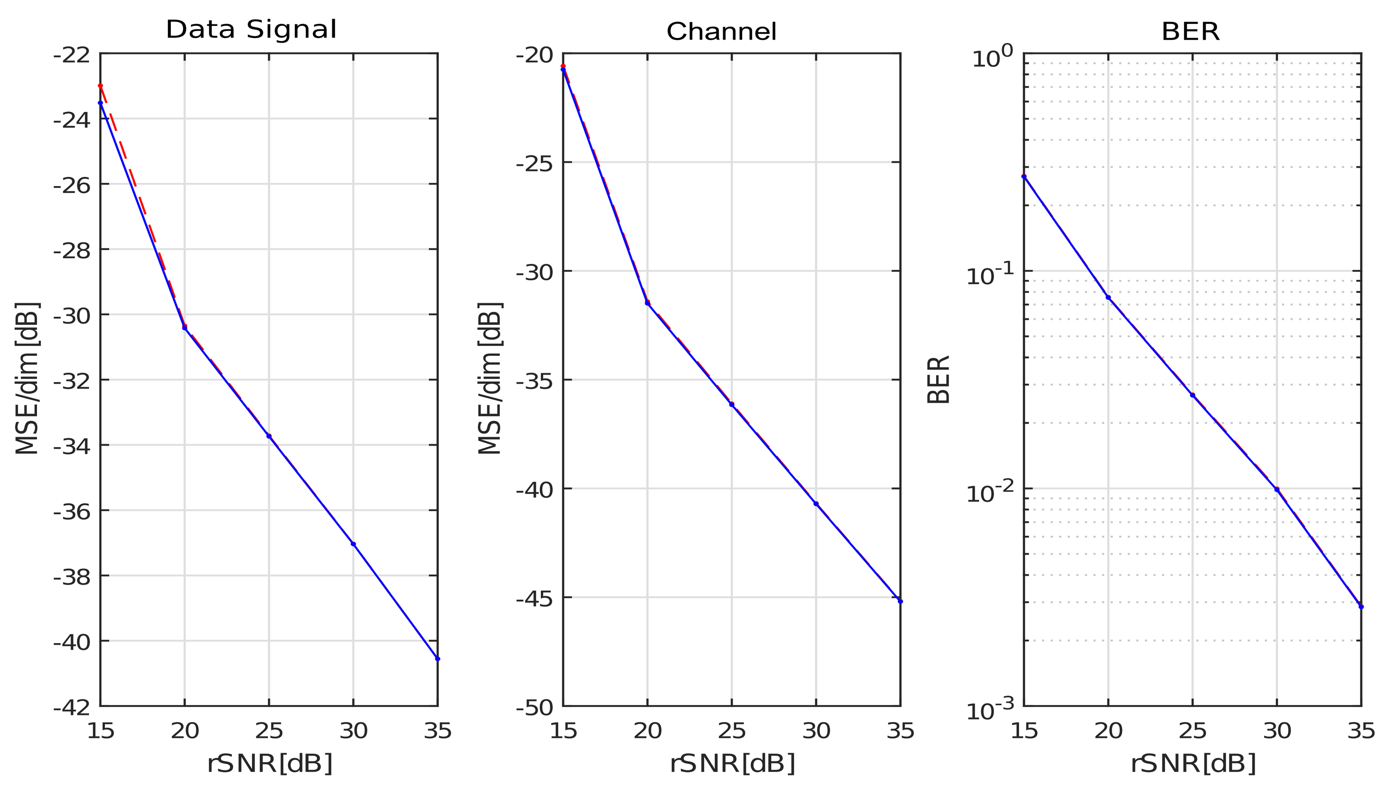


Figure: MSE for 13000 runs with $L = 32$ and $K = 8$, and BER over rSNR. Red-dashed curve is with unknown and blue-solid with known E.

- ☺ special structure allows to estimate first E independent of h
- ☹ reduce PAPR $E \to 2$, robust zeros requires $E \gg 2$
- ... complexity in SDP, sequence detection and root finding

Conclusions: all methods so far seems to suffer from a PAPR problem

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