Optimization Algorithms
Weekly Exercises 6

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1 Problems involving ℓ_1- and ℓ_∞-norms

These exercises are from Boyd et al http://www.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf:
(Ex. 4.11, pdf page 207) Formulate the following problems as LPs. Explain in detail the relation between the optimal solution of each problem and the solution of its equivalent LP. (See norm definitions below.)

a) Minimize $\|Ax-b\|_\infty$

b) Minimize $\|Ax-b\|_1$

c) Minimize $\|Ax-b\|_1$ subject to $\|x\|_\infty \le 1$

In each problem, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ are given.

Recall the general definition of a p-norm as $\|x\|_p = \Big[\sum_i |x_i|^p\Big]^{1/p}$. In particular $\|x\|_1 = \sum_i |x_i|$ (sum of absolute values), and $\|x\|_\infty = \max_i |x_i|$ (largest absolute value (limit $p\to\infty$)).

2 Minimum fuel optimal control

[This is a modification of Boyd's Ex 4.16, pdf page 208.] We consider a linear dynamical system with state $x(t) \in \mathbb{R}^n, t = 0,..,N$, and actuator or input signal $u(t) \in \mathbb{R}$, for $t = 0,..,N-1$. The dynamics of the system is given by the linear recurrence

$$x(t+1) = A\ x(t) + b\ u(t)\ ,\quad t = 0,..,N-1\ ,$$

where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ are given. We assume that the initial state is zero, i.e., $x(0) = 0$.

The minimal fuel optimal control problem is to choose the inputs $u(0),..,u(N-1)$ so as to minimize the total fuel consumed, which is given by

$$F = \sum_{t=0}^{N-1} c(u(t))\ ,$$

subject to the constraint that $x(N) = x_{\text{des}}$, where N is the (given) time horizon, and $x_{\text{des}} \in \mathbb{R}^n$ is the (given) desired final or target state. The function $c : \mathbb{R} \to \mathbb{R}$ is the fuel use map for the actuator, and gives the amount of fuel used as a function of the actuator signal amplitude. In this problem [modified from Boyd] we use

$$c(u) = u^2\ .$$

a) Formulate the minimum fuel optimal control problem as a Mathematical Program, where both controls and states are the optimization variable, and the goal and dynamics of the system are constraints. This should be an NLP where the cost function is quadratic, and all constraints linear (aka. Quadratic Program).