Optimization Algorithms
Exercise 7

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(only small typos fixed relative to the DRAFT version)

These exercises are not about optimization algorithms, but about training you to *formulate* optimization problems. This is actually a major part of successfully tackling real-world problems. It is often surprising in how many ways one can formulate an optimization problem, and it requires some experience to know some tricks or conventions of how to formulate them.

The exercises are from Boyd et al `http://www.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf`:

1 Problems involving ℓ_1- and ℓ_∞-norms

(Ex. 4.11, pdf page 207) Formulate the following problems as LPs. Explain in detail the relation between the optimal solution of each problem and the solution of its equivalent LP. (See norm definitions below.)

a) Minimize $\|Ax - b\|_\infty$

b) Minimize $\|Ax - b\|_1$

c) Minimize $\|Ax - b\|_1$ subject to $\|x\|_\infty \le 1$

In each problem, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ are given.

Recall the general definition of a p-norm as $\|x\|_p = [\sum_i |x_i|^p]^{1/p}$. In particular $\|x\|_1 = \sum_i |x_i|$ (sum of absolute values), and $\|x\|_\infty = \max_i |x_i|$ (largest absolute value).

2 Minimum fuel optimal control

(Ex 4.16, pdf page 208) We consider a linear dynamical system with state $x(t) \in \mathbb{R}^n, t = 0, .., N$, and actuator or input signal $u(t) \in \mathbb{R}$, for $t = 0, .., N-1$. The dynamics of the system is given by the linear recurrence

$$x(t+1) = A\ x(t) + b\ u(t)\ , \quad t = 0, .., N-1\ ,$$

where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ are given. We assume that the initial state is zero, i.e., $x(0) = 0$.

The minimal fuel optimal control problem is to choose the inputs $u(0), .., u(N-1)$ so as to minimize the total fuel consumed, which is given by

$$F = \sum_{t=0}^{N-1} f(u(t))\ ,$$

subject to the constraint that $x(N) = x_{\text{des}}$, where N is the (given) time horizon, and $x_{\text{des}} \in \mathbb{R}^n$ is the (given) desired final or target state. The function $f : \mathbb{R} \to \mathbb{R}$ is the fuel use map for the actuator, and gives the amount of fuel used as a function of the actuator signal amplitude. In this problem we use

$$f(x) = \begin{cases} |a| & |a| \le 1 \\ 2|a| - 1 & |a| > 1 \end{cases}\ .$$

This means that fuel use is proportional to the absolute value of the actuator signal, for actuator signals between -1 and 1; for larger actuator signals the marginal fuel efficiency is half.

Formulate the minimum fuel optimal control problem as an LP.

3 Convexity and positive-definite Hessian

Prove that a (twice differentiable) function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if for every $x \in \mathbb{R}^n$, the Hessian matrix $\nabla^2 f(x)$ is positive semidefinite. (A matrix H is pos-semi-def iff $\delta^\top H \delta \geq 0$ for any $\delta \in \mathbb{R}^n$.)

Notes: This exercise is meant to establish the relation between convexity and our extensive discussion of Hessians in the first part of the lecture. In particular, in Exercise sheet 2 we proved exponential convergence of iterative line search when the Hessian's eigenvalues are positive bounded – one could call this strict (bounded) convexity.