Optimization Algorithms
Exercise 11: Exam Preparation Questions

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1 Constrained Optimization Problem

Consider the optimization problem

$$\begin{aligned} \min_{x_1,x_2} \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & \sqrt{x_1^2 + x_2^2} \le 2 \\ & x_1 \ge 0 \\ & x_2 \ge 0.5x_1 - 1 \end{aligned} \tag{1}$$

a) Solve the optimization problem by sketching the feasible set and the objective function. Provide the optimal x, and the optimal function value.

b) State the KKT conditions for this problem, i.e. explicitly write down the KKT conditions for the given optimization problem.

c) Solve the optimization problem analytically with the KKT conditions.

2 Taylor

Provide the second order Taylor approximation around x_0 for

$$f(x) = \sum_{i=1}^{k} \frac{r_i^2}{r_i^2 + a^2} \tag{2}$$

with $r_i = y_i - \beta_i^T x$ where $x \in \mathbb{R}^n$, $y_i \in \mathbb{R}$, $\beta_i \in \mathbb{R}^n, a \in \mathbb{R}$.

3 Lagrange Duality

Consider the following linear program, with $x, y, z \in \mathbb{R}$

$$\begin{aligned} & \min_{x,y,z} x + y - z \\ & \text{s.t. } x - y - 3z = 1;\ x, y, z \ge 0 \end{aligned} \tag{3}$$

a) Draw the feasible set and guess where the solution is.

b) Compute the dual function, and formulate the dual optimization problem. On how many variables does the dual optimization problem depend?

c) Solve the dual optimization problem.

4 Optimal stock discount

We have n different products in stock, each with q_i items left. The original prize of the items is p_i.

The end of the season is approaching, and we want to maximize our profits by applying some discount c_i (resutling in the offered price $(1 - c_i)p_i$) to the prices of the products. Assume that the demand for a product i at price p'_i is given by $f_i(p'_i)$.

a) Write down the optimization problem, with the necessary constraints. Make sure that the new prices can not be negative, and should not be higher than the original. In addition, it is not possible to sell more stock than we have.

b) Assume now that the demand is given by

$$f_i(p'_i) = 1 - 0.5p'_i \; 0 \leq p'_i \leq 2$$

Write down the new optimization problem as a quadratic program.

c) Typically, the discount factors c_i are chosen from a predefined set, such that we do not end up with weird discount factors. Assume $c_i \in \{0.1, 0.2, 0.25, 0.5\}$. How can we leverage solvers for quadratic programs to obtain the solution to the new problem? Give a one-sentence-answer.

d) In addition, we want to discount a maximum number of $m < n$ of our products. Write down the new optimization problem.

5 Log barrier method

We want to solve the following optimization problem (with $x \in \mathbb{R}^2$):

$$\min_x x^T x \text{ s.t. } (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1$$
$$x_1 \geq 1$$

a) What are the requirements for the point with which we initialize an interior point method?

b) Solve the problem graphically, and draw the central path approximately.

c) Assuming that the algorithm is currently located at x, and the log-barrier parameter is μ, derive the linesearch-direction of a Newton step analytically.