points:	grade:	exam viewed on:

Exam
Maths for Intelligent Systems, WS 17/18, U Stuttgart

Prof. Dr. Marc Toussaint

Feb 21, 2017

Name:
Matrikelnummer:

DO NOT OPEN THE EXAM BEFORE THE ANNOUNCEMENT

GOOD LUCK!

- You have 120 minutes.
- Write your name on all sheets.
- Put your *Studentenausweis* next to you, so we can check it during the exam.
- **You may only use your pen and the given paper.** Don't use pencil or red colour. No other materials (no textbooks, script, or mobiles) are allowed.
- In case you need extra paper, please raise your hand and you will be provided extra sheets. All extra sheets need to be handed in together with the exam.
- Please try to **answer only with equations**, no lengthy text. Of course, we will try to read it if necessary. But usually all answers are well defined in terms of equations.
- Also use the back of the sheets if necessary. Please indicate clearly when you use the back of sheets or extra sheets.

Question 1 — Singular and Eigen Values – PUBLIC (5Pts)

(i) Let A be a diagonalizable squared matrix. By definition an eigenvector e_i of A fulfills $Ae_i = \lambda v_i$. Show that $A = Q\Lambda Q^{-1}$, where Λ is a diagonal matrix containing the eigenvalues of A and the columns of Q consist of the eigenvectors of A. What is $Q\Lambda Q^{-1}$ called? [1]

(ii) For a matrix $B = A^T A$, show that $\sqrt{\lambda_i^B} = \sigma_i^A$, where λ_i^A are the eigen values of B and σ_i^A are the singular values of A. (If necessary, assume that the eigen values and singular values are ordered by magnitude.) [2]

(iii) Consider the linear system of equations $Ax = b$ where $A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and A has rank r. Discuss the dimensionality of the 4 fundamental spaces and their implication on the existance of a solution (exactly one or no solution, exactly one solution, at least one solution) for the following cases:

a) $r = n < m$

b) $r = m < n$

c) $r = n = m$

[2]

Question 2 — Taylor expansion – PUBLIC (5Pts)

Consider a data set $D = \{(x_i, y_i)\}_{i=1}^n$, with $x_i \in \mathbb{R}^d$ and $y_i \in \mathbb{R}$. For parameters $\beta \in \mathbb{R}^d$ we define the ("Forsyth") cost function

$$L(\beta) = \sum_{i=1}^n \frac{r_i^2}{r_i^2 + a^2} \;, \quad r_i \triangleq y_i - x_i^\top \beta$$

Provide the 2nd-order Taylor approximation of $L(\beta)$.

Question 3 — Lagrangian Method – PUBLIC (4Pts)

Consider the problem

$$\min_{x \in \mathbb{R}} x^2 - 2x \quad \text{s.t.} \quad (x-2)(x-4) \le 0 \ .$$

(i) Provide the optimal solution x^* and the optimal value $p^* = f(x^*)$ simply by "sketching" the problem. [2]

(ii) Derive the optimal solution x^* and the optimal value $p^* = f(x^*)$ using the method of Lagrange multipliers. [2]

Question 4 — Gradient Descent, Newton Method, KKT & Log Barrier – PUBLIC (5Pts)

Consider the minimization problem

$$\min_{x\in\mathbb{R}^n} \underbrace{x^TAx + b^Tx + c}_{f(x)}$$

(i) Write down the update steps of Newton's method for this problem. Explicitly compute the derivitaves of $f(x)$ if they appear. Consider a constant step size η. [1]

(ii) Provide the analytic solution knowing that A is symmetric positive definite. [1]

(iii) Now assume the additional constraint on the norm of x is bounded:

$$\min_{x\in\mathbb{R}^n} x^TAx + b^Tx + c \quad \text{s.t.} \quad ||x||_C \le d$$

Explicitly state the 4 KKT conditions for THIS SPECIFIC problem, and provide explicit derivitaves if they appear. Note that the norm of x (not the squared norm!!!) is taken with respect to metric tensor C. [2]

(iv) We now want to solve this constrained optimization problem with the log barrier method. For a fixed outer iteration t with multipier μ_t, state the log barrier objective and write down the gradient descent update rule with a fixed step size of η. Provide explicit derivatives if they appear. [1]

TOTAL POINTS=19