Maths for Intelligent Systems
Exercise 8

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1 Optimize a constrained problem

Consider the following constrained problem

$$\min_x \sum_{i=1}^n x_i \quad \text{s.t.} \quad g(x) \le 0 \tag{1}$$

$$g(x) = \begin{pmatrix} x^\top x - 1 \\ -x_1 \end{pmatrix} \tag{2}$$

a) First, assume $x \in \mathbb{R}^2$ is 2-dimensional, and draw on paper what the problem looks like and where you expect the optimum.

b) Find the optimum analytically using the Lagrangian. Here, assume that you know apriori that all constraints are active! What are the dual parameters $\lambda = (\lambda_1, \lambda_2)$?

Note: Assuming that you know a priori which constraints are active is a huge assumption! In real problems, this is the actual hard (and combinatorial) problem. More on this later in the lecture.

c) Implement a simple the Log Barrier Method. Tips:

- Initialize $x = (\frac{1}{2}, \frac{1}{2})$ and $\mu = 1$
- First code an inner loop:
  - In each iteration, first compute the gradient of the log-barrier function. Recall that

$$F(x; \mu) = f(x) - \mu \sum_i \log(-g_i(x)) \tag{3}$$

$$\nabla F(x; \mu) = \nabla f - \mu \sum_i (1/g_i(x)) \nabla g_i(x) \tag{4}$$

  - Then make a small step along the negative gradient $x \leftarrow x - \alpha \nabla F(x, \mu)$, for $\alpha = 1/100$.
  - Iterate until convergence; let's call the result $x^*(\mu)$. Further, compute $\lambda^*(m) = -(\mu/g_1(x), mu/g_2(x))$ at convergence.
- Decrease $\mu \leftarrow \mu/2$, recompute $x^*(\mu)$ (with the previous x^* as initialization) and iterate this.

Does x^* and λ^* converge to the expected solution?

Note: The path $x^*(\mu) = \text{argmin}_x F(x; \mu)$ (the optimum in dependence of μ) is called *central path*.

2 Network flow problem

Comment: Solving problems in the real world involves 2 parts:

1) formulating the problem as an optimization problem ($\to$ human)
2) the actual optimization problem ($\to$ algorithm)

The first part is perhaps more interesting – that's the science part. But understanding the algorithms helps to better formalize problems. This and the following exercises are from Boyd & Vandenberghe's book `http://www.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf`. They are only about the first part.

Solve Exercise 4.12 (pdf page 207) from Boyd & Vandenberghe, *Convex Optimization*.

3 Minimum fuel optimal control

Solve Exercise 4.16 (pdf page 208) from Boyd & Vandenberghe, *Convex Optimization.*

4 Voluntary: Reformulating Norms

Solve Exercise 4.11 (pdf page 207) from Boyd & Vandenberghe, *Convex Optimization.*

5 Voluntary: Some more examples

a) Grocery Shopping: You're at the market and you find n offers, each represented by a set of items A_i and the respective price c_i. Your goal is to buy at least one of each item for as little as possible. Formulate as an integer LP.

b) Facility Location: There are n facilities with which to satisfy the needs of m clients. The cost for opening facility j is f_j, and the cost for servicing client i through facility j is c_{ij}. You have to find an optimal way to open facilities and to associate clients to facilities. Formulate as an ILP.