# Maths for Intelligent Systems
# Exercise 1

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## 1 Examples for Vector Spaces (voluntary, discussed on Oct 23rd)

(a) Find 3 interesting examples of vector spaces that we haven't covered in class. What are their dimensions? What bases are commonly used to represent them? (It's okay to look this up in a textbook or online—just find interesting examples.)

(b) Come up with one example of a set which is *almost* a vector space, i.e. it satisfies some of the core requirements of a vector space, but not all.

## 2 Basis

Given a linear transform $f : \mathbb{R}^2 \to \mathbb{R}^2$,

$$f(x) = Ax = \begin{pmatrix} 7 & -10 \\ 5 & -8 \end{pmatrix} x \ .$$

Consider the basis $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$, which we also simply refer to by the matrix $B = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$. Given a vector $x$ in the vector space $\mathbb{R}^2$, we denote its coordinates in basis $\mathcal{B}$ with $x^B$.

(a) Show that $x = Bx^B$.

(b) What is the matrix $F^B$ of $f$ in the basis $\mathcal{B}$, i.e., such that $[f(x)]^B = F^B x^B$? Prove the general equation $F^B = B^{\text{-}1} A B$.

(c) Provide $F^B$ numerically

Note that for a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $M^{\text{-}1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

## 3 From the Robotics Course (voluntary)

You have a book lying on the table. The edges of the book define the basis $B$, the edges of the table define basis $A$. Initially $A$ and $B$ are identical (also their origins align). Now we rotate the book by 45° counter-clock-wise about its origin.

(a) Given a dot $p$ marked on the book at position $p^B = (1,1)$ in the book coordinate frame, what are the coordinates $p^A$ of that dot with respect to the table frame?

(b) Given a point $x$ with coordinates $x^A = (0,1)$ in table frame, what are its coordinates $x^B$ in the book frame?

(c) What is the *coordinate* transformation matrix from book frame to table frame, and from table frame to book frame?

# 4 Bases for Polynomials

Consider the set $V$ of all polynomials $\sum_{i=0}^{n} \alpha_i x^i$ of degree $n$, where $x \in \mathbb{R}$ and $\alpha_i \in \mathbb{R}$ for each $i = 0, \dots, n$.

(a) Is this set of functions a vector space? Why?

(b) Consider two different bases

$$\mathcal{A} = \{1,\ x,\ x^2,\ \dots,\ x^n\}$$

and

$$\mathcal{B} = \{1,\ 1+x,\ 1+x+x^2,\ \dots,\ 1+x+\dots+x^n\}.$$

Let $f(x) = 1 + x + x^2 + x^3$. (This function $f$ is a vector in the vector space $\mathcal{V}$, so from here on we refer to it as a vector rather than a function.)

What are the coordinates $[f]^A$ of this vector in basis $\mathcal{A}$?

What are the coordinates $[f]^B$ of this vector in basis $\mathcal{B}$?

(c) What matrix $I^{BA}$ allows you to convert between coordinates $[f]^A$ and $[f]^B$, i.e. $[f]^B = I^{BA}[f]^A$? Which matrix $I^{AB}$ does the same in the opposite direction, i.e. $[f]^A = I^{AB}[f]^B$? What is the relationship between $I^{AB}$ and $I^{BA}$?

(d) What does the difference between coordinates $[f]^A - [f]^B$ represent?

(e) Consider the linear transform $t$ that takes

$$\begin{aligned} &1 \to 1 \\ &x \to 1 + x \\ &x^2 \to 1 + x + x^2 \\ &\vdots \end{aligned}$$

(This transform takes basis elements of $\mathcal{A}$ directly to basis elements of $\mathcal{B}$.)

- What is the matrix $T^A$ for the linear transform in the basis $\mathcal{A}$, i.e., such that $[tf]^A = T^A[f]^A$? (Basis $\mathcal{A}$ is used for both, input and output spaces.)
- What is the matrix $T^B$ for the linear transform in the basis $\mathcal{B}$, i.e., such that $[tf]^B = T^B[f]^B$? (Basis $\mathcal{B}$ is used for both, input and output spaces.)
- What is the matrix $T^{BA}$ if we use $\mathcal{A}$ as input space basis, and $\mathcal{B}$ as output space basis, i.e., such that $[tf]^B = T^{BA}[f]^A$?
- What is the matrix $T^{AB}$ if we use $\mathcal{B}$ as input space basis, and $\mathcal{A}$ as output space basis, i.e., such that $[tf]^A = T^{AB}[f]^B$?
- Show that $T^B = I^{BA}T^AI^{AB}$ (cp. exercise 2(b)). Also note that $T^{AB} = T^AI^{AB}$ and $T^{BA} = I^{BA}T^A$.