Artificial Intelligence

Search

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Winter 2018/19

(slides based on Stuart Russell's AI course)

Motivation:

Search algorithms are a core tool for decision making, especially when the domain is too complex to use alternatives like Dynamic Programming. With the increase in computational power search methods became a standard method of choice for complex domains, like the game of Go, or certain POMDPs. Recently, they are combined with machine learning methods which learn heuristics or evaluation functions to guide search.

Learning about search tree algorithms is an important background for several reasons:

- The concept of decision trees, which represent the space of possible future decisions and state transitions, is generally important for thinking about decision problems.
- In probabilistic domains, tree search algorithms are a special case of Monte-Carlo methods to estimate some expectation, typically the so-called Q-function. The respective Monte-Carlo Tree Search algorithms are the state-of-the-art in many domains.
- Tree search is also the background for backtracking in CSPs as well as forward and backward search in logic domains.

We will cover the basic tree search methods (breadth, depth, iterative deepening) and eventually A*

Outline

- Problem formulation & examples
- Basic search algorithms

Problem Formulation & Examples

Example: Romania

On holiday in Romania; currently in Arad.

Flight leaves tomorrow from Bucharest

Formulate goal:

be in Bucharest, $\mathcal{S}_{\text{goal}} = \{\text{Bucharest}\}$

Formulate problem:

states: various cities, $\mathcal{S} = \{\text{Arad, Timisoara, } \dots\}$

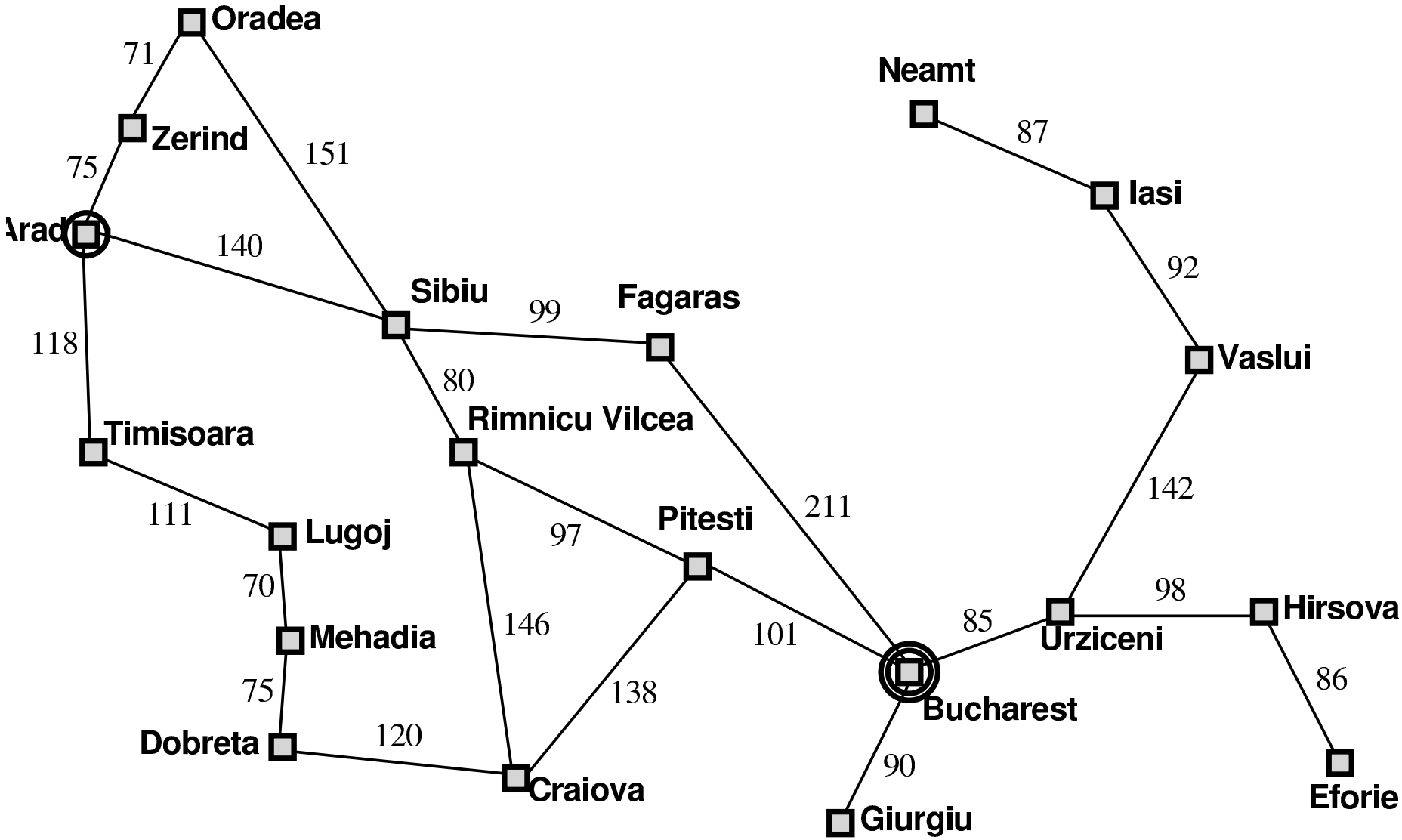
actions: drive between cities, $\mathcal{A} = \{\text{edges between states}\}$

Find solution:

sequence of cities, e.g., Arad, Sibiu, Fagaras, Bucharest

minimize costs with cost function, $(s, a) \mapsto c$

Example: Romania



Deterministic, fully observable search problem

A deterministic, fully observable search problem is defined by four items:

initial state $s_0 \in \mathcal{S}$ e.g., $s_0 = \text{Arad}$

successor function $succ : \mathcal{S} \times \mathcal{A} \to \mathcal{S}$

e.g., $succ(\text{Arad,Arad-Zerind}) = \text{Zerind}$

goal states $\mathcal{S}_{\text{goal}} \subseteq \mathcal{S}$

e.g., $s = \text{Bucharest}$

step cost function $cost(s, a, s')$, assumed to be ≥ 0

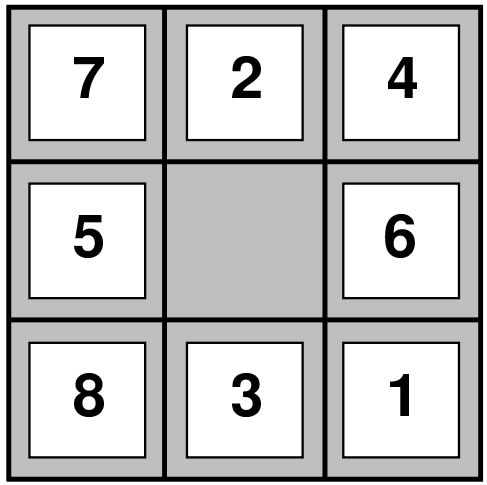
e.g., traveled distance, number of actions executed, etc.

the path cost is the sum of step costs

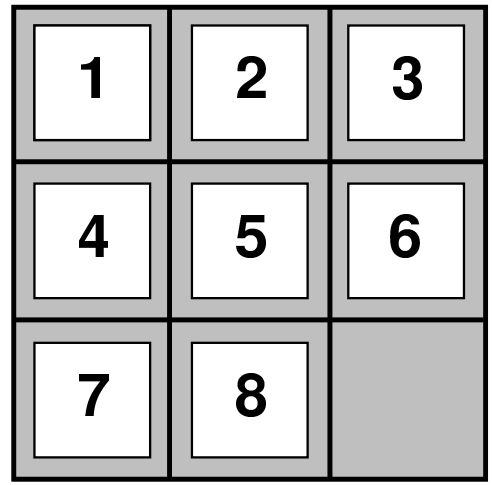
A solution is a sequence of actions leading from s_0 to a goal

An optimal solution is a solution with minimal path costs

Example: The 8-puzzle



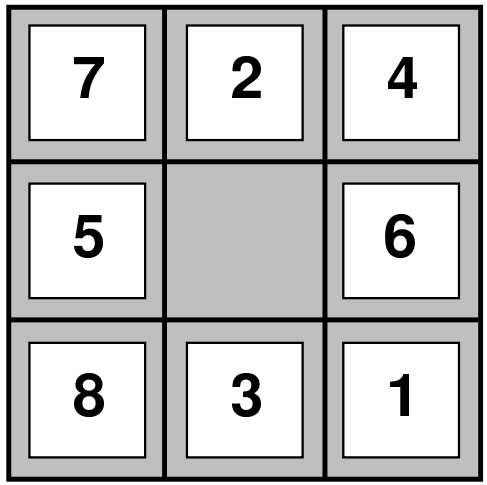
Start State



Goal State

states??:

Example: The 8-puzzle



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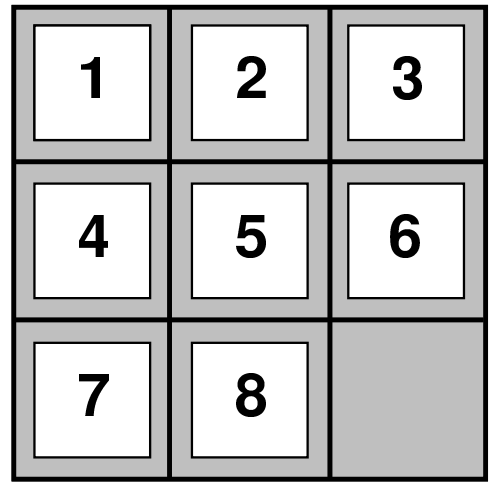
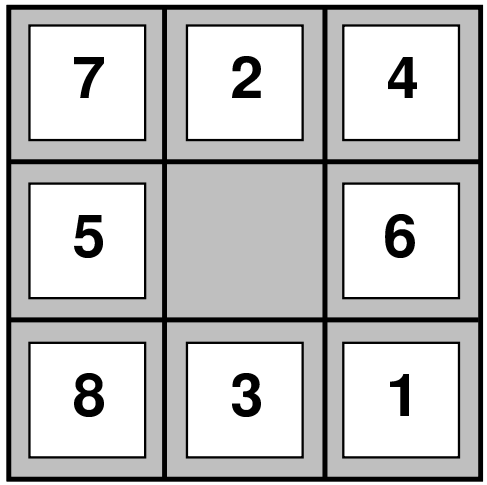


Goal State

states??: integer locations of tiles (ignore intermediate positions)

actions??:

Example: The 8-puzzle



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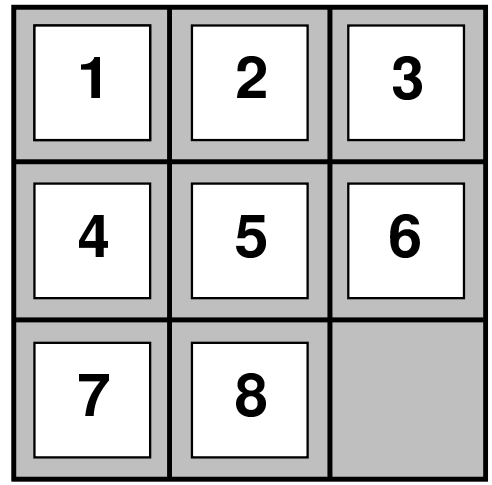


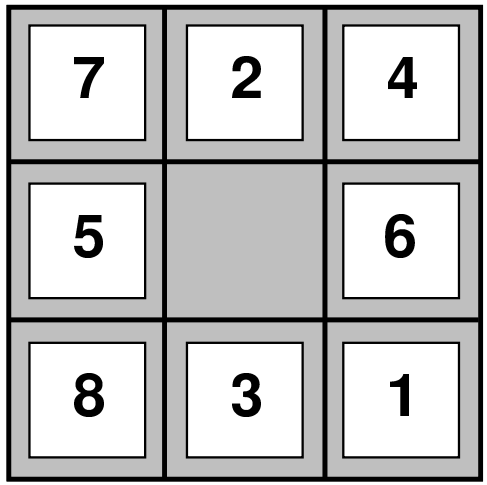
Goal State

states??: integer locations of tiles (ignore intermediate positions)

actions??: move blank left, right, up, down (ignore unjamming etc.)

goal test??:

Example: The 8-puzzle



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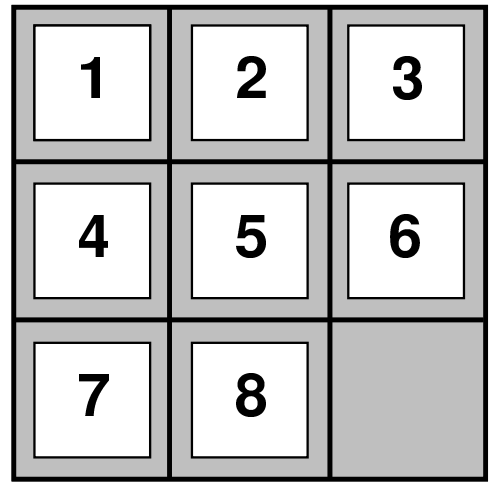

Goal State

states??: integer locations of tiles (ignore intermediate positions)

actions??: move blank left, right, up, down (ignore unjamming etc.)

goal test??: = goal state (given)

path cost??:

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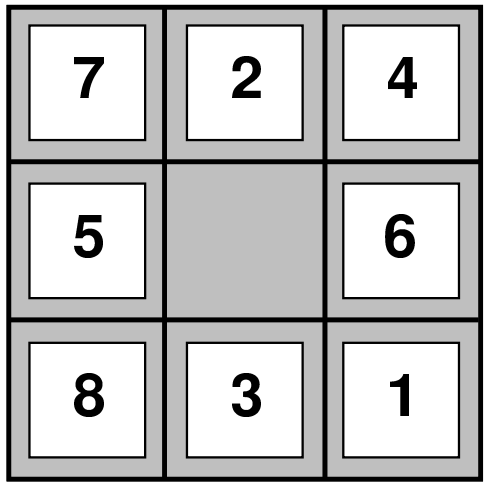

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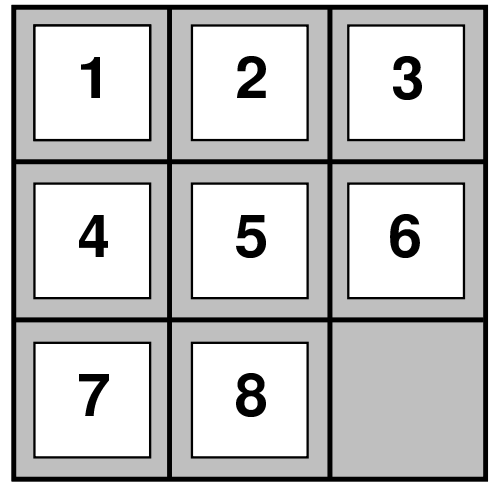


Goal State

states??: integer locations of tiles (ignore intermediate positions)

actions??: move blank left, right, up, down (ignore unjamming etc.)

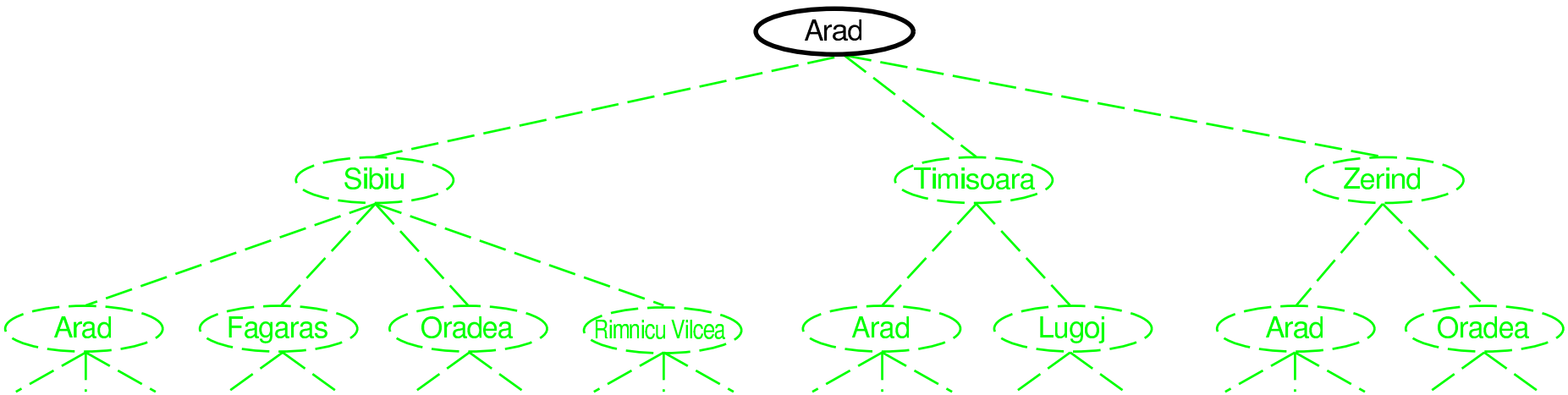
goal test??: = goal state (given)

path cost??: 1 per move

[Note: optimal solution of n-Puzzle family is NP-hard]

Basic Tree Search Algorithms

Tree search example



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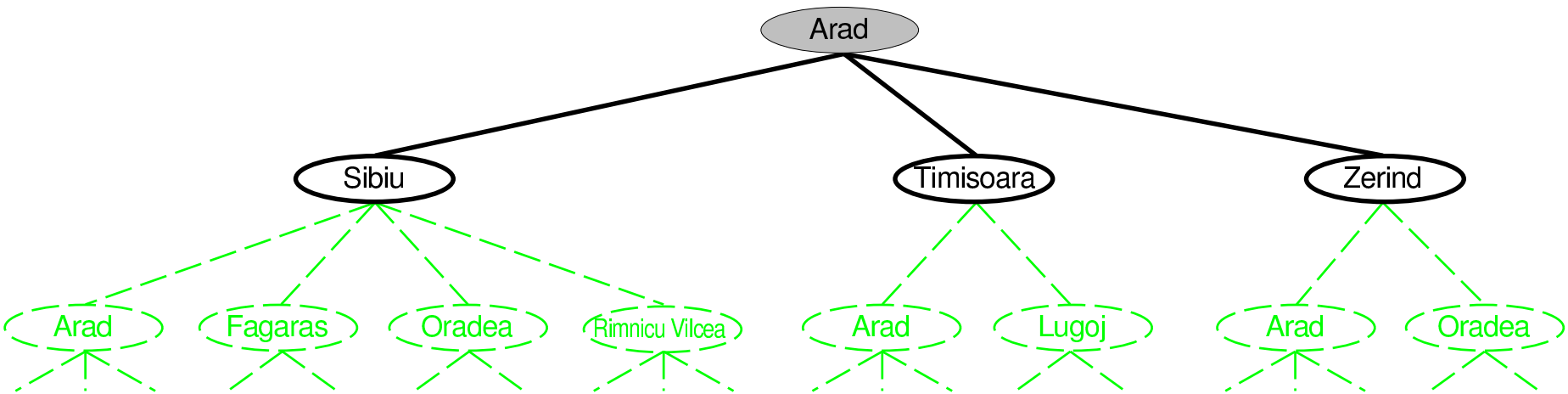

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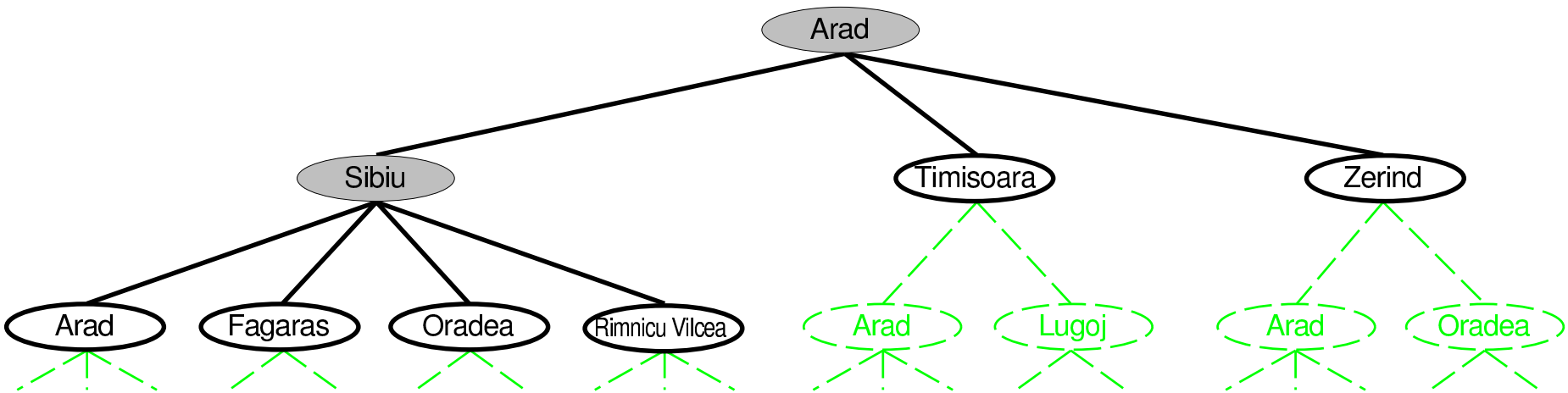
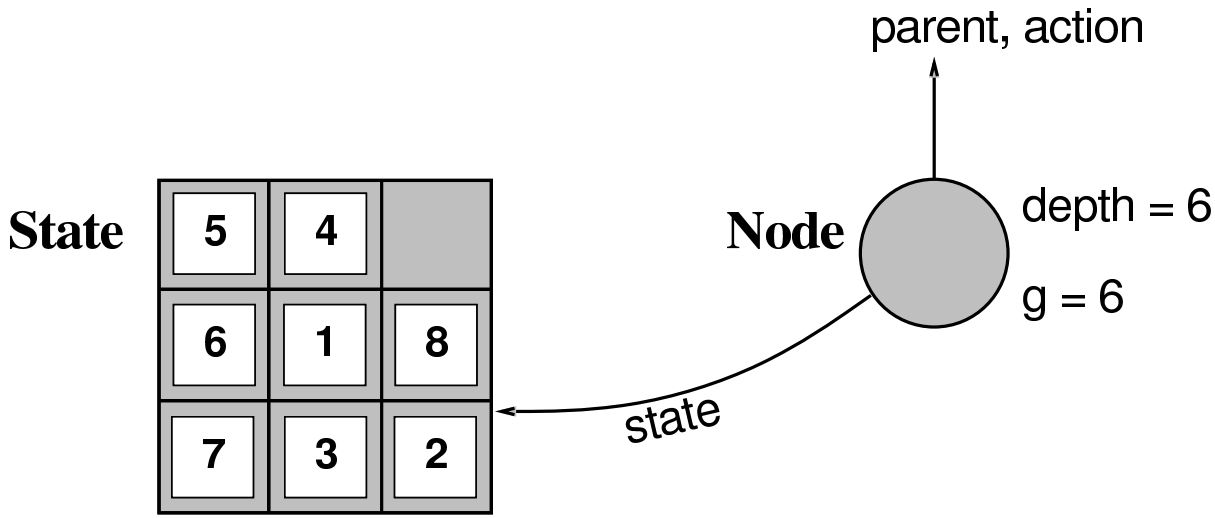


Implementation: states vs. nodes

- A state is a (representation of) a physical configuration
- A node is a data structure constituting part of a search tree
  includes parent, children, depth, path cost $g(x)$
  (States do not have parents, children, depth, or path cost!)



- The EXPAND function creates new nodes, filling in the various fields and using the SUCCESSORFN of the problem to create the corresponding states.

Implementation: general tree search

```
function TREE-SEARCH( problem, fringe) returns a solution, or failure
    fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
    loop do
        if fringe is empty then return failure
        node ← REMOVE-FRONT(fringe)
        if GOAL-TEST(problem, STATE(node)) then return node
        fringe ← INSERTALL(EXPAND(node, problem), fringe)
```

```
function EXPAND( node, problem) returns a set of nodes
    successors ← the empty set
    for each action, result in SUCCESSOR-FN(problem, STATE[node]) do
        s ← a new NODE
        PARENT-NODE[s] ← node;  ACTION[s] ← action;  STATE[s] ← result
        PATH-COST[s] ← PATH-COST[node] + STEP-COST(STATE[node], action, result)
        DEPTH[s] ← DEPTH[node] + 1
        add s to successors
    return successors
```

Search strategies

- A strategy is defined by picking the *ordering of the fringe*
- Strategies are evaluated along the following dimensions:
  - completeness—does it always find a solution if one exists?
  - time complexity—number of nodes generated/expanded
  - space complexity—maximum number of nodes in memory
  - optimality—does it always find a least-cost solution?
- Time and space complexity are measured in terms of
  - b = maximum branching factor of the search tree
  - d = depth of the least-cost solution
  - m = maximum depth of the state space (may be ∞)

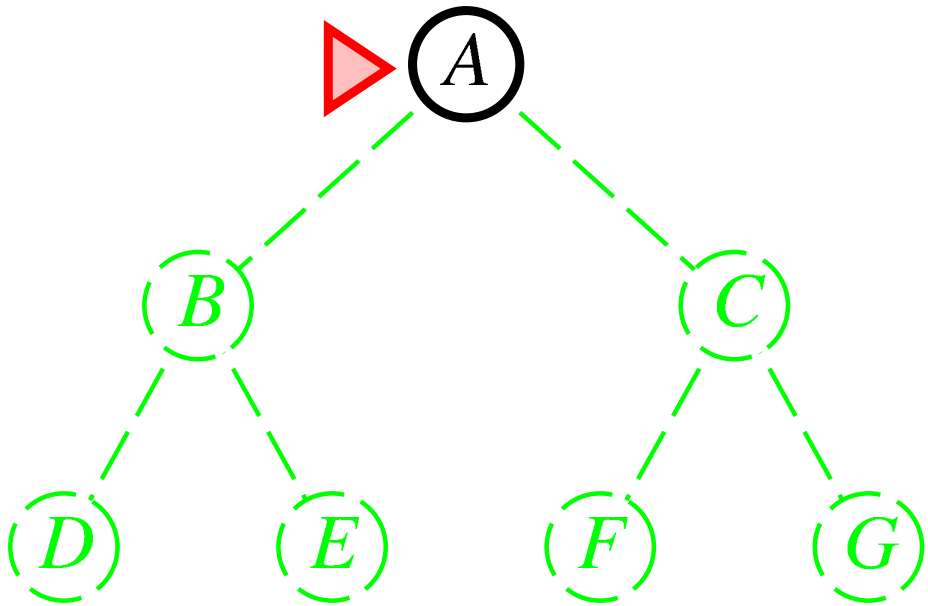
Summary of Search Strategies

- Breadth-first: fringe is a FIFO
- Depth-first: finge is a LIFO
- Iterative deepening search: repeat depth-first for increasing depth limit
- Uniform-cost: sort fringe by g
- A*: sort by $f = g + h$

Breadth-first search

- Pick shallowest unexpanded node
- *Implementation*:

  fringe is a **FIFO** queue, i.e., new successors go at end



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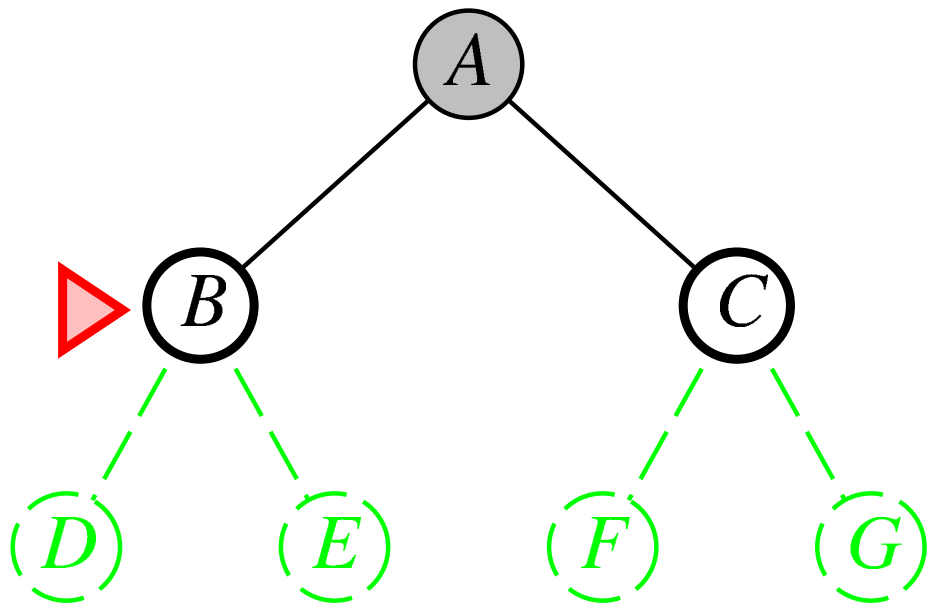

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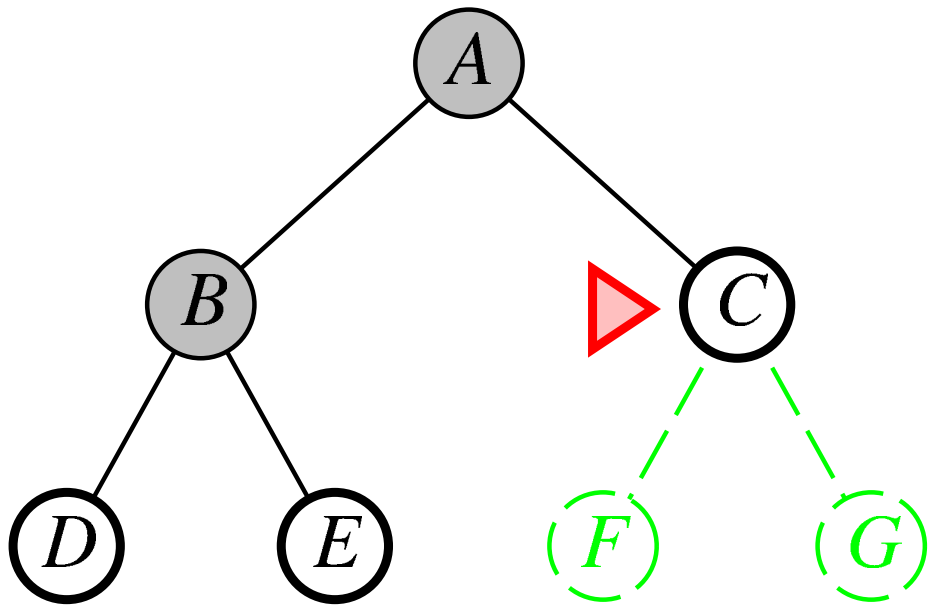

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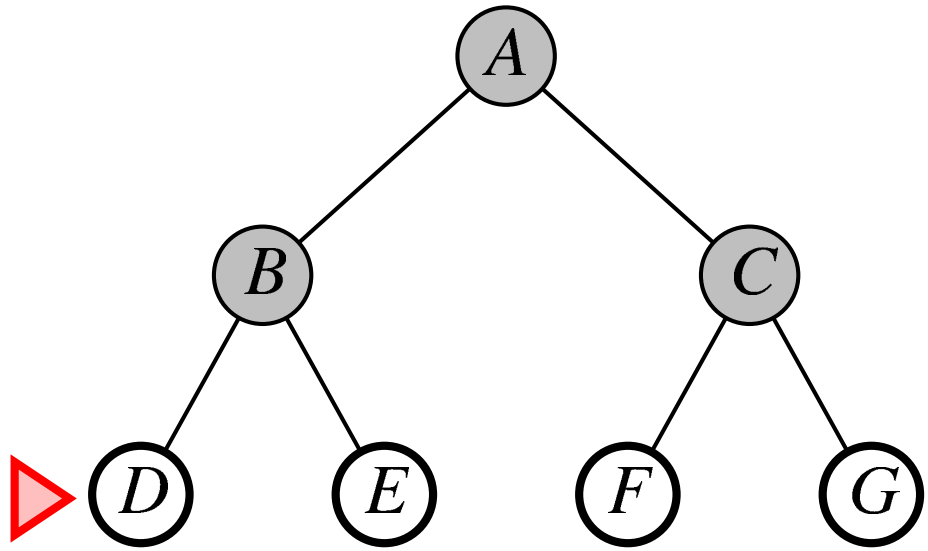

Properties of breadth-first search

Complete??

Properties of breadth-first search

Complete?? Yes (if b is finite)

Time??

Properties of breadth-first search

Complete?? Yes (if b is finite)

Time?? $1 + b + b^2 + b^3 + \ldots + b^d + b(b^d - 1) = O(b^{d+1})$, i.e., exp. in d

Space??

Properties of breadth-first search

Complete?? Yes (if b is finite)

Time?? $1 + b + b^2 + b^3 + \ldots + b^d + b(b^d - 1) = O(b^{d+1})$, i.e., exp. in d

Space?? $O(b^{d+1})$ (keeps every node in memory)

Optimal??

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Complete?? Yes (if b is finite)

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Space?? $O(b^{d+1})$ (keeps every node in memory)

Optimal?? Yes, if cost-per-step=1; not optimal otherwise

Space is the big problem; can easily generate nodes at 100MB/sec
so 24hrs = 8640GB.

Uniform-cost search

- *"Cost-aware BFS":* Pick least-cost unexpanded node
- *Implementation*:

  fringe = queue ordered by path cost, lowest first
- Equivalent to breadth-first if step costs all equal

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- Equivalent to breadth-first if step costs all equal

Complete?? Yes, if step cost $\geq \epsilon$

Time?? # of nodes with $g \leq$ cost-of-optimal-solution, $O(b^{\lceil C^*/\epsilon \rceil})$

where C^* is the cost of the optimal solution

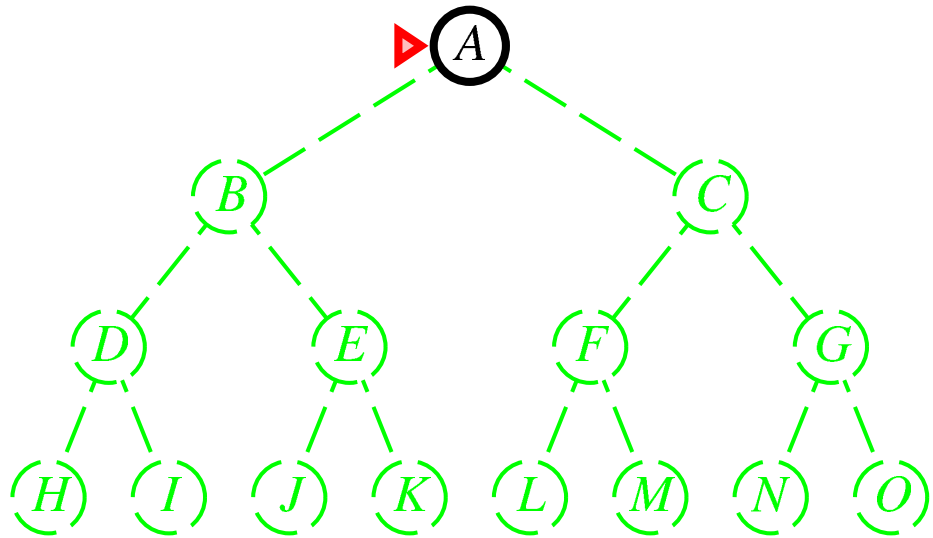
Space?? # of nodes with $g \leq$ cost-of-optimal-solution, $O(b^{\lceil C^*/\epsilon \rceil})$

Optimal?? Yes: nodes expanded in increasing order of $g(n)$

Depth-first search

- Pick deepest unexpanded node
- *Implementation*:

  fringe = **LIFO** queue, i.e., put successors at front



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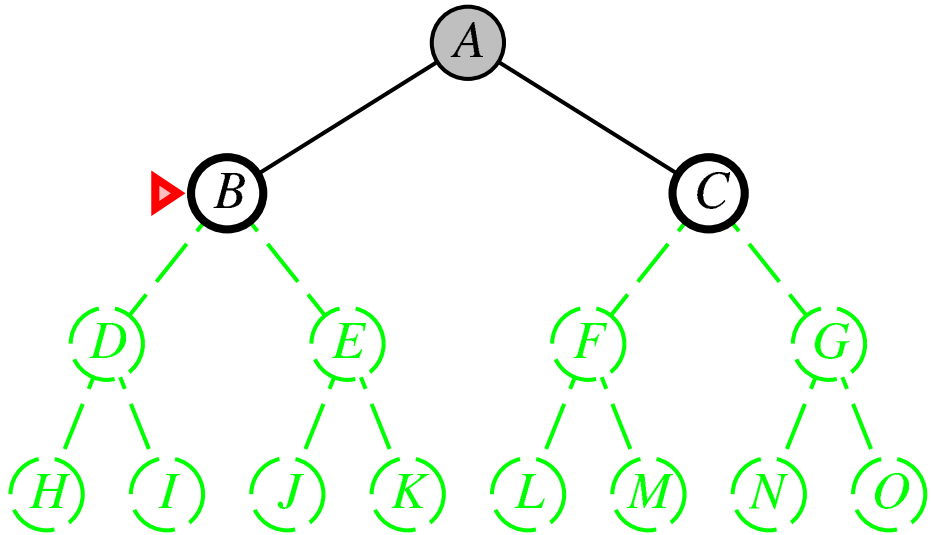

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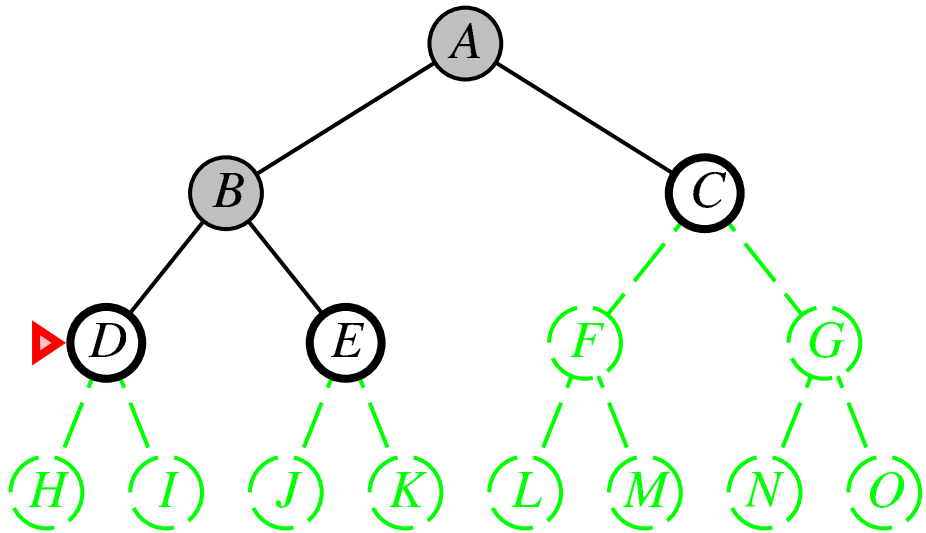

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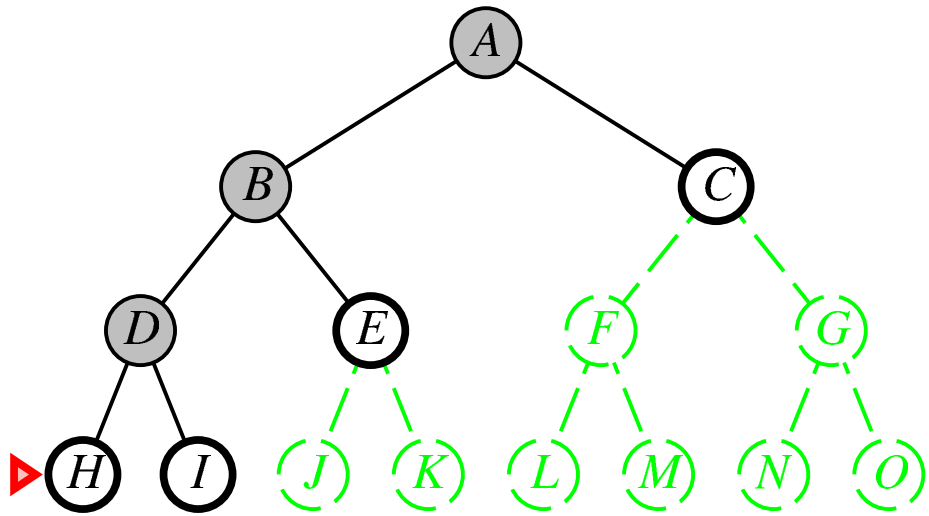

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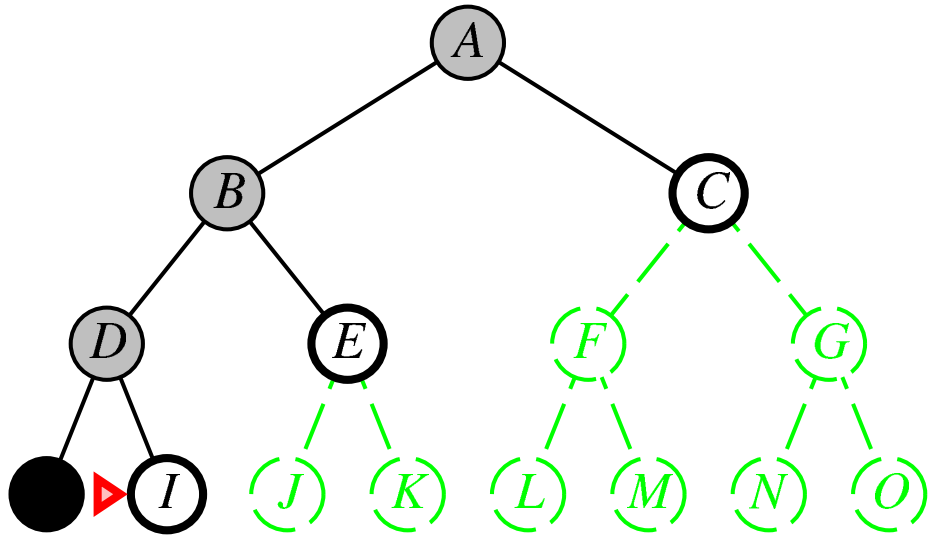

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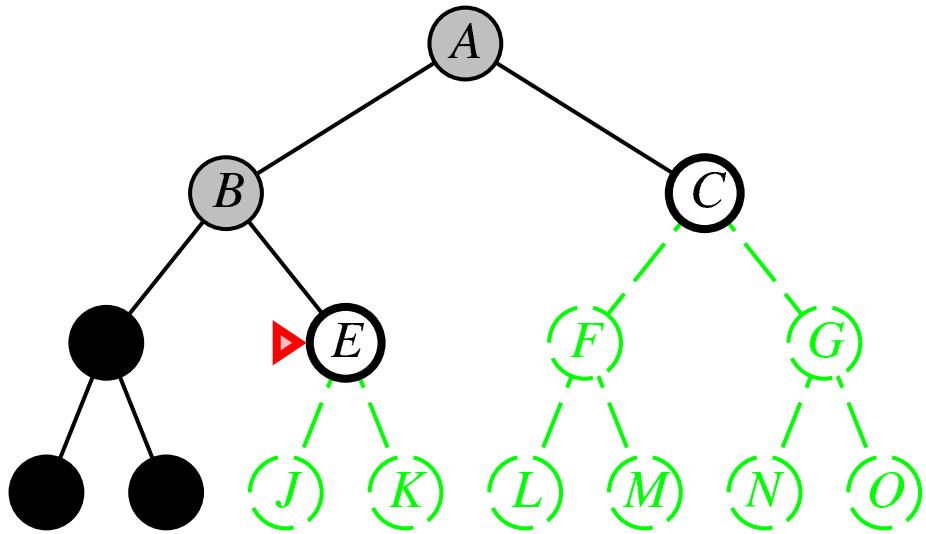

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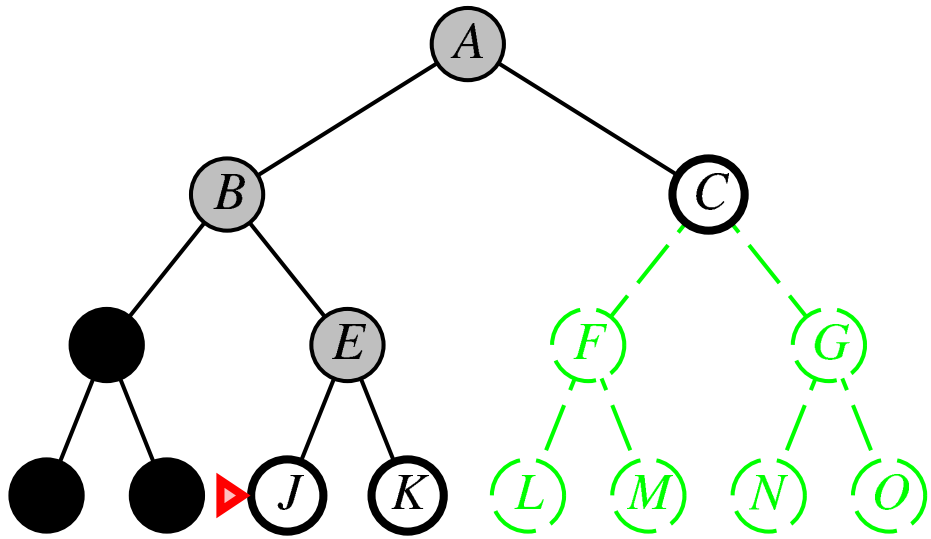

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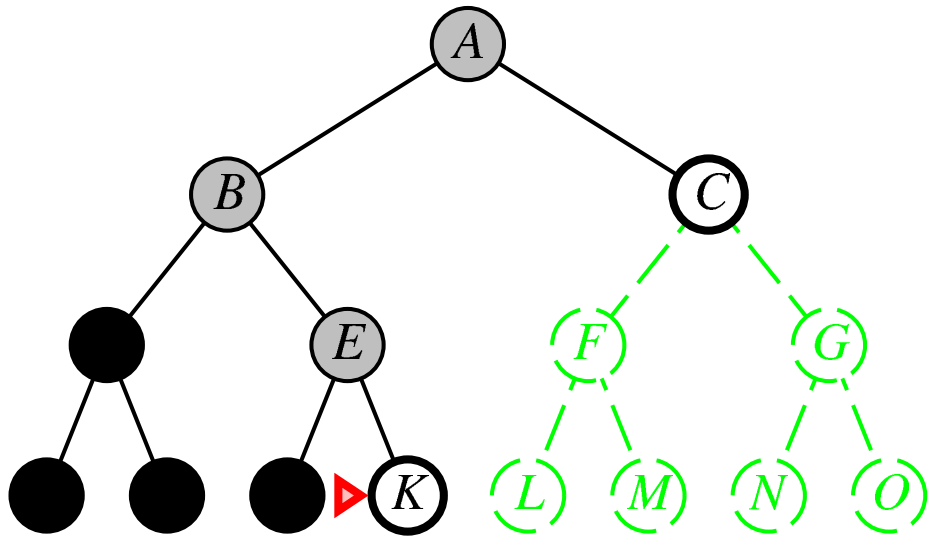

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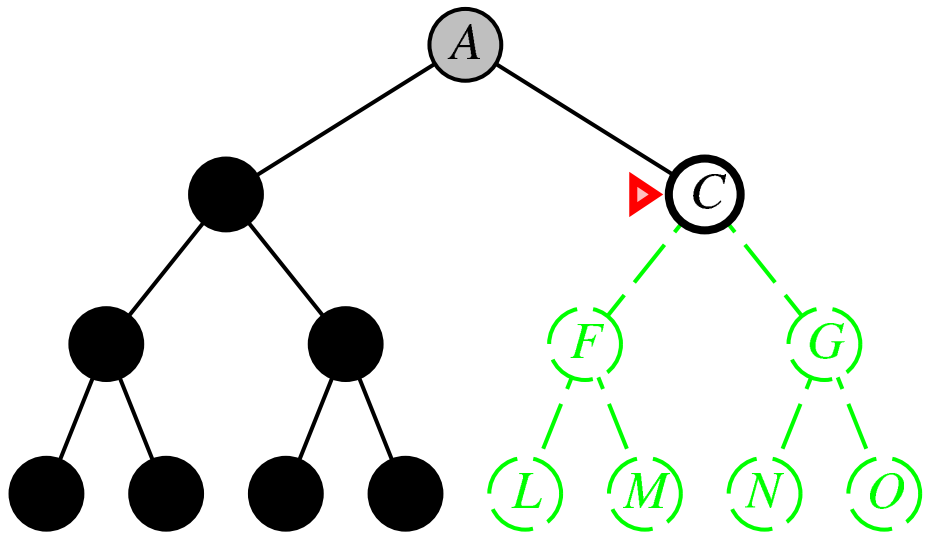

Properties of depth-first search

Complete??

Properties of depth-first search

Complete?? No: fails in infinite-depth spaces, spaces with loops

Modify to avoid repeated states along path $\Rightarrow$ complete in finite spaces

Time??

Properties of depth-first search

Complete?? No: fails in infinite-depth spaces, spaces with loops

Modify to avoid repeated states along path $\Rightarrow$ complete in finite spaces

Time?? $O(b^m)$: terrible if m is much larger than d

but if solutions are dense, may be much faster than breadth-first

Space??

Properties of depth-first search

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Time?? $O(b^m)$: terrible if m is much larger than d

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Space?? $O(bm)$, i.e., linear space!

Optimal??

Properties of depth-first search

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Time?? $O(b^m)$: terrible if m is much larger than d

but if solutions are dense, may be much faster than breadth-first

Space?? $O(bm)$, i.e., linear space!

Optimal?? No

Depth-limited search

- depth-first search with depth limit l,
  i.e., nodes at depth l have no successors
- *Recursive implementation* using the stack as LIFO:

```
function DEPTH-LIMITED-SEARCH( problem, limit) returns soln/fail/cutoff
  RECURSIVE-DLS(MAKE-NODE(INITIAL-STATE[problem]), problem, limit)

function RECURSIVE-DLS(node, problem, limit) returns soln/fail/cutoff
  cutoff-occurred? ← false
  if GOAL-TEST(problem, STATE[node]) then return node
  else if DEPTH[node] = limit then return cutoff
  else for each successor in EXPAND(node, problem) do
      result ← RECURSIVE-DLS(successor, problem, limit)
      if result = cutoff then cutoff-occurred? ← true
      else if result ≠ failure then return result
  if cutoff-occurred? then return cutoff else return failure
```

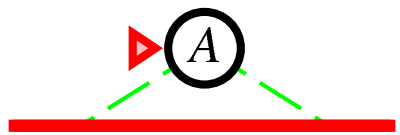
Iterative deepening search

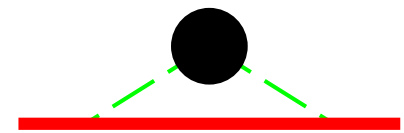
```
function ITERATIVE-DEEPENING-SEARCH( problem) returns a solution
    inputs: problem, a problem

    for depth ← 0 to ∞ do
        result ← DEPTH-LIMITED-SEARCH( problem, depth)
        if result ≠ cutoff then return result
    end
```

Iterative deepening search

Limit = 0

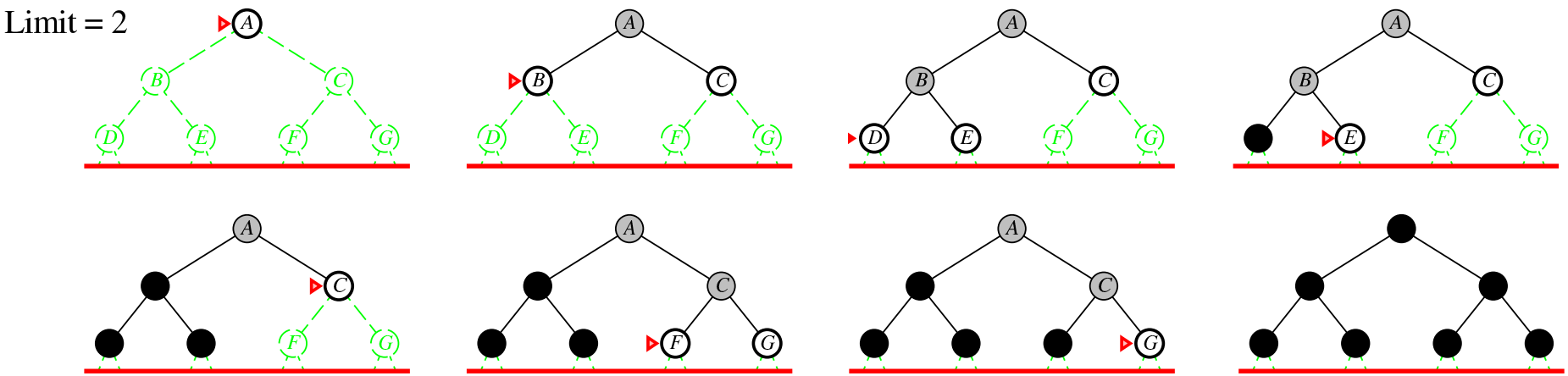




Iterative deepening search

Limit = 1

Iterative deepening search



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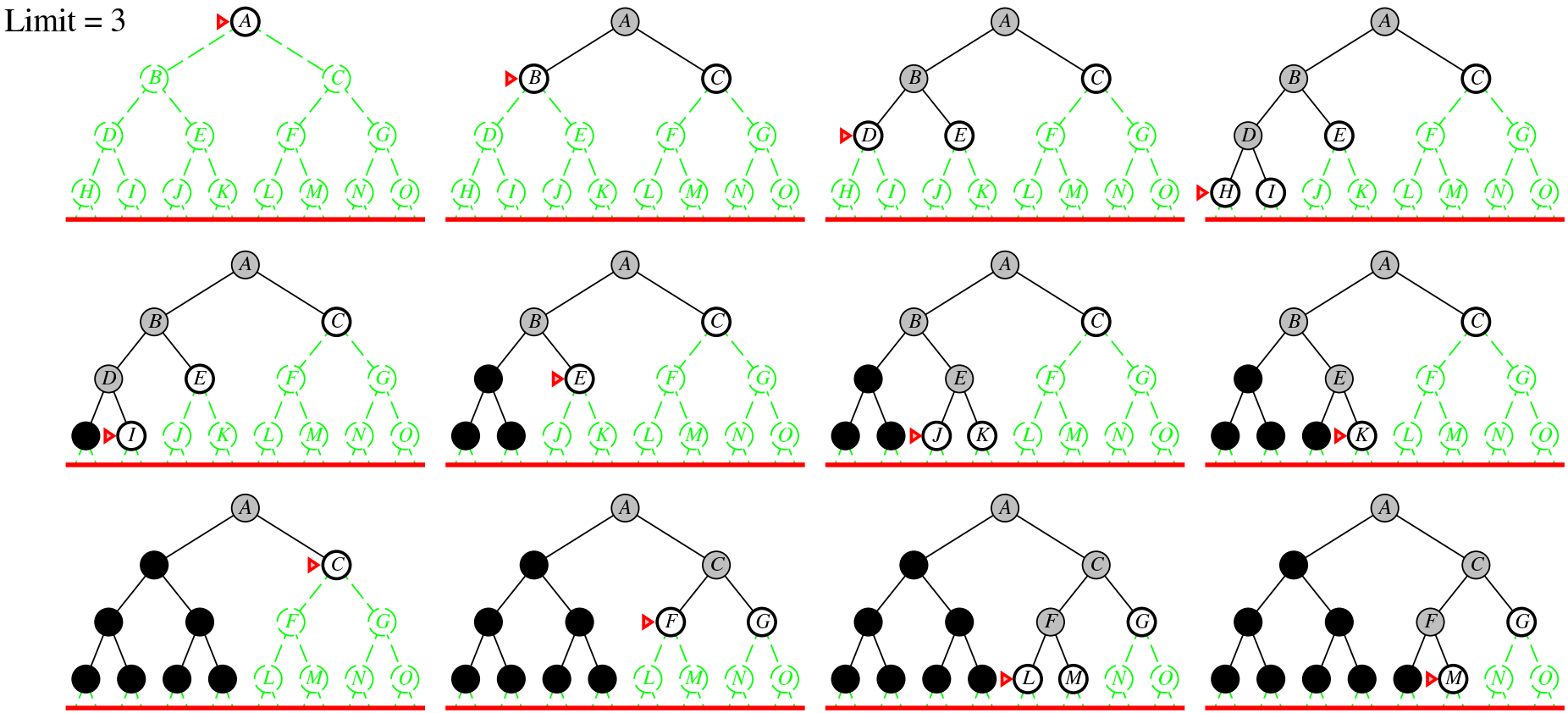


Properties of iterative deepening search

Complete??

Properties of iterative deepening search

Complete?? Yes

Time??

Properties of iterative deepening search

Complete?? Yes

Time?? $(d+1)b^0 + db^1 + (d-1)b^2 + \ldots + b^d = O(b^d)$

Space??

Properties of iterative deepening search

Complete?? Yes

Time?? $(d+1)b^0 + db^1 + (d-1)b^2 + \ldots + b^d = O(b^d)$

Space?? $O(bd)$

Optimal??

Properties of iterative deepening search

Complete?? Yes

Time?? $(d+1)b^0 + db^1 + (d-1)b^2 + \ldots + b^d = O(b^d)$

Space?? $O(bd)$

Optimal?? Yes, if step cost = 1

Can be modified to explore uniform-cost tree

- Numerical comparison for $b = 10$ and $d = 5$, solution at far left leaf:

$$N(\text{IDS}) = 50 + 400 + 3\,000 + 20\,000 + 100\,000 = 123\,450$$
$$N(\text{BFS}) = 10 + 100 + 1\,000 + 10\,000 + 100\,000 + 999\,990 = 1\,111\,100$$

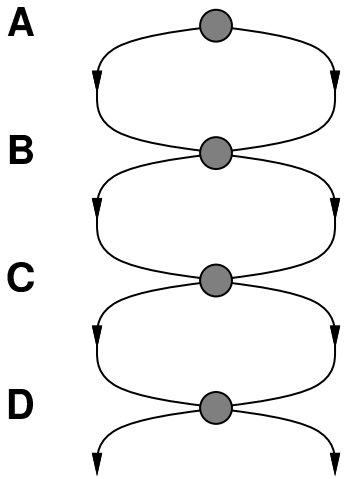
- IDS does better because other nodes at depth d are not expanded
- BFS can be modified to apply goal test when a node is *generated*

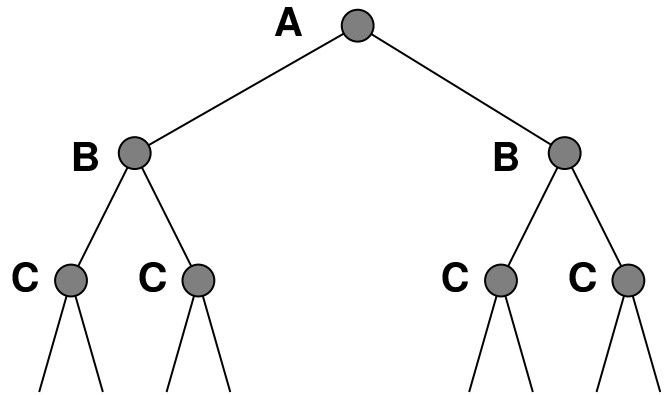
Summary of algorithms

Criterion	Breadth-First	Uniform-Cost	Depth-First	Depth-Limited	Iterative Deepening
Complete?	Yes*	Yes*	No	Yes, if $l \geq d$	Yes
Time	b^{d+1}	$b^{\lceil C^*/\epsilon \rceil}$	b^m	b^l	b^d
Space	b^{d+1}	$b^{\lceil C^*/\epsilon \rceil}$	bm	bl	bd
Optimal?	Yes*	Yes	No	No	Yes*

Loops: Repeated states

- Failure to detect repeated states can turn a linear problem into an exponential one!





Graph search

```
function GRAPH-SEARCH( problem, fringe) returns a solution, or failure

    closed ← an empty set
    fringe ← INSERT(MAKE-NODE(INITIAL-STATE[problem]), fringe)
    loop do
        if fringe is empty then return failure
        node ← REMOVE-FRONT(fringe)
        if GOAL-TEST(problem, STATE[node]) then return node
        if STATE[node] is not in closed then
            add STATE[node] to closed
            fringe ← INSERTALL(EXPAND(node, problem), fringe)
    end
```

But: storing all visited nodes leads again to exponential space complexity (as for BFS)

Summary

- In BFS (or uniform-cost search), the fringe propagates layer-wise, containing nodes of similar distance-from-start (cost-so-far), leading to optimal paths but exponential space complexity $O(b^{d+1})$
- In DFS, the fringe is like a deep light beam sweeping over the tree, with space complexity $O(bm)$. Iteratively deepening it also leads to optimal paths.
- Graph search can be exponentially more efficient than tree search, but storing the visited nodes leads to exponential space complexity as BFS.

A* Search

Best-first search

- Idea: use an arbitrary priority function $f(n)$ for each node
  - actually $f(n)$ is neg-priority: nodes with lower $f(n)$ have higher priority
- $f(n)$ should reflect which nodes *could* be on an optimal path
  - *could* is optimistic – the lower $f(n)$ the more optimistic you are that n is on an optimal path

  $\Rightarrow$ Pick the node with highest priority
- Implementation:

  fringe is a queue sorted with decreasing priority (increasing f-value)
- Special cases:
  - uniform-cost search ($f = g$)
  - greedy search ($f = h$)
  - A* search ($f = g + h$)

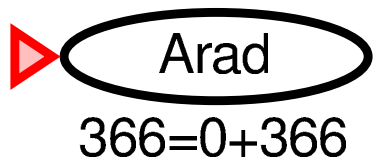
Uniform-Cost Search as special case

- Define $g(n)$ = cost-so-far to reach n
- Then Uniform-Cost Search is Prioritized Search with $f = g$

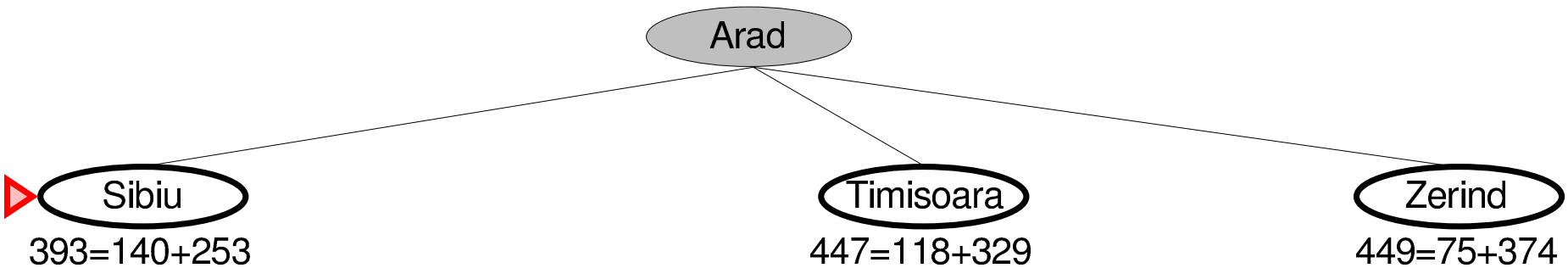
A* search

- Idea: *combine information from the past and the future*
  - neg-priority = cost-so-far + estimated cost-to-go
- The evaluation function is $f(n) = g(n) + h(n)$, with
  - $g(n)$ = cost-so-far to reach n
  - $h(n)$ = estimated cost-to-go from n
  - $f(n)$ = estimated total cost of path through n to goal
- A* search uses an admissible (=optimistic) heuristic
  - i.e., $h(n) \leq h^*(n)$ where $h^*(n)$ is the *true* cost-to-go from n.
  - (Also require $h(n) \geq 0$, so $h(G) = 0$ for any goal G.)
- E.g., $h_{\mathrm{SLD}}(n)$ never overestimates the actual road distance
- Theorem: A* search is optimal (=finds the optimal path)

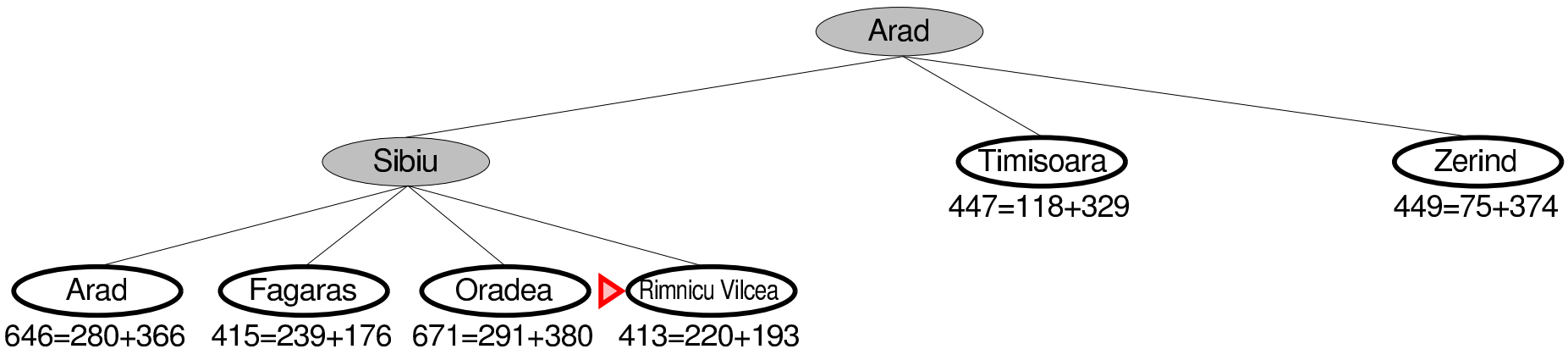
A* search example



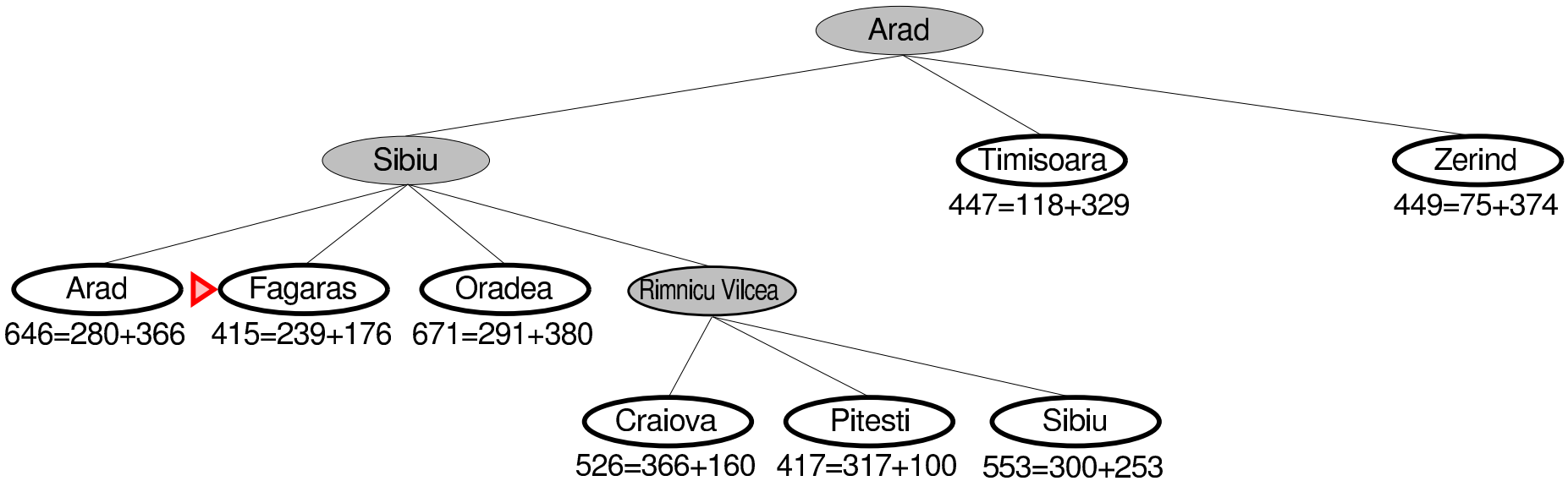
A* search example



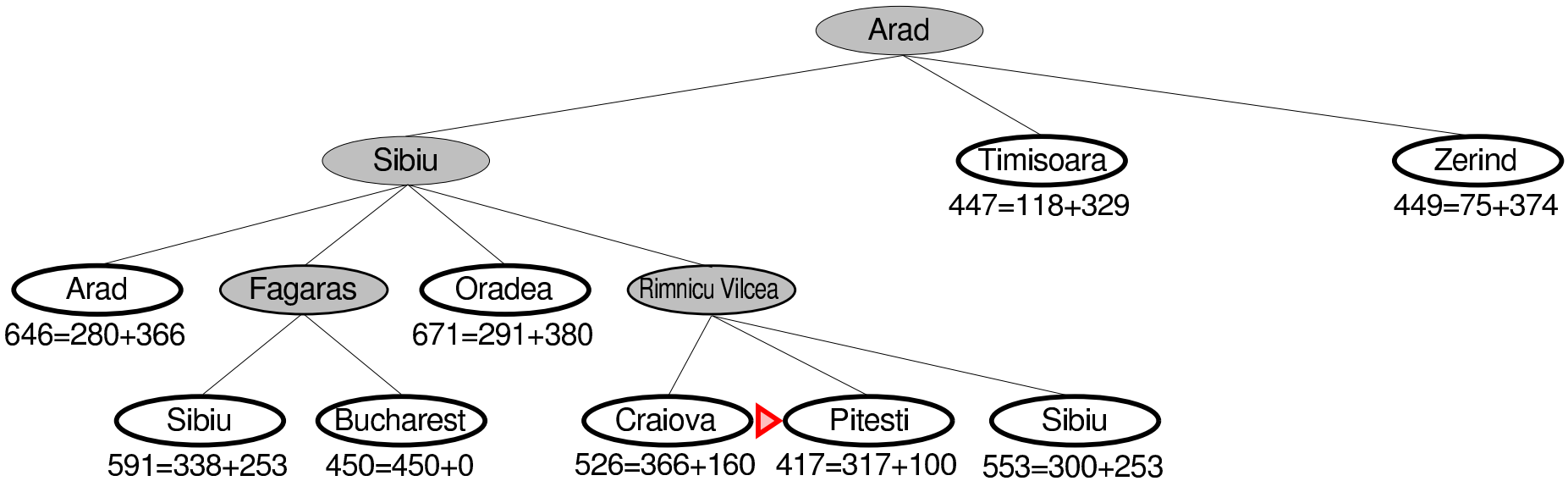
A* search example



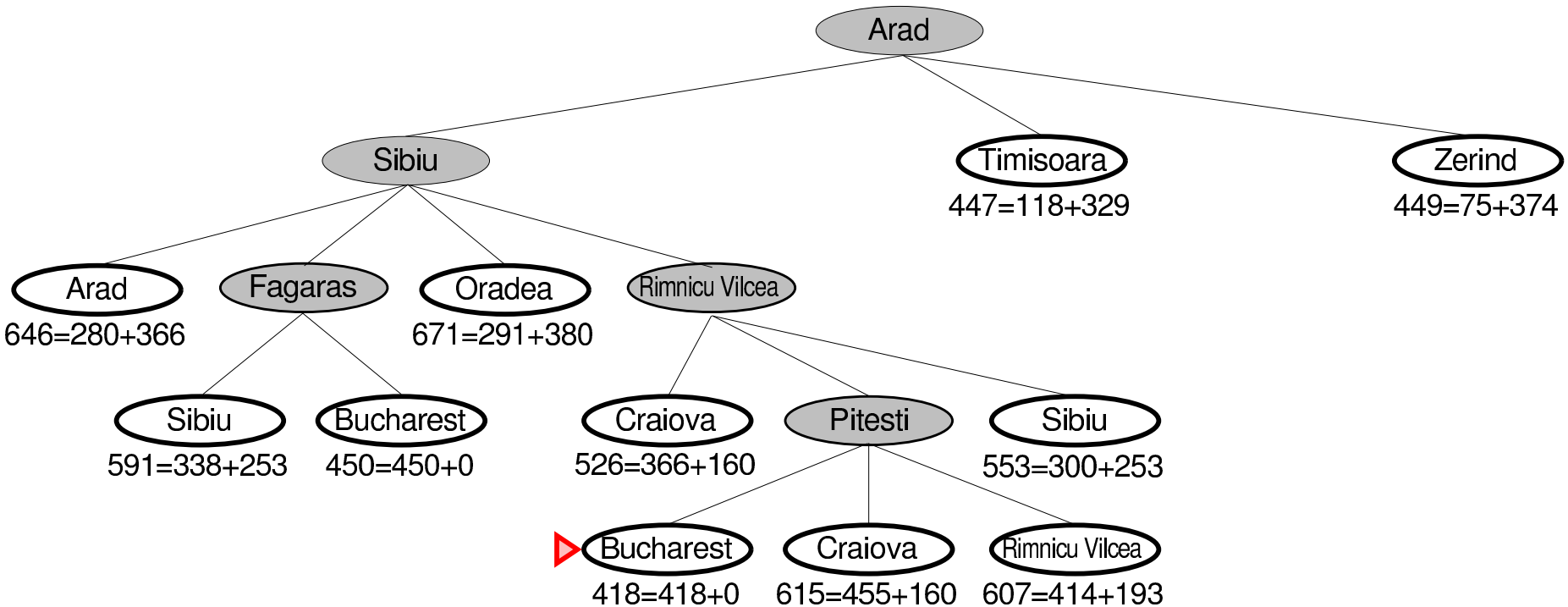
A* search example



A* search example



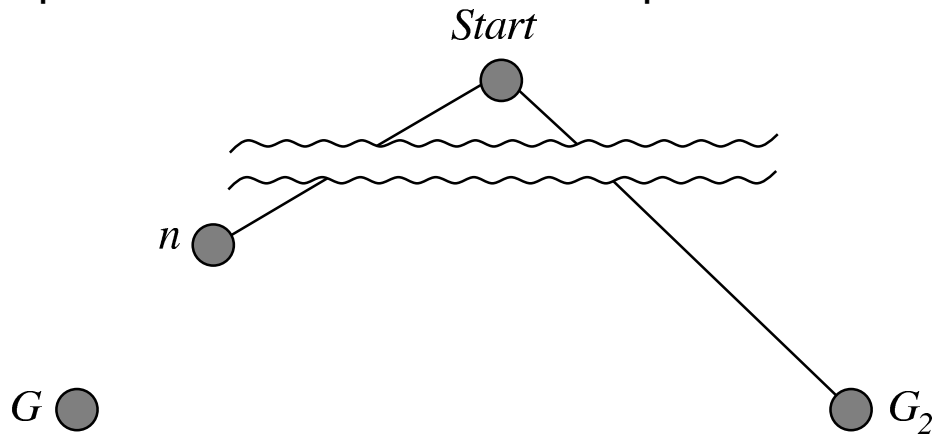
A* search example



Proof of optimality of A*

- Suppose some suboptimal goal G_2 has been generated and is in the fringe (but has not yet been selected to be tested for goal condition!). We want to proof: *Any node on a shortest path to an optimal goal G will be expanded before G_2.*

- Let n be an unexpanded node on a shortest path to G.



$$
\begin{aligned}
f(G_2) &= g(G_2) && \text{since } h(G_2) = 0 \\
&> g(G) && \text{since } G_2 \text{ is suboptimal} \\
&\geq f(n) && \text{since } h \text{ is admissible}
\end{aligned}
$$

- Since $f(n) < f(G_2)$, A* will expand n *before* G_2. This is true for any n on the shortest path. In particular, at some time G is added to the fringe, and since $f(G) = g(G) < f(G_2) = g(G_2)$ it will select G *before* G_2 for goal testing.

Properties of A*

Complete??

Properties of A*

Complete?? Yes, unless there are infinitely many nodes with $f \leq f(G)$

Time??

Properties of A*

Complete?? Yes, unless there are infinitely many nodes with $f \leq f(G)$

Time?? Exponential in [relative error in h $\times$ length of soln.]

Space??

Properties of A*

Complete?? Yes, unless there are infinitely many nodes with $f \leq f(G)$

Time?? Exponential in [relative error in h $\times$ length of soln.]

Space?? Exponential. Keeps all nodes in memory

Optimal??

Properties of A*

Complete?? Yes, unless there are infinitely many nodes with $f \leq f(G)$

Time?? Exponential in [relative error in $h \times$ length of soln.]

Space?? Exponential. Keeps all nodes in memory

Optimal?? Yes

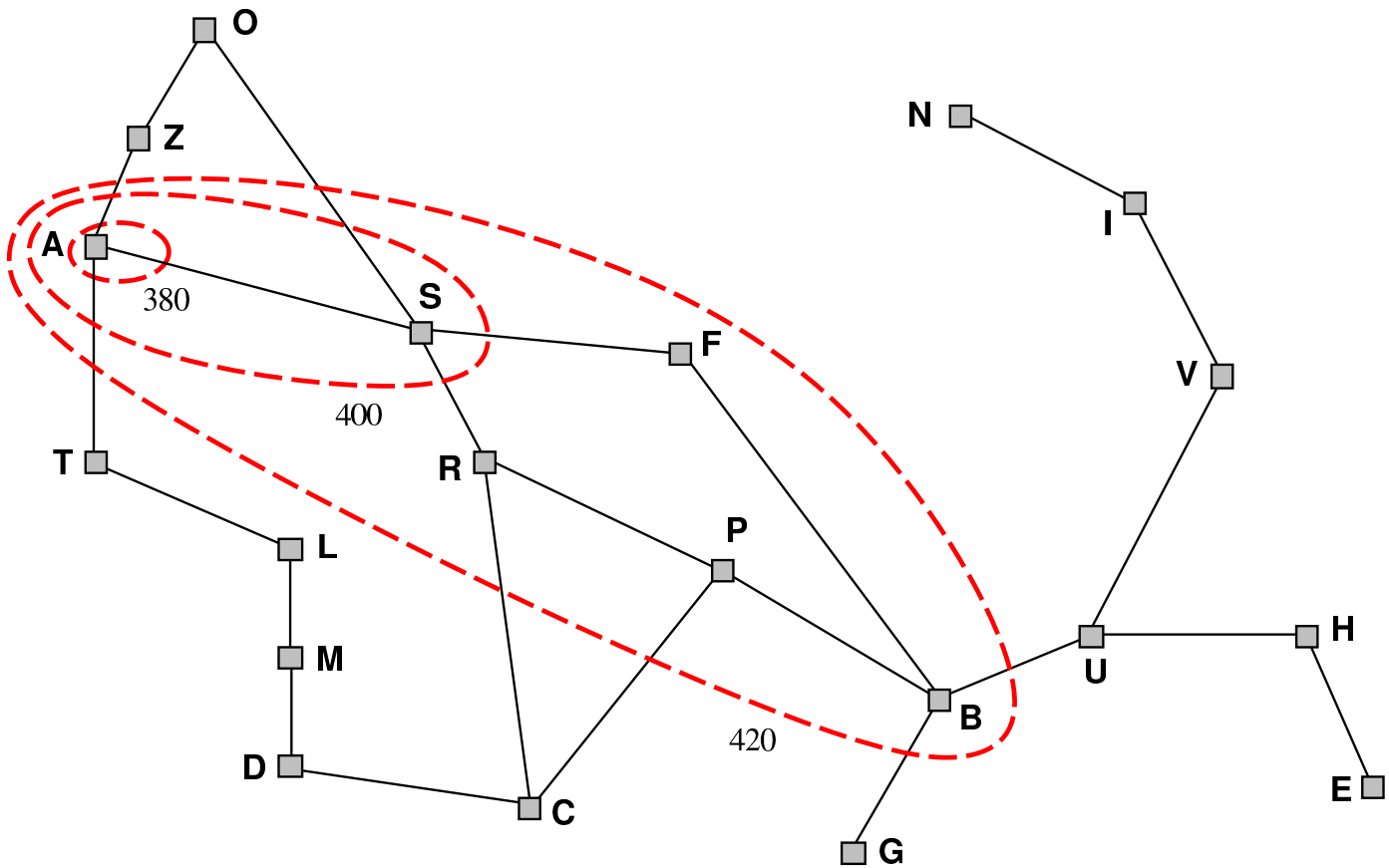
A* expands all nodes with $f(n) < C^*$

A* expands some nodes with $f(n) = C^*$

A* expands no nodes with $f(n) > C^*$

Optimality of A* (more useful)

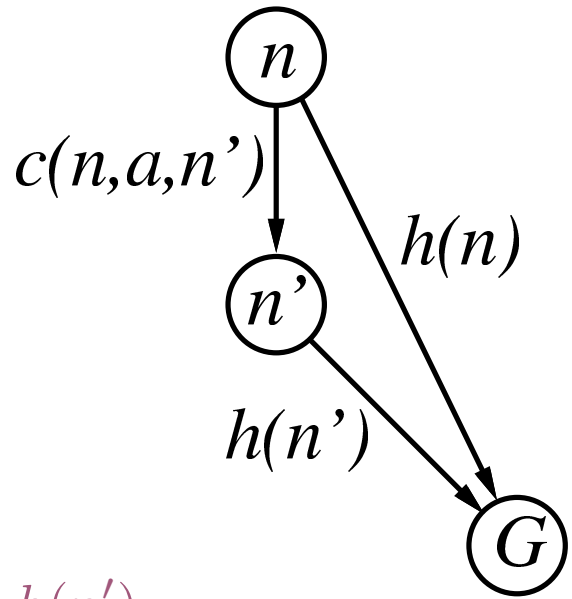
- Lemma: A* expands nodes in order of increasing f value*
  Gradually adds "f-contours" of nodes (cf. breadth-first adds layers)
  Contour i has all nodes with $f = f_i$, where $f_i < f_{i+1}$



Proof of lemma: Consistency

- A heuristic is consistent if

$$h(n) \leq c(n, a, n') + h(n')$$



- If h is consistent, we have

$$\begin{aligned} f(n') &= g(n') + h(n') \\ &= g(n) + c(n, a, n') + h(n') \\ &\geq g(n) + h(n) \\ &= f(n) \end{aligned}$$

I.e., $f(n)$ is nondecreasing along any path.

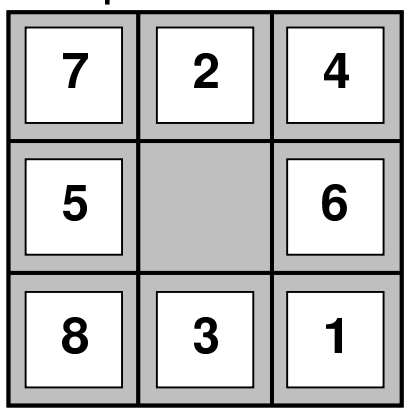
Admissible heuristics

E.g., for the 8-puzzle:

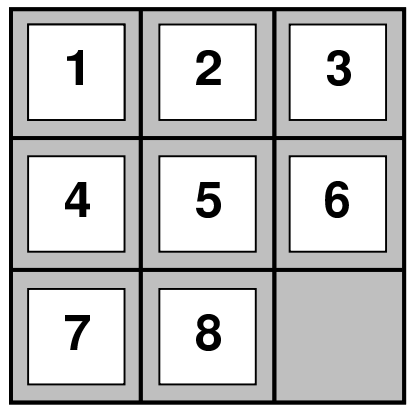
$h_1(n)$ = number of misplaced tiles

$h_2(n)$ = total Manhattan distance

(i.e., no. of squares from desired location of each tile)



Start State



Goal State

$h_1(S)$ =??

$h_2(S)$ =??

Admissible heuristics

E.g., for the 8-puzzle:

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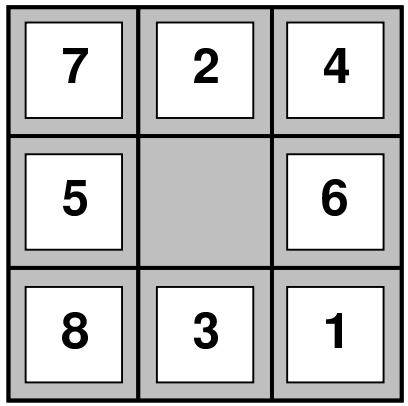
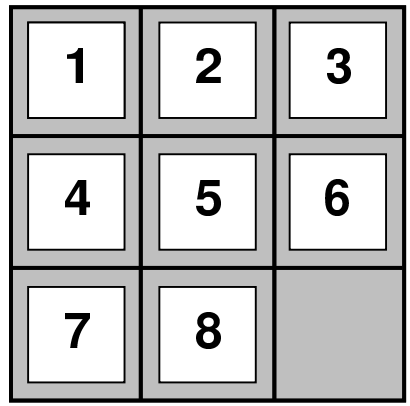


Start State



Goal State

$h_1(S) =$?? 6

$h_2(S) =$?? 4+0+3+3+1+0+2+1 = 14

Dominance

If $h_2(n) \geq h_1(n)$ for all n (both admissible)
then h_2 dominates h_1 and is better for search
Typical search costs:

$d = 14$ IDS = 3,473,941 nodes

A*(h_1) = 539 nodes

A*(h_2) = 113 nodes

$d = 24$ IDS $\approx$ 54,000,000,000 nodes

A*(h_1) = 39,135 nodes

A*(h_2) = 1,641 nodes

Given any admissible heuristics h_a, h_b,

$$h(n) = \max(h_a(n), h_b(n))$$

is also admissible and dominates h_a, h_b

Relaxed problems

- Admissible heuristics can be derived from the *exact* solution cost of a *relaxed* version of the problem
- If the rules of the 8-puzzle are relaxed so that a tile can move *anywhere*, then $h_1(n)$ gives the shortest solution
- If the rules are relaxed so that a tile can move to *any adjacent square*, then $h_2(n)$ gives the shortest solution
- Key point: the optimal solution cost of a relaxed problem is no greater than the optimal solution cost of the real problem

Memory-bounded A*

- As with BFS, A* has exponential space complexity
- Iterative-deepening A*, works for integer path costs, but problematic for real-valued
- (Simplified) Memory-bounded A* (SMA*):
  - Expand as usual until a memory bound is reach
  - Then, whenever adding a node, remove the *worst* node n' from the tree
  - worst means: the n' with highest $f(n')$
  - To not loose information, *backup* the measured step-cost $cost(\tilde{n}, a, n')$ to improve the heuristic $h(\tilde{n})$ of its parent

  SMA* is complete and optimal if the depth of the optimal path is within the memory bound

Summary

- Combine information from the past and the future
- A heuristic function $h(n)$ represents information about the future
  it estimates cost-to-go optimistically
- Good heuristics can dramatically reduce search cost
- A* search expands lowest $f = g + h$
  - neg-priority = cost-so-far + estimated cost-to-go
  - complete and optimal
  - also optimally efficient (up to tie-breaks, for forward search)
- Admissible heuristics can be derived from exact solution of relaxed problems
- Memory-bounded startegies exist

Outlook

- Tree search with *partial observations*
  - we discuss this in a fully probabilistic setting later
- Tree search for *games*
  - minimax extension to tree search
  - probabilistic Monte-Carlo tree search methods for games