Introduction to Optimization
Exercise 3

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1 Lagrangian and dual function

(Taken roughly from 'Convex Optimization', Ex. 5.1)
A simple example. Consider the optimization problem

$$\min x^2 + 1 \quad \text{s.t.} \quad (x-2)(x-4) \le 0$$

with variable $x \in \mathbb{R}$.
a) Give the feasible set, the optimal solution x^*, and the optimal value $p^* = f(x^*)$.
b) Write down the Lagrangian $L(x, \lambda)$. Plot (using gnuplot or so) $L(x, \lambda)$ over x for various values of $\lambda \ge 0$. You verify the lower bound property $\min_x L(x, \lambda) \le p^*$, where p^* is the optimum value of the primal problem.
c) Derive the dual function $l(\lambda)$ and plot it (for $\lambda \ge 0$). Derive the dual optimal solution $\lambda^* = \text{argmax}_\lambda\, l(\lambda)$. Is $\max_l l(\lambda) = p^*$ (strong duality)?

2 Phase I & Log Barriers

We again consider a constraint optimization problem very similar to the last exercise:

$$\min_x \sum_{i=1}^n x_i \quad \text{s.t.} \quad g(x) \le 0 \tag{1}$$

$$g(x) = \begin{pmatrix} x^\top x - 1 \\ -x_1 \end{pmatrix} \tag{2}$$

a) In the last exercise you've implemented basic constraint optimization methods (penalty or log barrier). Use these to find a feasible initialization (*Phase I*). Do this by solving the $n+1$-dimensional problem

$$\min_{(x,s) \in \mathbb{R}^{n+1}} s \quad \text{s.t.} \quad \forall_i :\ g_i(x) \le s,\ s \ge 0$$

Initialize this with the infeasible point $(1, 1) \in \mathbb{R}^2$.
b) Once you've found a feasible point, use the standard log barrier method to find the solution to the original problem (1). Start with $\mu = 1$, and decrease it by $\mu \leftarrow \mu/10$ in each iteration. In each iteration also report $\lambda_i := \frac{\mu}{g_i(x)}$ at the solution to the unconstraint problem $\min_x f(x) - \mu \sum_i \log(-g_i(x))$.