Convex Optimisation

Inofficial lecture notes
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Contents

Introduction

This course follows the book "Convex Optimisation" by Boyd and Vandenberghe.

We are interested in optimisation problems of the following form:

$$\begin{aligned} v^* = \inf_{\in \in \mathbb{R}} f_0(x) & \qquad \text{objective function} \\ \forall i = 1, \dots, n : f_i(x) \leq 0 & \qquad \text{constraints} \end{aligned}$$

Further x is the decision variable and v^* is the optimal value. We write inf instead of min to allow unbounded problems as well.
In general these problems are hard to solve, but we will focus on a restricted version. We start with Conic optimisation is a generalisation of linear programming.

$$\begin{aligned} v^* &= \inf c^T x = \sum_{i=1}^n c_i x_i \\ Ax &= b \\ x_i &\geq 0 \\ x &\succ_K 0 & \qquad x \in K, \text{ where } K \text{ is some cone} \end{aligned}$$

1 Preliminaries

1.1 Notation. We will use the following notation

$$\begin{aligned} [n] &= \{1, \dots, n\} \\ \mathbb{R}_+ &= \{x \in \mathbb{R} : x \geq 0\} \\ \mathbb{R}_{++} &= \{x \in \mathbb{R} : x > 0\} \\ \mathbb{S}^n &= \{X \in \mathbb{R}^{n \times n} : X^T = X\} \\ \mathbb{S}^n_+ &= \{X \in \mathbb{S}^n : X \text{is positive semidefinite}\} \\ \mathbb{S}^n_{++} &= \{X \in \mathbb{S}^n : X \text{is positive definite}\} \\ x \leq y &\Leftrightarrow \forall i \in [n]. x_{\leq} y_i \\ \mathbf{1} &= (1, \dots, 1)^T \\ \operatorname{Im} A &= \{Ax : x \in \mathbb{R}^n\} \\ \operatorname{Ker} A &= \{x \in \mathbb{R}^n : Ax = 0\} & \qquad \text{also called Nullspace} \\ \langle x, y \rangle &= \sum_{i=1}^n x_i y_i = x^T y & \qquad \text{if } x, y \in \mathbb{R}^n \\ \langle X, Y \rangle &= \sum_{i,j} X_{i,j} Y_{i,j} = \operatorname{tr}(X^T Y) & \qquad \text{for } X, Y \in \mathbb{R}^{n \times n} \\ \|x\| &= \sqrt{\langle x, x \rangle} \\ \|X\|_F &= \sqrt{\langle X, X \rangle} = \sqrt{\operatorname{tr} X^2} & \qquad \text{Frobenius norm} \end{aligned}$$

1.2 Definition. A *linear/affine function* is a function $f : \mathbb{R}^n \to \mathbb{R}$ of the form $f(x) = a^T x + b$. If $b = 0$, we say that f is a *linear form.*

1.3 Definition. A *quadratic function* is a function of the form $q(x) = x^T Q x + a^T x + b$. We can always assume that Q is symmetric, otherwise put $Q' = \frac{1}{2}(Q + Q^T)$, which still yields the same function.
A *quadratic form* is a function $q(x) = x^T Q x$.

1.4 Definition. • If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, we write $\nabla f : x \mapsto \left(\frac{\partial f}{\partial x_i}(x)\right)_{i\in[n]}$.

• If f is twice differentiable, $\nabla^2 f : x \mapsto \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right) \in \mathbb{S}^n$.

We usually will not distinguish between ∇f (function) and $\nabla f(x)$ (term).

1.5 Example. Given $u, v \in \mathbb{R}^n$. Then the map $X \mapsto u^T X v$ is a linear function of $X \in \mathbb{S}^n$. But we can also write

$$u^T X v = \operatorname{tr}(u^T X v) \overset{\operatorname{tr}(AB)=\operatorname{tr}(BA)}{=} \operatorname{tr}(X v u^T) = \langle X, v u^T \rangle$$

1.6 Remark (Random vectors). If X is a random vector with values in $\mathbb{R}^n$, then we have the *expected value* of X as $\mathbb{E}[X] = (\mathbb{E}X_i)_{i\in[n]}$. Recall from probability that $\mathbb{E}X = \int_{x\in\mathbb{R}^n} \mathbb{P}_X(\mathrm{d}x)$. The *variance-covariance matrix* of X is

$$\mathbb{V}[X] = \mathbb{E}[XX^T] - (\mathbb{E}X)(\mathbb{E}X)^T = \mathbb{E}\left[(X - \mathbb{E}X)(X - \mathbb{E}X)^T\right]$$

The latter expression shows that this matrix always is positive semidefinite. On the diagonal, we have the variance of X_i. All the other entries are the covariances of X_i and X_j.

1.7 Proposition. *If X is a random vector of $\mathbb{R}^n$, then $\mathbb{E}[AX + b] = A \cdot \mathbb{E}X + b$ and $\mathbb{V}[AX + b] = A \cdot \mathbb{V}[X] \cdot A^T$.*

Proof. by computation □

1.8 Example. let x be a random vector. The function $f(Q) = \mathbb{E}\left[X^T Q X\right]$ is linear in $Q \in \mathbb{S}^n$. We use $X^T Q X = \langle Q, XX^T \rangle$, so $f(Q) = \langle Q, \mathbb{E}[XX^T] \rangle = \langle Q, \Sigma + \mu\mu^T \rangle$.

missing lecture

Now assume we have a set $S = \{x_1, \ldots, x_4\}$ of 4 points in the plane and one interior point x. We want to write $x = \sum_{i=1}^k \lambda_i x_i$ with $k > m + 1$. We can find some vector $\mu \neq 0$ such that

$$\sum_{i=1}^k \mu_i x_i = 0 \qquad \sum_{i=1}^k \mu_i = 0$$

For all $\alpha \in \mathbb{R}$ we have

$$x = \sum_{i=1}^k (\lambda_i - \alpha\mu_i) x_i$$

If we choose $\alpha = \min\left\{\frac{\lambda_i}{\mu_i} : i \in [k], \mu_i > 0\right\}$, then by choosing $\lambda_i' := \lambda_i - \alpha\mu_i$ one coefficient vanishes and we have reduced their number. This we can iterate, until we found a minimum simplex.
IMPORTANT: This can be used for a more controlled way to find a cover!!

1.9 Corollary. *Let $\subseteq \mathbb{R}^n$. Then*

$$\operatorname{conv}(S) = \left\{\sum_{i=0}^n \lambda_i x_i : x_0, \ldots, x_n \in S, \sum \lambda_i = 1, \lambda_i \geq 0\right\}$$

I.e. the points in the convex hull are combinations of at most $n + 1$ points.

1.10 Theorem (Caratheodory's Theorem for Convex Cones). *Let $S \subseteq \mathbb{R}^n$ with $\dim S = m \leq n$. Then any $x \in \operatorname{cone} S$ can be expressed as a combination of at most m points of S.*

1.11 Example. Let $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.

- The set $H = \{x : a^T x = b\}$ is a hyperplane. It is affine, so also convex.
- $H^+ := \{x : a^T x \geq b\}$ is a *half-space*. It is convex, but not affine.
- For any norm $B := \{x : \|x - a\| \leq b\}$ is a *ball*, which is convex.
- The set $\{(x, b) \in \mathbb{R}^{n+1} : \|x - a\| \leq b\}$ is a *norm cone.*
- As a particular case we have the Lorentz cone $\{(x, b) \in \mathbb{R}^{n+1} : \|x\|_2 \leq b\}$.
- The *unit simplex* of $\mathbb{R}^n$ is $\Delta_n := \{X \in \mathbb{R}^n_+ : \sum x_i \leq 1\} = \operatorname{conv}(0, e_1, \ldots, e_n)$.
- The *probability simplex* is $\Delta_n^= := \{X \in \mathbb{R}^n_+ : \sum x_i = 1\} = \operatorname{conv}(e_1, \ldots, e_n)$. Note that we have replaced the inequality by an equality an dropped the zero in the hull. This is a convex set of dimension $n - 1$.
- The set $\mathbb{R}^n_+$ is the non-negative orthant, which is a convex cone.
- The set of positive definite matrices $\mathbb{S}^n_+$ is a convex cone of (affine) dimension $\binom{n+1}{2}$.

2 Operations that preserve convexity

To allow us some kind of inductive treatment of convex sets, we are interested, which operations preserve convexity.

Intersection: if A, B convex, then $A \cap B$ convex. More generally, if A_i convex, then $\bigcap_{i \in I} A_i$ is convex.

As an example we can write

$$\mathbb{S}^n_+ = \left\{X : \forall x \in \mathbb{R}^n . x^T X x \geq 0\right\} = \bigcap_{x \in \mathbb{R}^n} \left\{X : \langle X, xx^T \rangle \geq 0\right\}$$

as an intersection of infinitely many half-spaces.

Cartesian product: if A, B convex, then $A \times B$ is convex.

Affine transformation: If S is convex, $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, then $\{Ax + b : s \in S\}$ is convex. But also the preimage $\{x : Ax + b \in S\}$ is convex. Particular cases include

- rotations, A is orthogonal
- scaling $b = 0$
- translations $A = I_n$
- Minkowski sum $A + B = \{x + y : x \in A, y \in B\}$.

Closure/Interior: If S is convex, then also $\operatorname{cl}(S)$ and $\operatorname{int}(S)$ are convex.

Perspective transformation: Define $P : \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}^n$ via $(x, t) \mapsto \frac{x}{t}$. If S is convex, then $P(S)$ is convex. If S is convex, then $P^{-1}(S)$ is a convex cone. (Basically, $P^{-1}(S)$ are all scalings of S.)

3 Positive Semidefinite Matrices

3.1 Proposition. *TFAE*

1. $X \in \mathbb{S}^n_+$ *(or* $X \succeq 0$*, i.e.* X *is positive semidefinite)*
2. $\forall u \in \mathbb{R}^n . u^T X u \geq 0$
3. *all eigenvalues of* X *are non-negative.*
4. *There is some matrix* H *such that* $X = HH^T$.
5. $X \in \operatorname{conv}(\{xx^T : x \in \mathbb{R}^n\})$

Proof. **item 1⇔item 2** by definition

item 2⇒item 3 Let u be an eigenvector of X with corresponding eigenvalue $\lambda \in \mathbb{R}$. Then $0 \leq u^T X u = \lambda u^T u = \lambda \|u\|^2$, so $\lambda \geq 0$.

item 3⇒item 4 We can decompose $X = Q \operatorname{Diag}(\lambda_1, \ldots, \lambda_n) Q^T$ with Q orthogonal. Then take $H^T := Q \operatorname{Diag}(\sqrt{\lambda_i})$.

item 4⇒item 5 Let h_i be the i-th column of H. Then we can write $X = \sum h_i h_i^T$. Thus $X \in \operatorname{cone}(\{u^T : u \in \mathbb{R}^n\})$. But the form

$$X = \sum_{i=1}^{n} \frac{1}{n} (\sqrt{n} h_i)(\sqrt{n} h_i)^T$$

is a convex combination of the form uu^T with $u \in \mathbb{R}^n$. So X lies in the convex hull.

item 5⇒item 2 Let $X = \sum \lambda_i x_i x_i^T$ with $\lambda_i \geq 0$. Then

$$\forall u \in \mathbb{R}^n . u^T X u = \sum_{i=1}^{n} \lambda_i u^t x_i x_i^t u = \sum_{i=1}^{n} \lambda_i (u^T x_i)^2 \geq 0$$ □

The interior of $\mathbb{S}^n_+$ is $\mathbb{S}^n_{++}$, the set of positive definite matrices.

3.2 Proposition. *TFAE*

1. $X \in \mathbb{S}^n_{++}$*, written* $X \succ 0$
2. $\forall u \in \mathbb{R}^n \setminus \{0\} . u^T X u > 0$
3. *All eigenvalues of* X *are positive.*
4. *(Sylvester criterion) All leading principal minors of* X *are positive.*

3.3 Lemma. *If* X *is positive semidefinite, then*

a) All principal submatrices are positive semidefinite.

b) $\forall i, j \in [n] . |X_{ij}| \leq \sqrt{X_{ii} \cdot X_{jj}}$

c) $X_{ii} = 0 \implies \forall j . X_{ij} = 0$

d) AXA^T *is positive semidefinite for every* $A \in \mathbb{R}^{n \times n}$.

3.4 Proposition. *If X is positive semidefinite, then X has a positive semidefinite square root, i.e. $\exists Z \in \mathbb{S}^n_+ . X = Z^2$.*

Proof. See proof of Proposition 3.1. $\square$

3.5 Proposition (Cholesky). *If $X \succeq 0$, then $X = LL^T$ for some lower triangular matrix L.*

4 Generalised inequalities and Dual cone

4.1 Definition. A cone K is *proper*, if

- K is convex and closed
- K has non-empty interior
- K is pointed, i.e. it contains no line. Equivalently $\pm x \in K \Leftrightarrow x = 0$.

4.2 Definition. We write $x \preceq_K y$ for $y - x \in K$ and $x \prec_K y$ for $y - x \in \operatorname{int} K$.

4.3 Example.
- $x \preceq_{\mathbb{R}^n_+} y \Leftrightarrow \forall i \in [n]. x_i \leq y_i$, which we will write as $x \leq y$.
- $X \preceq_{\mathbb{S}^n_+} Y \Leftrightarrow Y - X$ is positive semidefinite. Note that this does not imply an elementwise inequality.

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \succeq 0$$

  Also we will omit the index at $\preceq$, when we mean psd-matrices.

- Let $K = \left\{ \alpha \in \mathbb{R}^{d+1} : \forall x \in [0,1]. \sum_{i=0}^d \alpha_i x^i \geq 0 \right\}$ Then $\alpha \preceq_K \beta$ iff $\forall x \in [0,1]. \sum \alpha_i x^i \leq \sum \beta_i x^i$. We also say the polynomial defined by β dominates over the one defined by α over $[0,1]$.

  K is a proper cone.

  cone: $\sum \alpha_i x^i \geq 0 \implies \sum t\alpha_i x x^i \geq 0$.

  closed: since it is defined by $\leq$

  convex: Let $\alpha, \beta \in K$, then $\sum (t\alpha_i + (1-t)\beta_i) x^i \geq 0$.

  nonempty interior: Choose $\alpha = (1, 0, \ldots)$. Then any perturbation by an $\varepsilon \leq \frac{1}{d}$ still remain non-negative.

  pointed: Let $\alpha, -\alpha \in K$. This means $\sum \alpha_i x^i \geq 0$ and $\sum \alpha_i x^i \leq 0$. By interpolation we get that all $\alpha_i = 0$.

Remark. *Recall that a matrix X is psd iff $X = HH^T$ for some matrix H. In particular there are two special decompositions for that:*

- $H = X^{\frac{1}{2}}$
- *H is lower triangle (Cholesky)*

4.4 Proposition. *Let K be a proper cone. Then the order $\preceq_K$ is*

- *transitive, reflexive and antisymmetric*
- *preserved under addition:* $x \preceq_K y$ *and* $u \preceq v$ *implies* $x + u \preceq y + v$.
- *non-negative scaling:* $x \preceq y$ *and* $\alpha \geq 0$ *implies* $\alpha x \preceq \alpha y$

4.5 Remark. For a cone K, $\preceq_K$ in general is it not a total order. For the most simple example take $(1, 0$ and $(0, 1)$ in $\mathbb{R}^2_+$.

4.6 Definition. Let K be a cone. The *dual cone* of K is

$$K^* := \{y : \forall x \in K. \langle y, x \rangle \geq 0\}$$

Figure 1 illustrates this concept.

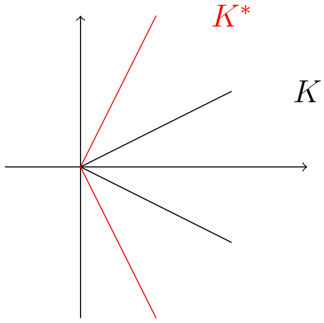


Figure 1: cone K and its dual K^*, between the lines we have right angles

4.7 Proposition. *The dual cone has the following properties*

- K^* *is a convex cone*
- K^* *is closed (even if* K *is not).*
- *If* $K_1 \subseteq K_2$, *then* $K_2^* \subseteq K_1^*$
- *If* $\operatorname{int} K \neq \emptyset$, *then* K^* *is pointed.*
- *If* $\operatorname{cl} K$ *is pointed, then* $\operatorname{int} K^* \neq \emptyset$.
- *Duality:* $\operatorname{conv} \operatorname{cl} K = K^{**}$.

In particular, if K *is proper, then* $K = K^{**}$.

4.8 Proposition. *Let* K *be a proper cone.*

$$x \preceq_K y \Leftrightarrow \forall \lambda \succeq_{K^*} 0. \lambda^T x \leq \lambda^T y \tag{1}$$
$$x \prec_K y \Leftrightarrow \forall \lambda \succeq_{K^*} 0 \setminus \{0\}. \lambda^T x < \lambda^T y \tag{2}$$

Proof of eq. (1). We have the following chain of reasoning

$$y - x \in K \Leftrightarrow y - x \in K^{**} \Leftrightarrow \forall \lambda \in K^*. \langle \lambda y - x \rangle \geq 0 \Leftrightarrow \forall \lambda \in K^*. \langle \lambda, y \rangle \geq \langle \lambda, x \rangle$$ □

4.9 Proposition. *The cones* $\mathbb{R}^n_+$, $\mathbb{S}^n_+$ *and* $\mathcal{L}^n = \{(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R} : \|x\| \leq t\}$ *are self-dual.*

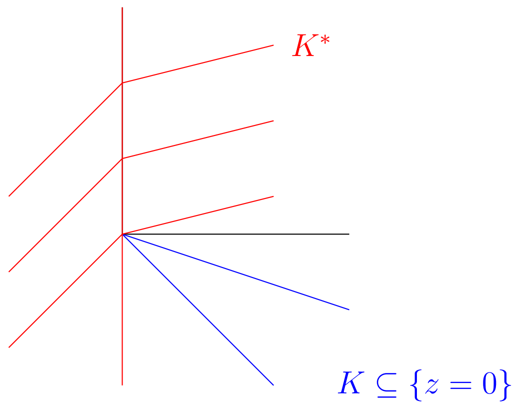


Figure 2: Example of the dual cone, where K does not have full dimension

5 Separating Hyperplane Theorem

5.1 Theorem. *Let X, Y be disjoint non-empty convex sets. Then there exist $v \in \mathbb{R}^n \setminus \{0\}$ and $c \in \mathbb{R}$ such that the hyperplane given by v and c separates X and Y, i.e.*

$$\forall x \in X. \langle v, x\rangle \leq c \qquad\qquad \forall y \in Y. \langle v, x\rangle \geq c$$

We will divide the proof into several smaller results.

5.2 Lemma. *Let $S \subseteq \mathbb{R}^n$ be a closed convex set. Then there is a* unique *vector $v \in S$ of minimal norm.*

Proof. Take some $v_0 \in S$ with $r := \|v_0\|$. Then $S' := S \cap B_r$ is compact, so we minimise a continuous function over a compact set. From analysis we know that S' attain this minimum δ by a vector v.
Now assume we have two vector $x, y \in S$ with $\|x\| = \|y\| = \delta$. By convexity, this means $\frac{x+y}{2} \in S$, so $\left\|\frac{1}{2}(x+y)\right\| \geq \delta$. By the parallelogram law we get

$$\|x-y\|^2 = \underbrace{2\|x\|^2}_{2\delta^2} + \underbrace{2\|y\|^2}_{2\delta^2} - \underbrace{4\|x+y\|^2}_{\geq 4\delta^2} \leq 0 \implies x = y \qquad \square$$

Proof of Theorem 5.1. First we introduce $S := X - Y$, which is another convex set. So $\operatorname{cl} S$ has a unique vector v of minimal norm. Let $z \in S$ and define $f(t) := \|v + t(z-v)\|^2$ (which describes a beam from v to z). We have $f(0) = \|v\|^2$ and $\forall t \in [0,1]. f(t) \geq f(0)$ since v has minimal norm.

$$\begin{aligned} \frac{1}{t}\left(f(t) - f(0)\right) &= \frac{1}{t}\left(\|v\|^2 + t^2\|z-v\|^2 + 2\langle v, z-v\rangle - \|v\|^2\right) \\ &\in 2v^T z - 2\|v\| + \mathcal{O}(z) \end{aligned}$$

which is ≥ 0 for all $t \in (0,1)$. Taking the limit as $t \to 0$, we get $v^T z \geq \|v\|$ for all $z \in S$.
Assume $v \neq 0$. Then for all $x \in X, y \in Y$ we get $v^T(y-x) \geq \|v\|^2 \geq 0$, so $v^T y \geq v^T x$. This yields the theorem by setting $c := \inf\{\langle v, y\rangle : y \in Y\}$.

case $v = 0$

$\square$

5.3 Theorem (Strict Separating Hyperplane). *Let X, Y be disjoint non-empty convex sets. If X is closed and Y is compact, then there exist $c \in \mathbb{R}$ and $v \in \mathbb{R}^n \setminus \{0\}$ such that*

$$\forall x \in X, y \in Y. v^T x < c < v^T y$$

Remark. *We need that one of the sets is bounded. Otherwise take* $X = \{(t, \frac{1}{t} : t \in \mathbb{R}\}$ *and* $Y = \{(t, 0) : t \in \mathbb{R}\}$.

5.4 Theorem. *Let K be a closed convex cone of $\mathbb{R}^n$ and $x \in \mathbb{R}^n \setminus \{K\}$. Then there exists $v \in \mathbb{R}^n$ such that $\forall z \in K. \langle z, v\rangle \geq 0$ but $\langle x, v\rangle < 0$.*

5.5 Definition. Let $S \subseteq \mathbb{R}^n$ be non-empty. Then $H = \{x : v^T x = c\}$ is a *supporting hyperplane* of S if

- $H \cap \partial S \neq \emptyset$
- S is included in one of the two half-spaces defined by H.

5.6 Theorem (Supporting Hyperplane Theorem). *Let $S \subseteq \mathbb{R}^n$ be convex. Let $x_0 \in \partial S$. Then there exist $v \in \mathbb{R}^n \setminus \{0\}$ such that $\forall x \in S. \langle v, x\rangle \leq \langle v, x_0\rangle$.*

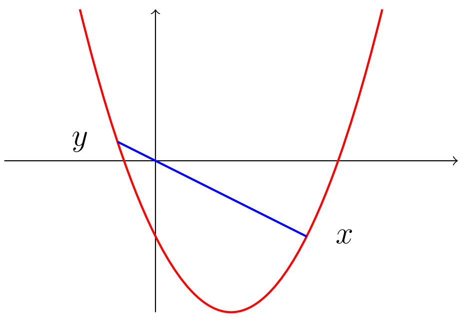
6 Convex Functions

6.1 Convex Functions

6.1 Definition. A function $f : S \to \mathbb{R}$ with $S \subseteq \mathbb{R}^n$ is *convex* if

- $S = \operatorname{dom} f$ is convex
- $\forall x, y \in S, \alpha \in [0, 1]. f((1-\alpha)x + \alpha y) \leq (1-\alpha)f(x) + \alpha f(y)$

Intuitively, the line form x to y lies above the function on the interval $[x, y]$.
A function $f : S \to \mathbb{R}$ is *concave*, if $-f$ is convex.



The convexity inequality can be understood in terms of extended value function

$$\widetilde{f}(x) = \begin{cases} f(x) & : x \in \operatorname{dom} f \\ \infty & : \text{else} \end{cases}$$

6.2 Proposition. *Let $S \subseteq \mathbb{R}^n$. A function $f : S \to \mathbb{R}$ is convex iff the function $g : t \mapsto f(x_0 + tu)$ is convex for any choice of x_0, u with $x_0 \in \operatorname{dom} f$ and $\operatorname{dom} g = \{t : x_0 + zu \in \operatorname{dom} f\}$.*

Proof. If $\operatorname{dom} f$ is convex, then $\operatorname{dom} g$ is an interval. So $\operatorname{dom} g$ is convex in $\mathbb{R}$. Furthermore

$$g((1-\alpha)r + \alpha s) = f(x_0 + ((1-\alpha)t + \alpha s)u) = f((1-\alpha)(x_0 + ru) + \alpha(x_0 + su)) \leq (1-\alpha)g(r) + \alpha g(s)$$

For the other direction let $x, y \in \operatorname{dom} f$. We write $x_0 = x$ and $u = y - x$. So $g(0) = x$ and $g(1) = y$. By convexity of g, we have $[0,1] \subseteq \operatorname{dom} g$. Therefore $(1-\alpha)x + \alpha y \in \operatorname{dom} f$ for all $\alpha \in [0,1]$. Furthermore

$$f((1-\alpha)x + \alpha y) = g((1-\alpha)\cdot 0 + \alpha \cdot 1) \leq (1-\alpha)g(0) + \alpha g(1) = (1-\alpha)f(x) + \alpha f(y) \qquad \square$$

6.3 Definition. The α-sublevel set of f is

$$C_\alpha(f) = \{x \in \operatorname{dom} f : f(x) \leq \alpha\}$$

The α-superlevel set of f is

$$C^\alpha(f) = \{x \in \operatorname{dom} f : f(x) \geq \alpha\}$$

The *epigraph* of f is “everything above the curve”, i.e.

$$\operatorname{epi} f = \{(x,t) \in \operatorname{dom} f \times \mathbb{R} : f(x) \leq t\}$$

The *hypograph* of f is “everything below the curve”, i.e.

$$\operatorname{hypo} f = \{(x,t) \in \operatorname{dom} f \times \mathbb{R} : f(x) \geq t\}$$

6.4 Proposition.

- *If f is convex, then the c_α are convex.*
- *If f is concave, then the $C^\alpha(f)$ are convex.*
- *f is convex iff* $\operatorname{epi} f$ *is convex*
- *f is concave iff* $\operatorname{hypo}(f)$ *is concave.*

6.5 Example. For example take $f(x,y) = xy$ with $\operatorname{dom} f = \mathbb{R}^2_+$. Then $C^\alpha(f) = \left\{(x,y) \geq 0 : y \geq \frac{\alpha}{x}\right\}$.

6.2 Jensen’s Inequality

Let f be convex and $x_1, \ldots, x_n \in \operatorname{dom} f$. Let $\lambda_i \geq 0$ with $\sum \lambda_i = 1$. Then $f\left(\sum \lambda_i x_i\right) \leq \sum \lambda_i f(x_i)$. More generally, if X is a random variable, which takes value in $S \subseteq \mathbb{R}^n$ and $f : S \to \mathbb{R}$ is convex, then $f(\mathbb{E}x) \leq \mathbb{E}\left(f(x)\right)$.

6.3 First and Second Order Condition for Convexity

6.6 Theorem. *Let $f : S \to \mathbb{R}$ with $S \subseteq \mathbb{R}^n$ be differentiable over S. Then f is convex iff $f(y) \geq f(x) + \nabla f(x)^T(y-x)$ for all $x, y \in S$, i.e. the tangent line is below the curve.*

6.7 Theorem. *Let $f : S \to \mathbb{R}$ with $S \subseteq \mathbb{R}^n$ be twice differentiable over S. Then f is convex iff $\nabla^2 f(x) \succeq 0$ for all $x \in S$.*
If $\nabla^2 f(x) \succ 0$, then f is strictly *convex.*

6.4 Examples of Convex Functions

Usually we will show convexity by building functions out of known convex functions.

- $x \mapsto e^{ax}$ is convex over $\mathbb{R}$
- $x \mapsto x^a$ is concave if $a \in (0,1)$ and convex otherwise, (demand $x > 0$ for $a < 0$)

- $x \mapsto \log x$ is concave over $\mathbb{R}_{++}$
- $x \mapsto x \log x$ is convex over $\mathbb{R}_{+}$ (put $0 \log 0 := 0$)
- $x \mapsto \|x\|$ is convex over $\mathbb{R}^n$ for any norm
- $x \mapsto \max(x_1, \ldots, x_n)$ is convex over $\mathbb{R}^n$
- $x \mapsto \log\left(e^{x_1} + \ldots + e^{x_n}\right)$ is convex over $\mathbb{R}^n$
- $x \mapsto x^T Q x + a^T + b$ is convex iff $Q \succeq 0$
- $x \mapsto \prod x_i^{\lambda_i}$ with $\sum \lambda_i = 1$ is concave (in particular the geometric mean is concave)
- $X \mapsto \operatorname{tr}\left(X^{-1}\right)$ is concave over $\mathbb{S}^n_{++}$
- $X \mapsto \log\det(X)$ is concave over $\mathbb{S}^n_{++}$
- $X \mapsto (\det X)^{\frac{1}{n}}$ is concave over $\mathbb{S}^n_{++}$

Proof. We will show this for $\log\det$ using Proposition 6.2. Let $Z \succ 0$ and $V \in \mathbb{S}^n$. We define $g(t) := \log\det(Z + tV)$. This we write as

$$\begin{aligned} g(t) &= \log\det\left(\sqrt{Z}\left(I + tZ^{-\frac{1}{2}} V Z^{-\frac{1}{2}}\right)\sqrt{Z}\right) = \log\det Z + \log\det\left(I + tZ^{-\frac{1}{2}} V Z^{-\frac{1}{2}}\right) \\ &= \log\det Z + \log\prod(1 + t\lambda_i) = \log\det Z + \sum \log(1 + t\lambda_i) \end{aligned}$$

using that the eigenvalues of $I + tM$ are $1 + t\lambda_i$ where λ_i are the eigenvalues of $M \in \mathbb{S}^n$.
The last line is just a sum of concave functions (note that Z and λ_i are fixed), so it is concave as well. Alternatively, take the second derivative

$$g''(t) = -\sum \frac{\lambda_i^2}{(1 + t\lambda_i)^2} \leq 0$$

□

6.5 Operations that preserve Convexity

Assume f, f_i is convex, then the following are convex as well i

- non-negative scaling: if $\alpha \geq 0$ then αf convex
- sum $f_1 + f_2$
- composition with affine map: $x \mapsto f(Ax + b)$ is convex
- pointwise maximum $x \mapsto \max(f_1(x), \ldots, f_n(x))$.
  If for all $y \in Y \subseteq \mathbb{R}^m$ the function $x \mapsto f(x, y)$ is convex, then $x \mapsto \sup_{y \in Y} f(x, y)$ is convex.
- Minimisation: If $f : \mathbb{R}^n \times \mathbb{R}^m$ is convex (jointly convex wrt x and y), then $x \mapsto \inf_{y \in \mathbb{R}^m} f(x, y)$ is convex.

6.8 Example.

- Let $A = (a_1^T, \ldots, a_m^T)^T \in \mathbb{R}^{m \times n}$. Regard the functions

$$\operatorname{dom} f = \{x : Ax > b\} \qquad\qquad f(x) = -\sum_{i=1}^{m} \log(a_i x - b_i)$$

  Since log is concave, $-\log$ is convex. Composed with an affine mapping, this is still convex, so f as a sum of convex functions is convex.

- Let $\operatorname{dom} F = \mathbb{S}^n$ with

$$f(X) = \lambda_{\max}(X) = \sup\left\{\frac{v^T X v}{\|v\|} : v \in \mathbb{R}^n, v \neq 0\right\} = \sup\left\{u^T X u : \|u\| = 1\right\} = \sup\left\{\langle X, uu^T\rangle : \|u\| = 1\right\}$$

  and the latter is a linear function in X. So f is the pointwise maximum of a family of linear functions. Hence f is convex.

- $g(x) = \operatorname{dist}(x, S)$ is a convex function if S is convex.

  First note that $f(x, y) = \|x - y\|$ is jointly convex in x and y, i.e. convex in $\mathbb{R}^{2n}$. We alter this function to

$$\widetilde{f} = \begin{cases} \|x - y\| & : y \in S \\ \infty & : \text{else} \end{cases}$$

  If S is convex, then $\widetilde{f}$ is convex. By the minimisation rule g is convex. (We need $\widetilde{f}$, because minimisation is stated only for $\mathbb{R}^{n+m}$.)

- For $f : \mathbb{R}^n \to \mathbb{R}$ its perspective function is $Pf : \mathbb{R}^n \times \mathbb{R}_{++} \to \mathbb{R}$ by $(x, t) \mapsto t \cdot f\left(\frac{x}{t}\right)$. If f is convex, then Pf is convex.

  In particular $(x, t) \mapsto \frac{\|x\|}{t}$ is convex over $\mathbb{R}^n \times \mathbb{R}_{++}$. Even more special $(x, y) \mapsto \frac{x^2}{y}$ is convex over $\mathbb{R} \times \mathbb{R}_{++}$.

6.9 Lemma (Composition Rules). *For composition we have the following rules.*

- *Let $g : \mathbb{R}^n \to \mathbb{R}^k$ and $h : \mathbb{R}^k \to \mathbb{R}$. Assume h is convex, non-decreasing and all g_i convex, then $f := h \circ g$ (note $f(x) = h(g(x))$) is convex.*

- *h cvx, $\searrow$, g_i ccv $\implies$ f convex*

- *h ccv, $\nearrow$, g_i ccv $\implies$ f concave*

- *h ccv, $\searrow$, g_i cvx $\implies$ f concave*

Proof. For simplicity regard the case $n = k = 1$, and g, h are twice differentiable. Then

$$f''(x) = g''(x) \cdot h'(g(x)) + g'(x)^2 \cdot h''(x)$$

Then the claim follows by checking the signs of each term. □

Remark. *Again, we are restricted to some $\mathbb{R}^n$ as domain. But take $f(x) = \|x\|^3$ as composition of $h(x) = x^3$ and $g(x) = \|x\|$, then we cannot conclude convexity, because h is not convex. However, h is convex over $\mathbb{R}_+$, and its argument is positive. So we use $h(x) = 0$ for $x < 0$ instead. Still $f = h \circ g$ so f is convex.*

6.10 Remark. If we don't use extended functions, the statement changes a bit:
Let $S \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^k$. Let $g : S \to T$ and $h : T \to \mathbb{R}$. Assume h convex, all g_i convex.

Approach cancelled in lecture

7 Conjugate Function

7.1 Definition. Let $f : \mathbb{R}^n \to \mathbb{R}$. The (Fenchel) conjugate of f is

$$f^* : \{y \in \mathbb{R}^n : f^*(y) < \infty\} \qquad\qquad y \mapsto \sup_{x \in \mathbb{R}^n} \langle x, y \rangle - f(x, y)$$

7.2 Proposition.

- *f^* is convex, even if f is not. (by the supremum rule)*
- *If* $\operatorname{epi} f$ *is closed and f is convex, then $f = f^{**}$.*
- *Fenchel-Young-inequality: $\forall x, y. \langle x, y \rangle \leq f(x) + f^*(y)$*

7.3 Example.

- Let $Q \succ 0$ and $f(x) = \frac{1}{2} x^T Q x$. To obtain f^* we compute

  $$\nabla \left(y^T x - \frac{1}{2} x^T Q x \right) = y - Qx$$

  Hence the maximiser satisfies $x^* = Q^{-1} y$. So we get

  $$f^*(y) = y^T x^* - \frac{1}{2}(x^*)^T Q x^* = y^T Q^{-1} y - \frac{1}{2} y^T Q^{-1} Q Q^{-1} y = \frac{1}{2} y^T Q^{-1} y$$

  Then the FY-inequality yields

  $$\langle x, y \rangle \leq \frac{1}{2} x^T Q x + \frac{1}{2} y^T Q^{-1} y$$

- Let $f(x) = a^T x + b$. Then

  $$f^*(y) = \sup_x x^T y - a^T x - b = \begin{cases} -b & : y = a \\ \infty & : \text{else} \end{cases}$$

  or short $f^*(y) = -b$ with $\operatorname{dom} f^* = \{a\}$.

- Let $f(x) = \max_i a_i x + b_i$ over $\mathbb{R}$. We assume $a_1 < \ldots < a_n$.

  $$f^*(y) = \sup_x xy - \max_i (a_i x + b_i) = \sup_x \min_i x(y - a_i) - b_i$$

  Assume that $y \in (a_j, a_{j+1})$. Then $y - a_j > 0$ and $y - a_{j+1} < 0$. The maximiser x^* is at the intersection of pieces corresponding to indices j and $j+1$.

  $$x^*(y - a_j) - b_j = x^*(y - a_{j+1}) - b_{j+1} \Leftrightarrow x^* = \frac{b_j - b_{j+1}}{a_j - a_{j+1}}$$

  $$f^*(y) = \frac{b_j - b_{j+1}}{a_j - a_{j+1}} (y - a_j) - b_j$$

  f^* is a piecewise linear function with $f^*(a_j) = b_j$.

8 Convex Optimisation

8.1 Definition. A non-linear optimisation problem (NLP) has the form

$$\begin{aligned} &\min f_0(x) \\ &\text{s.t.}\, \forall i \in [m]. f_i(x) \leq 0 \end{aligned} \tag{P}$$

- f_0 is the *objective function*
- the inequalities $f_i(x) \leq 0$ are called *constraints*
- $x \in \mathbb{R}^n$ is the *decision variable*
- $\mathcal{F} := \{x \in \mathbb{R}^n : \forall i \in [m]. f_i(x) \leq 0\}$ is called *feasible set*, we say x *is feasible* if $x \in \mathcal{F}$
- $p^* = \inf \{f_0(x) : x \in \mathcal{F}\} \in \mathbb{R} \cup \{\infty, -\infty\}$ is the *optimal value*
- If $f(x^*) = p^*$ for some $x^* \in \mathcal{F}$, we say x is *optimal* or x^* solves eq. (P)
- If f_0 is a constant function, then any $x \in \mathcal{F}$ is optimal. This is a *feasibility problem.*
- If $x \in \mathcal{F}$ with $f(x) \leq p^* + \varepsilon$, we say x is *ε-suboptimal.*

8.2 Remark. From the Definition 8.1 we get

- We say $p^* = \infty$ iff $\mathcal{F} = \emptyset$, meaning eq. (P) is *infeasible.*
- We say $p^* = -\infty$ iff eq. (P) is *unbounded* from below.
- The constraints $x \in \operatorname{dom} f_i$ are the implicit constraints of the problem. So we have $\mathcal{F} \subseteq \bigcap_{i=1}^m \operatorname{dom} f_i$ and wlog we can assume $\mathcal{F} \subseteq \bigcap_{i=0}^m \operatorname{dom} f_i$.
- The problem $\max f_0(x)$ is equivalent to $\min -f_0(x)$, so the theory also solves maximisation.

8.3 Example. The problem $\min \frac{1}{x}$ such that $x \geq 0$ has optimal value $p^* = 0$ but no optimal solution.

8.4 Definition. A vector $x \in \mathcal{F}$ is *locally optimal* if there is some $R > 0$ such that $\forall z \in B_R(x) \cap \mathcal{F}. f_0(x) \leq f_0(z)$.

8.5 Definition. eq. (P) is a *convex optimisation problem* if all f_i (including f_0) are convex.

8.6 Theorem. *If eq.* (P) *is a convex optimisation problem, then any locally optimal solution is globally optimal.*

Proof. Let x^* be locally optimal with radius R. Assume we have $y \in \mathcal{F}$ with $f_0(y) < f_0(x^*)$. Put $\theta = \frac{R}{\|x^*-y\|}$, then $0 < \theta < 1$ since $y \notin B_R(x^*)$. But also $z := (1-\theta)x^* + \theta y \in B_R(x^*)$. Convexity now yields

$$f_0(x^*) \leq f_0(z) \geq (1-\theta) f_0(x^*) + \theta f_0(y) < f_0(x^*) \lightning \qquad \square$$

8.7 Remark. It is possible to include equality $f(x) = 0$ by demanding $f(x) \leq 0$ and $-f(x) \leq 0$. However, for convex optimisation this is only possible if f is affine, i.e. $f(x) = a^T x + b$.

8.1 Problem Reformulations

8.8 Definition. We say eq. (P) and (Q) are equivalent if there is a “simple procedure” to transform an optimal solution of eq. (P) to an optimal solution of (Q) and vice-versa. We write *eq.* (P) $\dot{\sim}$ (Q).

add/remove affine equalities Consider the problem

$$(P) : \min\{f_0(x) : f_i(x) \le 0, Ax = b\}$$

with $A \in \mathbb{R}^{m \times n}$. This defines an affine set $L = \{x : Ax = b\} = \{Cz + d : z \in \mathbb{R}^r\}$ for some $C \in \mathbb{R}^{n \times r}, d \in \mathbb{R}^n$ and $r \in \mathbb{N}$. r is the (affine) dimension of L. If A has full rank, then $r = n - m$. Now regard

$$(Q) : \min\{f_0(Xz + d) : f_i(Cz + d) \le 0\}$$

Then $(P) \dot{\sim} (Q)$.

- If z^* is optimal, then $x^* := Cz^* + d$ is optimal for (P).
- If x^* is optimal for (P) and z^* solves $Cz^* = x^* - d$, then z^* is optimal for (Q).

slack variables An inequality $a^T x \le b$ can be transformed to $a^T x + s = b, s \ge 0$.

epigraph form Problem (P) can be written as

$$\min\{t : f_i(x) \le 0, f_0(x) \le t\}$$

The conversion is $x \leftrightarrow (x, f(x))$ Hence we can always assume that the objective is linear.

partial minimisation Sometimes we can partition out problem into blocks

$$\min\{f(x_1, x_2) : x_k \in \mathbb{R}^{n_k}, f_i(x_1) \le 0, g_j(x_2) \le 0\} \quad \dot{\sim} \quad \min\left\{\widetilde{f}(x_1) : f_i(x_1) \le 0\right\}$$

where $\widetilde{f}(x_1) = \inf\{f(x_1, x_2) : g_j(x_2) \le 0\}$. This is important if we can solve the latter case analytically.

8.2 First order optimality conditions

8.9 Theorem. *Consider the convex optimisation problem* $\min\{f_0(x) : x \in \mathcal{F}\}$. *Assume* f_0 *is differentiable over* $\mathcal{F}$. *Then* $x^* \in \mathcal{F}$ *is optimal iff*

$$\forall y \in \mathcal{F}. \nabla f_0(x^*)^T (y - x^*) \ge 0$$

If the global minimum of f_0 *lies in* $\mathcal{F}$, *then we have equality. Otherwise putting* $a := \nabla f_0(x^*)$, *the condition describes a supporting hyperplane* $\forall y \in \mathcal{F}. a^T \ge y^T x^*$.

Proof. The first-order condition for convexity of f_0 is

$$\forall y \in \mathcal{F}. f_0(y) \ge f_0(x^*) + \underbrace{\nabla f_0(x^*)^T (y - x^*)}_{\ge 0} \ge f_0(x^*)$$

So x^* is optimal.

Conversely assume x^* is optimal and there exists $y \in \mathcal{F}$ such that $\nabla f_0(x^*)^T(y - x^*) < 0$. Then

$$\nabla f_0(x^*)^T(y - x^*) = Df_0(x^*)[y - x^*] := \frac{\mathrm{d}}{\mathrm{d}t}\left[t \mapsto f_0(x^* + t(y - x^*))\right]$$

Going to $t = 0$ we get

$$\nabla f_0(x^*)^T(y - x^*) = \lim_{t \to 0} \frac{f_0(x^* + t(y - x^*)) - f_0(x^*)}{t}$$

So for $t > 0$ small enough we have

$$f_0(\underbrace{x^* + t(y - x^*)}_{\in \mathcal{F}}) < f_0(x^*)$$

which contradicts minimality of x^*. □

We continue with the problem

$$\min\{f(x) : x \in \mathcal{F}\}$$

where f is differentiable, convex and $\mathcal{F}$ is convex. We know

$$x^* \in \mathcal{F} \text{ optimal} \Leftrightarrow \forall y \in \mathcal{F}. \nabla f(x^*)^T(y - x^*) \geq 0$$

A special case of this problem is

$$\min\{f(x) : Ax = b\}$$

Here we have the equivalence of the following

- x^* is optimal
- $\nabla f(x^*)$ orthogonal to $\{x : Ax = b\}$
- $\nabla f(x^*) \in \operatorname{Im} A^T$

8.10 Theorem. *Regard* $\min\{f(x) : x \geq 0\}$. *Then* x^* *is optimal iff all of the following*

- $x \geq 0$
- $\nabla f(x^*) \geq 0$
- $\forall i. x_i^* = 0 \vee \frac{\partial f}{\partial x_i}(x^*) = 0$

Proof. **⇒:** Let x^* be optimal. Then for all $y \geq 0$ we have $\nabla f(x^*)^T(y - x^*) \geq 0$. Assume $\frac{\partial f}{\partial x_i}(x^*) < 0$ for some $i \in [n]$. Let $y = \alpha e_i$ with $\alpha \to \infty$. Then we get a contradiction. So we already have $x^* \geq 0$ and $\nabla f(x^*) \geq 0$. For $y = 0$ we have

$$\sum_i \frac{\partial f(x^*)}{\partial x_i} \cdot x_i \leq 0 \implies \forall i. \frac{\partial f(x^*)}{\partial x_i} \cdot x_i = 0$$

since it is a sum of non-negative terms.

⇐: Let $y \geq 0$. For all i we have $\frac{\partial f}{\partial x_i}(x^*) \geq 0$, because either $\frac{\partial f}{\partial x_i} = 0$ or $\frac{\partial f}{\partial x_i} \geq 0$ and $(x_i - x_i) = y_i \geq 0$. Summing over i we get $\nabla f(x^*)^T(y - x^*) \geq 0$. □

9 Conic Programming

9.1 Definition. Let K be a proper cone. A *conic programme in standard form* is an optimisation problem of the form

$$\min\{c^T x : Ax = b, x \in K\}$$

We have some special cases which have their own names

- $K = \mathbb{R}^n_+$ is a linear programme (LP).
- $K = K_1 \times \ldots \times K_m$ where

  $$K_i = \mathbb{L}^{n_i}_+ = \left\{x : \sqrt{x_1^2 + \ldots + x_{n_i-1}^2} \leq x_{n_i}\right\}$$

  called Lorentz-cone. This is second-order-cone-programming (SOCP).
- $K = \mathbb{S}^n_+$ is semidefinite programming (SDP).

9.2 Definition. Let V_i be a vector space of dimension n_i and $f_i : \mathbb{R}^n \to V_i$ an affine function for $i \leq q$. The following problem is equivalent to a conic programme in standard form

$$\min\{c^T x : Ax = b, \forall i \leq q. f_i(x) \in K_i\}$$

We call this a *conic programme in general form.*

For our special classes we get the generalised forms

LP $f(x) \in K \Leftrightarrow Ax \geq 0$ for some A, b

SOCP $f(x) \in K \Leftrightarrow \|Ax + b\| \leq c^T x + d$

SDP we have the form $f(x) = x_1 m_1 + \ldots + x_n M_n - M_0$ for some $M_i \in \mathbb{S}^n$. If the variable x is a matrix, note that functions are of the form $x \mapsto PxP^T$ or $x \mapsto Ax + x^T A^T$. The inequality $f(x) \succeq_{\mathbb{S}^n_+} 0$ is called a *linear matrix inequality.*

9.3 Example. If we have the general LP

$$\min\{c^T x : Fx = f, Hx \geq h\}$$

then we can rewrite it into standard form via

- $Hx \geq h \Leftrightarrow \exists s \geq 0. Hx - s = h$
- $x \in \mathbb{R}^n \Leftrightarrow \exists x_1, x_2 \in \mathbb{R}^n_+ . x = x_1 - x_2$

This example can be generalised to prove the remark during Definition 9.2 of the equivalence.

Proof. Start with the general problem

$$\min\{c^T x : Ax = b, \forall i. H_i x \succeq_{K_i} g_i\}$$

We introduce slack variables s_i, demanding $s_i \in K_i$ and decompose $x = x_1 - x_2$ as above. Putting

$$K = K_1 \times \ldots \times K_m \times \mathbb{R}^{2n}$$

with variable $\overline{x} = (s_1, \ldots, s_m, x_1, x_2)$ we have the standard form

$$\min\{c^T x : Ax = b, \forall i. H_i x - s_i = g_i, \overline{x} \in K\}$$

$\square$

9.4 Lemma. *For semidefinite we can use the equivalence*

$$\begin{pmatrix} X_1 & 0 & \dots & \\ 0 & X_2 & 0 & \dots \\ \dots & 0 & \ddots & \\ & \dots & 0 & X_n \end{pmatrix} \succeq_{\mathbb{S}_+} 0 \qquad \sim \qquad \begin{cases} X_1 \succeq_{\mathbb{S}_+} 0 \\ \vdots \\ X_n \succeq_{\mathbb{S}_+} 0 \end{cases}$$

9.1 What can be expressed with LP?

We review convex functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the inequalities $f(x) \le t$ can be written as a system of linear inequalities in x, t.

(a) Let $f(x) = \max\{a_i^T x + b_i . i = 1, \dots, m\}$. Then

$$f(x) \le t \Leftrightarrow \forall i \in [n]. a_i^T x + b_i \le t$$

(b) For the 1-norm we have

$$f(x) = \|x\|_1 \le t \Leftrightarrow \exists u \in \mathbb{R}^n_+ . -u \le x \le u \wedge \sum u_i \le t$$

(c) The maximum-norm can be written as

$$f(x) = \|x\|_\infty \le t \Leftrightarrow \forall i \in [n]. -t \le x_i \le t$$

(d) $f(x) = S_k(x)$ is the sum of the k largest elements of x.

$$S_k(x) \le t \Leftrightarrow \exists u \in \mathbb{R}, v \in \mathbb{R}^n . ku + \mathbf{1}^T u \le t, v \ge 0, v \ge x - u\mathbf{1} \tag{3}$$

By pushing up the dimension we greatly reduce the number of facets of the polytope. Using the approach from item a would yield a practically infeasible number of constraints.

To show that our new solution is correct, we show

$$S_k(x) = \min \left\{ u \in \mathbb{R} : \underbrace{ku + \sum_{i=1}^n \max(0, x_i - u)}_{:=h(x,u)} \right\}$$

Denote by $x_{(1)} \ge \dots \ge x_{(n)}$ the sorted entries of x. Let $u \in [x_{(k)}, x_{(k+1)}]$. Then

$$ku + \sum_{i=1}^n \max(0, x_i - u) = ku + \sum_{i=1}^k (x_{(i)} - u) = S_k(x)$$

So $\exists u . S_k(x) = h(x, u)$. Hence we have $S_k(x) \le p^*$.

Conversely let $u \in \mathbb{R}$. Then

$$\begin{aligned} S_k(x) &= \sum_{j=1}^k x_{(j)} = ku + \sum_{j=1}^k (x_{(j)} - u) \le ku + \sum_{j=1}^k \max(0, x_{(j)} - u) \\ &\le ku + \sum_{j=1}^n \max(0, x_{(j)} - u) = ku + \sum_{j=1}^n \max(0, x_j - u) = h(u, x) \end{aligned}$$

Therefore $\forall u. S_k(x) \leq h(x,u)$. This shows $S_k(x) \leq p^*(x)$, so we have equality.

Together this yields

$$S_k(x) \leq t \Leftrightarrow \exists u \in \mathbb{R}. ku + \sum_{i=1}^{n} \max(0, x_i - u) \leq t$$

and the latter we may rewrite via

$$\max(0, x_i - u) \leq v_i \Leftrightarrow \begin{cases} v_i \geq x_i - u \\ v_i \geq 0 \end{cases}$$

Combining everything we obtain the above formulation from eq. (3).

9.5 Remark. With the reformulation eq. (3) from above we have reduced the number of constraints from $\binom{n}{k}$ to $2n+1$.
The idea is that by going some dimensions up, we may greatly reduce the number of facets of a polytope.

9.6 Definition (K**-representability).** Let $K \subseteq \mathbb{R}^m$ be a proper cone and let $f : \mathbb{R}^n \to \mathbb{R}$. We say f is *K-representable* if there exists an affine function $F : \mathbb{R}^{n+k+1} \to \mathbb{R}^m$ such that

$$f(x) \leq t \Leftrightarrow \exists u \in \mathbb{R}^k. F(x,t,u) \succeq_K 0$$

9.7 Example. Consider the NLP

$$\min\{3\|x\|_1 + 7S_4(x) : x \in \mathbb{R}^n, Ax \leq b\}$$

This is equivalent to the LP

$$\min\left\{3\lambda_1 + 7\lambda_2 : \ \|x\|_1 \leq \lambda_1, Ax \leq b, -u \leq x \leq u, \mathbf{1}^T u \leq \lambda_1, v \geq 0, v \geq x - t\mathbf{1}, 4t + \mathbf{1}^T v \leq \lambda_2\right\}$$

We may even further reduce the problem to

$$\min\left\{3\mathbf{1}^T u + 7(4z + \mathbf{1}^t v) : Ax \leq b, -u \leq x \leq u, v \geq 0, v \geq x - t\mathbf{1}\right\}$$

9.2 What can be expressed by SOCP?

(a) The function $f : (x,t) \mapsto \frac{\|x\|^2}{t}$ is SOCP-representable over $\mathbb{R}^n \times \mathbb{R}_{++}$. More precisely

$$\|(2x, t-u)\| \leq 2 + u \Leftrightarrow \|x\| \leq tu \wedge t \geq 0 \wedge u \geq 0$$

Proof. We have

$$\begin{aligned} \|(2x, t-u)\| \leq t + u &\Leftrightarrow 4\|x\|^2 + (t-u)^2 \leq (2+u)^2, t+u \geq 0 \\ &\Leftrightarrow 4\|x\|^2 \leq 4ut, t+u \geq 0 \\ &\Leftrightarrow \|x\|^2 \leq ut, t, \geq 0, u \geq 0 \end{aligned}$$

because both $t+u$ and $t \cdot u$ must be non-negative. □

(b) Let $Q \in \mathbb{S}^n_+$. The convex quadratic $x \mapsto x^T Qx + a^t x + b$ is SOC-representable.

Proof. We decompose $Q = HH^T$. Then we have

$$x^T Qx + a^T x + b \leq t \Leftrightarrow x^T HH^T x \leq t - (a^T x + b) \Leftrightarrow \|(2H^T x, t - (a^T b + b) - 1)\| \leq t - (a^T x + b) + 1$$

where the last step uses item a, multiplied with 1. □

(c) The geometric mean $G(X) := \sqrt[n]{\prod x_i}$ and the harmonic mean $H(x) := \left(\sum x_i^{-1}\right)^{-1}$ are concave SOC-representable.

(d) Rational powers: Let $p \in \mathbb{Q}_+$, then $f : x \mapsto x^p$ is SOC-representable.

Proof. Write $p = \frac{\alpha}{\beta}$ with $\alpha, \beta \in \mathbb{N}$. Then

$$x^p \leq t \Leftrightarrow x \leq t^{\frac{\beta}{\alpha}} \Leftrightarrow x \leq G(\underbrace{t, \ldots, t}_{\beta}, \underbrace{1, \ldots, 1}_{\alpha - \beta})$$ □

9.3 What be be expressed by SDP?

9.8 Lemma (Schur-complement-Lemma). *Let*

$$M = \begin{pmatrix} A & C \\ C^T & b \end{pmatrix} \in \mathbb{S}^{n+m}$$

Then

$$M \succ 0 \Leftrightarrow B \succ 0, A - CB^{-1}C^T \succ 0 \Leftrightarrow A \succ 0, B - C^T A^{-1} C \succ 0$$

If $A \succ 0$ *then* $M \succeq 0 \Leftrightarrow B - C^T A^{-1} C \succeq 0$. *Conversely if* $B \succ 0$, *then* $M \succeq \Leftrightarrow A - CB^{-1}C^T \succeq 0$.

Proof. We just prove the third part. We assume $B \succ 0$. Then

$$M \succeq 0 \Leftrightarrow \forall (x, y) \in \mathbb{R}^{n+m}.(x^T, y^T) \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \geq 0 \Leftrightarrow \inf_{x,y} \left\{ x^T Ax + y^T By + 2y^T C^T x \geq 0 \right\}$$

By the partial minimisation rule this is equivalent to

$$\inf_x \left\{ x^T Ax + \widetilde{f}(x) \geq 0 \right\}$$

where

$$\widetilde{f}(x) = \inf_y \left\{ y^T By + 2y^T C^T x \right\}$$

This latter function we can solve analytically. Argument y^* minimises $\widetilde{f}(x)$ iff $2(By^* + C^T x) = 0$. Therefore

$$\widetilde{f}(x) = x^T CB^{-1} BB^{-1} C^T x - 2x^T CB^{-1} C^T x = -x^T CB^{-1} C^T x$$

So our original problem simplifies to

$$M \succeq 0 \Leftrightarrow \forall x. x^T (A + CB^{-1}C^T) x \geq 0 \Leftrightarrow A - CB^{-1}C^T \succeq 0$$ □

(a) Largest eigenvalue of a symmetric matrix: $\lambda_{\max} : \mathbb{S}^n \to \mathbb{R}$ is convex and SDR (semidefinite-representable).

Proof. We can reformulate the problem as

$$\lambda_{\max}(X) = \sup\left\{\frac{u^T X u}{u^T u} : u \neq 0\right\}$$

Using this characterisation we get

$$\begin{aligned}\lambda_{\max}(X) \leq t &\Leftrightarrow \forall u \neq 0. \frac{u^T X u}{u^T u} \leq t \Leftrightarrow \forall u. u^T X u \leq t u^T u \Leftrightarrow \forall u. u^T (X - tI_n) u \leq 0 \\ &\Leftrightarrow tI_n - X \succeq 0 \Leftrightarrow X \preceq tI_n\end{aligned}$$

□

(b) Analogously we have $\lambda_{\min}(X) \geq t \Leftrightarrow X \succeq tI_n$, so it is SDR.

(c) Let $M_0, \ldots, M_n \in \mathbb{S}^n_{++}$. So $F : x \mapsto M_0 + \sum x_i M_i$ is an affine mapping $\mathbb{R}^n_+ \to \mathbb{S}^n_{++}$. Then let $x \in \mathbb{R}^n$ and define $g(x) := c^T F(x)^{-1} c$. Then $g : \mathbb{R}_+ \to \mathbb{R}_+$ is SDR.

Proof. By Schur-complement-Lemma Lemma 9.8 we have

$$g(x) \leq t \Leftrightarrow x^T F(x)^{-1} c \leq t \Leftrightarrow \begin{pmatrix} F(x) & c \\ c^T & t \end{pmatrix} \succeq 0$$

□

(d) SOCP is a subclass of SDP via

$$\|x\| \leq t \Leftrightarrow \begin{pmatrix} t & x^T \\ x & tI_n \end{pmatrix} \succeq 0$$

Proof. Rewrite the first condition as $\|x\|^2 \leq t^2, t \geq 0$. This we rewrite as $x^T (tI_n)^{-1} x \leq t$ and $t > 0$ or $x = t = 0$. By Schur-complement-Lemma Lemma 9.8 this is equivalent to the right hand side. □

(e) For the determinant we have

$$\sqrt[n]{\det(X)} \geq t \Leftrightarrow \exists L \in \mathbb{R}^{n \times n}, u \in \mathbb{R}^n. \begin{pmatrix} X & L \\ L^T & \operatorname{diag}(u) \end{pmatrix} \succeq 0, \operatorname{diag}(L) = u, G(u) \geq t, L \text{ lower triangular}$$

where $G(u)$ denotes the geometric mean.

9.9 Example (Löwner-John-ellipsoid). Given points $x_i \in \mathbb{R}^n$, what is the ellipsoid of minimal volume, which contains all x_i? An ellipsoid is

$$\mathcal{E} = \{x : (x - x_0)^T Q (x - x_0) \leq 1\}$$

for some $Q \succ 0$ and $x_0 \in \mathbb{R}^n$. Since $Q \succ 0$, we can write $Q = A^T A$ for some $A \succ 0$, so the ellipsoid can be written as

$$\mathcal{E} = \{x : (x - x_0)^T X^T X (x - x_0) \leq 1\} = \{x : \|Ax - b\| \leq 1\}$$

where $b := Ax_0$.
The semi-axis are proportional to $\lambda^{-\frac{1}{2}} \cdot u_i$, where (λ_i, u_i) are eigenvalue and eigenvector of Q. So we want to minimise $\prod \frac{1}{\sqrt{\lambda_i}}$. This means we maximise $\prod \lambda_i = \det Q = \det(A)^2$. To make it concave, we instead maximise $\sqrt[n]{\det(A)}$. So we reformulated the problem

$$\max\{\sqrt[n]{\det A} : \forall i. \|Ax_i - b\| \leq 1, A \in \mathbb{S}^n_+, b \in \mathbb{R}^n\}$$

which can be transformed into an SDP with the above methods.

10 Application to statistics, Data analysis and Machine Learning

Assume that we have a collection $y_1, \dots, y_m \in \mathbb{R}$ of observations. Each observation is associated with a vector of *features* $x_i \in \mathbb{R}^n$. For an index i, the pair $(x_i, y_i) \in \mathbb{R}^{n+1}$ is a *sample*. We write the data as

$$X = \begin{pmatrix} x_1^T \\ \vdots \\ x_m^T \end{pmatrix} \in \mathbb{R}^{m \times n} \qquad\qquad y = \begin{pmatrix} y_1^T \\ \vdots \\ y_m^T \end{pmatrix} \in \mathbb{R}^m$$

We now search for a functions F, such that $y_i \approx F(x_i)$. Usually this is useful for

- understanding how each feature x_{ik} influences the outcome y_i.
- predicting y_* for some new input x_*.

10.1 Linear Regression

One of the simplest models is to assume a linear relationship $y_i \approx x_i^T \Theta$. In fact, we can also transform the vector to study non-linear models. First we can change it to affine relationship, by adding $x_{i0} = 1$. Furthermore we can add higher dimensions via

$$y_i \approx (1, x_i, x_i \otimes x_i)^T \cdot \Theta$$

for the cost of greatly expanding the problem size.
This justifies the linear approach, so we focus on the problem

$$y = X\Theta + \varepsilon$$

10.1.1 Least Squares

In this approach, we want to minimise the squared error, so

$$\min \left\{ \|X\Theta - y\|^2 : \Theta \in \mathbb{R}^n \right\}$$

This is an unconstrained convex optimisation problem. To solve it, we put its gradient to 0.

$$0 = \nabla_\Theta \|X\Theta - y\|^2 = 2X^T X\Theta - 2X^T y \implies \Theta^* = \left(X^T X\right)^{-1} X^T y$$

Note, if X has full (column) rank, then $X^T X$ is invertible. Otherwise some extra care has to be taken.

10.1.2 Best linear estimation

We will see that Θ^* satisfies a very strong property: It is the *best linear unbiased estimator* (BLUE) of Θ.
We assume that the model errors ε_i are random variables, identically and independently distributed. Moreover we assume $\mathbb{E}[\varepsilon_i] = 0$ and $\mathbb{V}[\varepsilon_i] = \sigma^2$ for all i. This implies that the observations y_i are realisations of a random variable Y_i with $\mathbb{E}[Y_i] = x_i^T \Theta$, so $\mathbb{E}[Y] = X\Theta$. We search a linear estimator $\widehat{\Theta} = L^T Y$. We say that $\widehat{\Theta}$ is unbiased if

$$\Theta = \mathbb{E}[\widehat{\Theta}] = L^T \left(\mathbb{E}[Y]\right) = L^T X \Theta$$

Hence $\widehat{\Theta}$ is unbiased off $L^T X = I_n$.

10.1 Theorem (Gauss-Markov). *Let $\widehat{\Theta} = l^T Y$ be an unbiased estimate for Θ. Then $\mathbb{V}\left[\widehat{\Theta}\right] \succeq \left(X^T X\right)^{-1}$ and the lower bound is attained for $L^* = X\left(X^T X\right)^{-1}$, which corresponds to the least-squares-estimate of Θ.*

Proof. We simplify the variance to

$$\mathbb{V}[\widehat{\Theta}] = \mathbb{V}\left[L^T Y\right] = L^T \mathbb{V}[Y] L = \sigma^2 L^T L$$

We only need to show

$$L^T X X = I \implies L^T L \succeq (X^T X)^{-1}$$

which is a consequence of the Schur-complement Lemma 9.8 on

$$0 \preceq \begin{pmatrix} X^T \\ L^T \end{pmatrix} \begin{pmatrix} X & L \end{pmatrix} = \begin{pmatrix} X^T X & I \\ I & L^T L \end{pmatrix}$$ □

Continue with the problem

$$\operatorname{argmin} \|X\Theta - y\|^2$$

There exists $\Theta \in \mathbb{R}^n$, y_i is the realisation of a random variable Y_i such that $\mathbb{E}Y_i = x_i^T \Theta$, $\mathbb{V}Y_i = \gamma^2$, $Y_i \perp Y_j$ for $i \neq j$. Putting $\widehat{\Theta} = \left(X^T X\right)^{-1} X^T Y$ we have $\mathbb{E}\widehat{\Theta} = \Theta$ and $\widehat{\Theta}$ is in an ellipsoid around Θ with high probability.

10.2 Beyond Least Squares

10.2.1 Ridge Regression

Instead of least squares, the ridge estimator of Θ solves

$$\min\left\{\|X\Theta - y\|^2 + \lambda\|\Theta\| : \Theta \in \mathbb{R}^n\right\}$$

for some fixes parameter λ. This has the optimal solution

$$\widehat{\Theta}_{\text{ridge}} = \left(X^T X + \lambda I\right)^{-1} X^T y$$

This is an unconstrained convex optimisation problem. The gradient of the objective is

$$\nabla_\Theta\left(\Theta^T X^T X\Theta - 2y^T X\Theta + y^T y + \lambda\Theta^T\Theta\right) = 2X^T X\Theta - 2X^T y + 2\lambda\Theta = 2\left((X^T X + \lambda I)\Theta - X^T y\right)$$

Putting this to zero yields the above formula for the optimum.
The ridge regression shifts all the eigenvalues by λ. Often least-squares has a bad condition number, due to small eigenvalues. The shift can solve this issue.
when λ is larger, it produces "smaller" estimates $\widehat{\Theta}_{\text{ridge}}$. And then, the variance of the estimates is smaller, too. More precisely we have

$$\mathbb{V}[\widehat{\Theta}_{\text{ridge}}] = \sigma^2\left(X^T X + \lambda I\right)^{-1} X^T X\left(X^T X + \lambda I\right)^{-1} \leq \sigma^2 (X^T X)^{-1}$$

which is the variance of LS (for $\lambda = 0$). The price is that we now have a bias, so the expected value is not the real one.

10.2.2 Lasso estimator

We have samples of the form

$$y_i = \Theta^T x_i + \varepsilon$$

In sparse regression we are looking for an estimate Θ with many zeros. The Lasso estimator minimises

$$\|X\Theta - y\|^2 + \lambda\|\Theta\|_1$$

over $\Theta \in \mathbb{R}^n$ with fixed λ. The idea is that $\|\Theta\|_1$ serves as an approximation for

$$\|\Theta\|_0 = |\{i \in [n] : \Theta_i \neq 0\}|$$

Even on its own, the 1-norm has its vertices on the axes, so optimal solutions are more likely to lie on an axis, so they have more zero-entries.

10.2.3 Elastic net estimator

Combining the previous approaches, we get

$$\min \|X\Theta - y\|^2 + \lambda_1\|\Theta\|_1 + \lambda_2\|\Theta\|^2$$

10.2.4 Huber Regression

One remaining problem is that we might have single faulty values, which lie far away from the rest of the sample. However, these may greatly influence the results for the worse. Hence we want to put less weight on the deviation, if it differs too far.
To solve this we define the Huber-loss-function

$$H(x) = \begin{cases} x^2 & : |x| \leq \delta \\ \delta(2|x| - \Delta) & \end{cases}$$

Then the problem we have is

$$\min \sum_{i=1}^{n} H(y_i - x_i^T\Theta)$$

10.3 Classification

We have a set of sample $x \in \mathbb{R}^{m \times n}$. Let $y_i \in \{-1, 1\}$, as known classification of some samples.

10.3.1 Support Vector machines

We say that the data is *linearly separable* if there exist $a \in \mathbb{R}^n, b \in \mathbb{R}$ such that $y_i^T(ax_i - b) > 0$.

Remark. *A set is linearly separable if the convex hulls of 0 and 1 do not intersect.*

The hard-margin-classifier is

$$\forall i \in [n]. y(x^T x_i - b_i = 0).$$

There also is a soft-version: The data is not always linearly separable. So we introduce a penalty function

$$\phi(x,y;a,b) = \begin{cases} 0 & : y_i(a^T x_i - b) \geq 1 \\ 1 - y_i((a^T x_i - b) & : \text{else} \end{cases}$$

So we want to solve

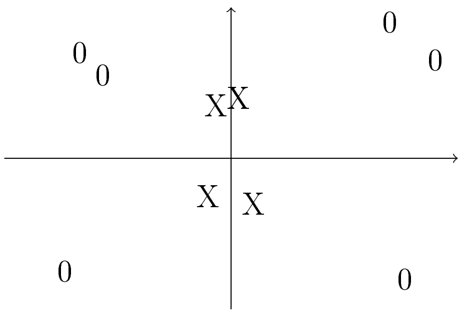
$$\min \left\{ \|a\|^2 + \sum_i \max(0, 1 - y_i(a^T x_i - b)) : a \in \mathbb{R}^n, b \in \mathbb{R} \right\}$$

This is equivalent to the SOCP

$$\min \left\{ t + \sum u_i : \|a\|^2 \leq t, u_i \geq 0, u_i \geq 1 - y(a^T x_i - b) \right\}$$

10.4 Kernel Trick

Sometimes our data is separated by a circle rather than a hyperplane. by lifting the dimension



with a third variable $x_3 = x_1^2 + x_2^2$ get now have the chance to find a separating hyperplane.
More general doing a non-linear transformation $x' = \varphi(x)$ can help. Unfortunately the dimension of $n' := \varphi(x)$ can become very large, so algorithms might become inefficient. This is one motivation for the "Kernel Trick".

10.2 Definition. A *positive semidefinite kernel* over $\mathbb{R}^n$ is a function $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that for all $N \in \mathbb{N}, x_1, \ldots, x_N \in \mathbb{R}_N$ the matrix $(k(x_i, x_j))_{i,j}$ is positive semidefinite.

Example.
- $K(x,y) = (1 + x^T y)^d$
- $K(x,y) = \exp\left(-\frac{1}{\sigma^2}\|x-y\|^2\right)$

10.3 Theorem (Mercer). *If K is a positive semidefinite kernel, then there exists a function φ such that $K(x,y) = \langle \varphi(x), \varphi(y) \rangle$ for all $x, y \in \mathbb{R}^n$.*

So the idea of the kernel trick is as follows

1. Assume a learning algorithm that depends only on scalar products $\langle x_i, x_j \rangle$
2. We want to apply the algorithm to the transformed features $x' = \varphi(x)$.
3. We choose a kernel K. Denote by φ its associated Mercer-function.
4. In the algorithm replace all occurrences of $\langle x_i, y_j \rangle$ by $K(x_i, x_j)$.

10.4 Example. Kernelise Ridge regression

$$\widehat{\Theta} = \left(X^T X + ambdaI\right)^{-1} X^T y$$

The prediction for a new sample x_* is

$$y_* = x_*^T \widehat{\Theta} = x_*^T \left(X^T X + \lambda I\right)^{-1} X^T Y$$

We can show that this is equal to

$$y_* = x_*^T X^T \left(X^T X + \lambda I\right)^{-1} Y$$

Equivalently we have

$$\widehat{y}_* = k(x_*)^T (K + \lambda I)^{-1} Y$$

where $k_i := x_i^T x_*$.

10.5 Design of Experiments

Given some features $x_i \in \mathbb{R}^n$ we ask whether there exists some $\Theta \in \mathbb{R}^n$ such that y_i is a realisation of random variable Y_i with

$$\mathbb{E} Y_i = x_i^T \Theta \qquad \mathbb{V} Y_i = \sigma^2 \qquad Y_i \perp Y_j$$

Often we are free to choose the features. So we want to find out, what the best choice is to gain as much information as possible.
In optimal Design of Experiment we want to select the features x_i that lead to the best possible estimator of Θ. (We have to pay to obtain new samples, so which samples should we select?)
We assume that experimental trials must be selected from $\{x_1, \dots, x_m\}$. We have a budget for N trials. Let n_i be the number of times we choose the i-th experimental trial. Then

$$y = (x_1, \dots, x_1, \dots, x_n, \dots, x_n)^T \Theta + \varepsilon$$

From the Gauss Markov Theorem (Theorem 10.1) we know that the variance of the BLUE of Θ is

$$V = \sigma^2 \left(X(\vec{n})^T X(\vec{n})\right)^{-1} = \sigma^2 \left(\sum_{i=1}^m n_i x_i x_i^T\right)^{-1} = \frac{\sigma^2}{N} \left(\sum_{i=1}^m \underbrace{\frac{n_i}{N}}_{=:w_i} x_i x_i^T\right)^{-1}$$

The weights w_i satisfy $\sum s_i = 1$ and represent the fraction of"experimental effort" at x_i. Formally, this was an integer optimisation problem. But for large N we are close to the continuous case. Thus we treat the w_i as real numbers and hence we have a convex optimisation problem.
The optimal DoE-problem is to "minimise" $\left(\sum w_i x_i^T x_i\right)^{-1}$, or to "maximise" the information matrix

$$M(w) = \sum_{i=1}^m w_i x_i^T x_i$$

but we have to choose according to which scalar function.

the D-optimal design problem is to find the weights that minimise the volume of the confidence ellipsoids. The axes of the ellipsoid are the eigenvectors with factors $\lambda_i^{\frac{1}{2}}$ (eigenvalues). This means we want to maximise

$$\psi_D(w) = \left(\prod \lambda_i\right)^{\frac{1}{n}} = \sqrt[n]{\det M(w)}$$

i.e. we solve the convex problem

$$\max\left\{\sqrt[n]{\sum w_i x_i^T x_i} : w \in \mathbb{R}^n_{\geq 0}, \sum w_i = 1\right\}$$

The A-optimal design is to minimise the diagonal of the bounding box of the confidence ellipsoid. Here we have target function

$$\phi_A(w) = \sum_{i=1}^{m} \frac{1}{\lambda_i} = \operatorname{tr}(M(w))^{-1}$$

To write this as an SDP first note

$$\operatorname{tr}(M(w))^{-1} \leq t \Leftrightarrow \exists Y \in \mathbb{S}^n. \begin{pmatrix} M(w) & I_n \\ I_n & Y \end{pmatrix} \succeq 0, \operatorname{tr}(Y) \leq t$$

Proof. **⇒:** We admit that the LMI implies $M(w) \succ 0$. Then from Lemma 9.8 $Y \succeq I_n(M(w))^{-1}I_n = M(w)^{-1}$. So $t \geq \operatorname{tr}(Y) \geq \operatorname{tr}\left(M(w)^{-1}\right)$.

⇐: $\operatorname{tr}\left(M(w)^{-1}\right) \leq t$, so $M(w)^{-1}$ exists, hence $M(w) \succ 0$. Let $Y = M(w)^{-1}$. From Lemma 9.8

$$\begin{pmatrix} M(w) & I_n \\ I_n & M(w)^{-1} \end{pmatrix} \succeq 0$$

and of course $\operatorname{tr}(Y) = \operatorname{tr}\left(M(w)^{-1}\right) \leq t$. □

To finally get the SDP formulation write

$$\min\left\{\operatorname{tr}(Y) : \begin{pmatrix} M(w) & I_n \\ I_n & Y \end{pmatrix} \succeq 0, w \geq 0, \sum w_i = 1, w \in \mathbb{R}^m, Y \in \mathbb{S}^n\right\}$$

11 Duality

Regard a convex optimisation problem. The primal view is to find the point farthest away in some given direction. The dual view is to find the closest supporting hyperplane in the opposite of the given direction.
We want to find a connection between

$$\min f(x) + \lambda g(x) \qquad\qquad \min\{f(x) : g(x) \leq \alpha\}$$

In this part, we will switch between to formulations

- Non-linear programme (NLP)

$$p^* := \inf\{f_0(x) : \forall i \in [m]. f_i(x) \leq 0, \forall j \in [p]. h_j(x) = 0\}$$

- conic programme (CP)

$$\min\{c^T x : A_0 x = b_0, A_i x \succeq_{K_i} b_i\}$$

11.1 Lagrangian Dual of NLP

The Lagrangian of (NLP) is the function $\mathcal{L} : \mathbb{R}^{n+m+p} \to \mathbb{R}$ via

$$\mathcal{L}(x, \lambda, \mu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \mu_j h_j(x)$$

The Lagrange dual function of (NLP) is $g : \mathbb{R}^{m+p} \to \mathbb{R}$ via

$$g(\lambda, \mu) = \inf\{\mathcal{L}(x, \lambda, \mu) : x \in \mathbb{R}^n\}$$

11.1 Theorem. *Let $\lambda \in \mathbb{R}^m_+$, $\mu \in \mathbb{R}^p$. Then $g(\lambda, \mu) \leq p^*$.*

Proof. If (NLP) is infeasible, we have $p^* = \infty$, so the theorem holds.
Let $\widetilde{X}$ be feasible for (NLP). Then

$$g(\lambda, \mu) = \inf_x \mathcal{L}(x, \lambda, \mu) \leq \mathcal{L}(\widetilde{x}, \lambda, \mu) = f_0(\widetilde{x}) + \sum_{i=1}^{m} \lambda_i f_i(\widetilde{x}) + \sum_{j=1}^{p} \mu_j h_j(\widetilde{x}) \leq f_0(\widetilde{x})$$

Taking the infimum over feasible $\widetilde{x}$ yields $g(\lambda, \mu) \leq p^*$. □

Note that this proof does not use convexity (nor does the theorem).

11.2 Theorem. *For the Lagrangian we have*

$$\sup_{\lambda \geq 0, \mu} \inf_{x \in \mathbb{R}^n} \mathcal{L}(x, \lambda, \mu) \leq \inf_{x \in \mathbb{R}^n} \sup \left\{\mathcal{L}(x, \lambda, \mu) : \lambda \in \mathbb{R}^m_+, \mu \in \mathbb{R}^p\right\}$$

The left hand side is the Lagrangian dual problem.

Proof. We need to show that the right hand side is p^*. Let $x \in \mathbb{R}^n$. We claim that

$$\sup\left\{\mathcal{L}(x, \lambda, \mu) : \lambda \geq 0, \mu \in \mathbb{R}^p\right\} = \begin{cases} f_0(x) & : x \text{ feasible} \\ \infty & : \text{else} \end{cases}$$

Let x be feasible. Then

$$\mathcal{L}(x, \lambda, \mu) = f_0(x) + \sum \lambda_i f_i(x) + \sum \mu_j h_j(x) \leq f_0(x)$$

So $\sup\{\mathcal{L}(x, \lambda, \mu) : \lambda \geq 0\} \leq f_0(x)$ and we have equality for $\lambda = \mu = 0$.
Let x be infeasible. Then we have some violated constraint, which means $f_i(x) > 0$ or $h_j(x) \neq 0$.
Sending this $\lambda_i \to \infty$ or $\mu_j \to \pm\infty$ yields a supremum of ∞.
The claim implies the value of the inf-sup problem is p^* so we are done. □

missing lecture

$$\inf\{c^T x : A_i x \succeq_{k_i} b_i\} = \inf_x c^T x + \begin{cases} 0 & : A_i x \succeq_{K_i} b_i \\ \infty & : \text{otherwise} \end{cases} = \inf c^T x + \sup_{y_i \succeq_{K_i} 0} \langle y_i, b_i - A_i x\rangle$$

For the last equality we have to be aware that

$$\sup\{a \cdot y : y \geq 0\} = \begin{cases} \infty & : a > 0 \\ 0 & : a \leq 0 \end{cases}$$

Let K be a proper cone. Then

$$x \in K \Leftrightarrow x \in K^{**} \Leftrightarrow \forall y \in K^*. \langle x, y\rangle \geq 0$$

11.3 Theorem. *We have*

$$p^* := \inf\{\langle x, y\rangle : y \in K^*\} = \begin{cases} 0 & : x \in K \\ -\infty & : \textit{else} \end{cases} \tag{4}$$

$$\sup\{\langle x, y\rangle : y \in K^*\} = \begin{cases} 0 & : x \preceq_K 0 \\ 0 & : \textit{else} \end{cases} \tag{5}$$

Proof. We just show the first. If $x \in K$, then $\forall y \in K^*.\langle x, y\rangle \geq 0$, so $p^* \geq 0$. Moreover $\langle x, 0\rangle = 0$, so $p^* = 0$. If $x \notin K$, there exists some y_0 such that $\langle x, y_0\rangle < 0$. Then $ty_0 \notin K$ for all $t > 0$, so

$$p^* \leq \lim_{t\to\infty} \langle x, ty_0\rangle = -\infty$$

□

11.3 Strong Duality

11.4 Definition. In a convex optimisation problem we say that the constraints are *qualified*, if they satisfy some property ensuring that strong duality holds.

11.3.1 Slater's condition for conic programming

Consider a pair of conic programmes.

$$p^* = \inf\left\{c^T x : A_0 x = b_0, \forall i \in [m].A_i x \succeq_{K_i} b_i\right\} \tag{6}$$

$$d^* = \sup\left\{\sum_{i=0}^{m} b_i^T y_i : \sum_{i=0}^{m} A_i^T y_i = c, \forall i \in [m].y_i \succeq_{K_i^*} 0\right\} \tag{7}$$

11.5 Definition. We say eq. (6) is strictly feasible if there exists some $x \in \mathbb{R}^n$ with $A_0 x = b_0$ and $A_i \succ_{K_i} b_i$. Then we say x is a Slater's point.
Equation (7) is strictly feasible if there exists some feasible y with $y_i \succ_{K_i^*} 0$.
More generally a conic problem is essentially strictly feasible, if there exists some feasible point that satisfies all *non-linear* conic inequalities strictly.

Remark. *For our cones we will usually have $K = \mathbb{R}^n$ (polytope) or $K = \coprod_+^n$, $K = \mathbb{S}_+^n$ (non-linear conic inequalities).*

11.6 Theorem.

1. *$d^* \leq p^*$ (Weak duality)*
2. *The dual of eq. (7) is eq. (6) (symmetry).*
3. *If one of eq. (6),eq. (7) is essentially strictly feasible and bounded, then strong duality holds, i.e. $p^* = d^*$ and the other problem is solvable (dual attainment). In particular, if both problems are strictly feasible, then there exists a pair (x^*, y^*) of primal and dual optimal solutions.*
4. *Assume (x^*, y^*) is a feasible pair. If strong duality holds, then TFAE*
   (a) *(x^*, y^*) is a pair of optimal solutions.*
   (b) *$c^T x^* ? b^T y^*$ (no duality gap)*
   (c) *$\forall i \in [m].\langle y_i^*, A_i x^* - b_i\rangle = 0$ (complementary slackness)*

Proof. 1. already done

2. later

3. Recall that all conic problems scan be put into the following form:

$$p^* = \inf \left\{ c^T : AX :_K self(b) \right\} d^* = \sup \left\{ b^T y : A^T y = c, \succeq_{K^*} 0 \right\}$$

   By symmetry we can assume that the primal problem is strictly feasible and bounded. Hence we have

$$p^* > -\infty \qquad \exists x \in \mathbb{R}^n : Ax \succ_K b$$

   We need to show that

$$\exists y \succeq_{K^*} 0. A^T y = c, b^y \geq p^*$$

   Let $M := \{Ax - b : c^T x \leq p^*\}$. We want to apply separating hyperplanes on M and $\operatorname{int} K$. Since p^* is some finite vector, we have $M \neq \emptyset$. $\operatorname{int} K \neq \emptyset$, because K is proper. To show that they do not intersect, assume there exists some x with $c^T x \leq p^*$ and $y = Ax - b$ for $y \in \operatorname{int} K$. Then $Ax \succ_K b$. But then $Ax' \succeq b$ for all x' in the vicinity of x. Hence there exists some x' with $Ax' \succeq b$ and $c^T x' < c^T x \leq p^*$. This contradicts optimality of p^*.

   Therefore by Theorem 8.9 there exist some hyperplane given by $0 \neq z \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$ where $z^T y \leq \mu$ for all $y \in M$ and $z^T, y\rangle \geq \mu$ for all $y \in \overline{int}(K)$. In fact we have $\forall y \in \operatorname{int} K. z^T y \geq 0$. Otherwise $z^T(ty) \to -\infty$ as $t \to \infty$. So wlog we can assume $\mu \geq 0$. Furthermore $z^T y \to 0$ as $y \to 0$, so $\mu \leq 0$. This is equivalent to

$$\forall y \in \operatorname{int} K. z^T y \geq 0 \tag{8}$$
$$\forall x : c^T x \leq p^*. z^T (Ax - b) \leq 0 \tag{9}$$

   Condition eq. (8) we rephrase as

$$\forall y \succ_K 0. \langle z, y \rangle \geq 0 \Leftrightarrow \forall y \succeq_K 0. \langle z, y \rangle \geq 0 \Leftrightarrow z \in K^*$$

   For eq. (9) we have $c^T x \leq p^*$ implies $\langle x, A^T z \rangle \leq b^T z$. This is a linear function of x bounded on a halfspace. Hence $A^T z = \mu c$ for some $\mu \geq 0$. We want to show that $\mu \neq 0$. Assume $\mu = 0$, then $A^T z = 0$, so $b^T z \geq 0$. By assumption there exists some $\widehat{x}$ with $A\widehat{x} - b \succ_K 0$. We have $z \succeq_{K^*} 0$ and $z \neq 0$. So $\langle z, A\widehat{x} - b \rangle > 0$, which means $\widehat{x}^T A^T z > b^T z = 0$. ↯

   This means $x \succeq_K 0$, $x \neq 0$ and $y \succ_{K^*} 0$, so $\langle x, y \rangle > 0$.

   We can conclude the proof by setting $y = \frac{z}{\mu}$. Then $y \succeq_{K^*} 0$ and $A^T y = c$.

   And from eq. (9) $c^T x \leq p^*$, so $x^T A^T y \leq b^T y$. So $b^T y \geq p^*$. And for $c = 0$, we get $p^* = 0$ and $d^* \geq 0$ (for $y = 0$).

4. Assume strong duality holds. So (x^*, y^*) is optimal iff $p^* = c^T x^* = b^T y^* = d^*$. And since y^* is feasible, the duality gap is

$$\underbrace{c^T}_{A^T y} x^* - b^T y^* = (y^*)^T (Ax^* - b) = 0 \text{ when slackness holds} \qquad \square$$

missing lecture

12 Sensitivity analysis

Regard the problem $P_{u,v}$ given by

$$p^*(u,v) := \min\{f_0(x) : x \in \mathbb{R}^n, \forall i \leq m. f_i(x) \leq u_i, \forall j \leq p. h_j(x) = v_j\}$$

We write $p^* := p^*(0,0)$ for the unperturbed problem and put $(P) := (P_{0,0})$.

12.1 Theorem (Global Sensitivity). *Assume that (P) is a convex optimisation problem and strong duality holds. If (λ^*, μ^*) is a pair of optimal Lagrange multipliers, then*

$$\forall (u,v) \in \mathbb{R}^m \times \mathbb{R}^p. a^*(u,v) \geq p^* - u^T\lambda^* - v^T\mu^*$$

Proof. By strong duality $p^* = d^* = g(\lambda^*, \mu^*)$. Let X be feasible for the perturbed problem.

$$g(\lambda^*, \mu^*) \leq \mathcal{L}(x, \lambda^*, \mu^*) = f_0(x) + \sum_i \lambda_i^* f_i(x) + \sum_j \mu_j^* h_j(x) \leq f_0(x) + (\lambda^*)^T u + (\mu^*)^T v$$

Taking the infimum over all feasible x yields

$$p^* \leq p^*(u,v) + (\lambda^*)^T u + (\mu^*)^T v$$ □

12.2 Theorem (Local Sensitivity). *Taking the assumptions from Theorem 12.1, and additionally assume $p^*(\cdot,\cdot)$ is differentiable at 0. Then $\frac{\partial p^*}{\partial u_i}(0,0) = -\lambda_i^*$ and $\frac{\partial p^*}{\partial v_j}(0,0) = -\mu_j^*$.*

Proof. By definition

$$\frac{\partial p^*}{\partial u_i}(0,0) = \lim_{t \to \infty} \frac{p^*(te_i, 0) - p^*}{t}$$

By the global sensitivity $\forall t \in \mathbb{R}. p^*(te_i, 0) \geq p^* - t\lambda_i^*$. So

$$\frac{p^*(te_i,0) - p^*}{t} \geq -\lambda_i^* \qquad \text{if } t > 0$$
$$\frac{p^*(te_i,0) - p^*}{t} \leq -\lambda_i^* \qquad \text{if } t < 0$$

Since p^* is differentiable, the limit must be the same. □

Note that differentiability is a rather hard assumption, since often an optimum is attained at the intersection of constraints. So at these points, it is not clear, that p^* is differentiable, even if all functions involved are smooth.

Dual of $\mathbb{R}_+^n$, $\mathbb{L}_+^n$, $\mathbb{S}_+^n$

Recall that for cone $K \subseteq \mathbb{R}^n$ its dual is

$$K^* = \{y \in \mathbb{R}^n : \forall x \in K. \langle x, y\rangle \geq 0\}$$

$K = \mathbb{R}_+^n$: For $y \in K$, we have $y \in K^*$, because we take the scalar product of only positive values. If $y \notin K$, then $y_i < 0$ for some i, so take $x = e_i$. Then $\langle x, y >= y_i < 0$, so $y \notin K$. This shows $K^* = K$.

$K = \mathbb{S}^n_+$**:** Assume $Y \in K$. Take some arbitrary $X \in K$. Write $X = HH^T$ and $Y = KK^T$. Then $\langle X, Y\rangle = \operatorname{tr}(K^T H H^T K) = \|H^T K\|_F^2 \geq 0$.

Let $Y \notin L$, then it has a negative eigenvalue $\lambda < 0$ with eigenvector $Yu = \lambda u$. Set $X = uu^T$. Then

$$\langle X, Y\rangle = \operatorname{tr}(uu^T Y) = u^T Y u = \lambda u^T u = \lambda \|u\|^2 < 0$$

$K = \mathbb{L}^{n+1}_+$**:** More explicitly

$$K = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \|x\| \leq t\}$$
$$K^* = \left\{(y,s) \in \mathbb{R}^{n+1} : \forall \|x\| \leq t . x^T y + st \geq 0\right\}$$

Let $(y,s) \in K$ and let $\|x\| \leq t$. Then

$$x^T y \geq -\|x\| \cdot \|y\| \geq -st \implies x^T y + st \geq 0$$

For the converse, suppose $(y,s) \notin K$, i.e. $\|y\| > s$. Set $x = y$ and $t = \|y\|$. Then $x^T y + st < -\|y\|^2 + \|y\|^2 = 0$, so $(y,s) \notin K^*$.

Geometric Programming

We define the exponential cone

$$K_{\text{exp}} = \left\{(x,y,z) \in \mathbb{R}^3 : y > 0, y e^{\frac{x}{y}} \leq z\right\}$$

Since this cone is not closed (due to $y > 0$) we take its closure

$$K_e = \operatorname{cl}(K_{\text{exp}}) = K_{\text{exp}} \cup \{(x,y,z) : x \leq 0, y = 0, z \geq 0\}$$

The last term is the epigraph of the exponential function.
We can show that it is *proper* (although technical) with

$$K_e^* = \left\{(u,v,w) \in \mathbb{R}^3 : u \leq 0, w \geq 0, u \log\left(-\frac{w}{u}\right) \leq v - u\right\}$$

The last term is called *relative entropy.*

12.3 Example (K_e-representable functions).

- $e^x \leq t \Leftrightarrow (x, 1, t) \in K_e$.
- $\log(x) \geq t \Leftrightarrow x \geq e^t \Leftrightarrow (t, 1, x) \in K_e$
- $\sum x_i \log \frac{x_i}{y_i} \leq t \Leftrightarrow \exists z_i . x_i \log \frac{x_i}{y_i} \leq z_i \wedge \sum z_i \leq t$. And with the above $x_i \log \frac{x_i}{y_i} \leq z_i \Leftrightarrow (-z_i, x_i, y_i) \in K_e$. The linear condition is $\sum z_i \leq t \Leftrightarrow (0, 0, t - \sum z_i) \in K_e$.
- $\operatorname{LSE}(x) := \log \sum \exp(x_i) \leq t$ also is K_e-representable.

12.4 Definition. A posynomial is an expression of the form

$$p = \sum \alpha_j x_1^{p_1} \cdot \ldots \cdot x_n^{p_n}$$

where $\alpha_j \in \mathbb{R}_+$ and $p_i \in \mathbb{R}$ (in particular may be negative, or non-integer).

12.5 Definition. A geometric programme in posynomial form is an optimisation problem of the form

$$\min\{f_0(x) : \forall i \le m. f_i(x) \le 1, \forall j \le p. h_j(x) = 1\}$$

where the f_i are posynomials, and the h_j are positive monomials.

12.6 Example. Regard the following GP

$$\begin{aligned} \min 3\frac{z}{y} + 2\sqrt{z}x^3y^{-7} + \frac{y}{x^2} & \\ \frac{x}{yz} + 3\sqrt{z} &\le 1 \\ 2x^3y^{-1} &= 1 \end{aligned}$$

With change of variables $x' = \log x$ etc. and taking the logarithm we obtain

$$\begin{aligned} \min \mathrm{LSE}\left(z' - y' + \log 3, \frac{1}{2}z' + 3x' - 7y' + \log 2, y' - 2x'\right) & \\ \mathrm{LSE}\left(x' - y' - z', \frac{1}{2}z' + \log 3\right) &\le 0 \\ 3x' - y' + \log 2 &= 0 \end{aligned}$$

Example. *Maximise volume of a box, i.e.* $\min w^{-1}d^{-1}h^{-1}$ *with constraints*

- *Area of walls* $2wh + 2dh \le W$
- *are of floor* $wd \le F$
- *bound on aspects ratios* $\frac{w}{h} \in [\alpha, \beta]$

This is the reason, why it is called Geometric Programme.

Example (Truss Design). *When constructing a bridge, we have some vector expressing the external forces. Each force creates a small displacement of the structure. But some of the nodes are fixed. When deciding the width of each bar, this can be expressed as a GP.*

13 SDP in Combinatorial Optimisation

13.1 Independent sets and graph colouring

The problem of finding a stable set of maximal size is a well-known NP-hard problem. It can be formulated as integer programme as follows:

$$\max\left\{\sum_{v\in V} x_v : \forall\{u,v\} \in E. x_u + x_v \le 1, \forall v \in V. x_v \in \{0,1\}\right\}$$

Just dropping the integrality constraint would yield the feasible solution $x_v = \frac{1}{2}$ for all v. This usually does not lift to to original problem. In case of the complete graph, the relaxation yields $\frac{1}{2}|V|$, but the solution is 1.

13.1 Definition (Clique-Cover). A *k-clique cover* is a partition of V in k sets such that each set forms a clique. The smallest $k \in \mathbb{N}$ such that a k-clique cover exists, is called the clique cover number.

13.2 Definition. Let $G = (V, E)$ be some graph. We denote

- $\alpha(G)$ maximal size of an independent set
- $\omega(G)$ maximal clique size
- $\chi(G)$ colouring number
- $\overline{\chi}(G)$ clique cover number.

13.3 Proposition. *Let $\overline{G}$ denote the complementary graph. Then $\alpha(G) = \omega(\overline{G})$ and $\chi(G) = \overline{\chi}(\overline{G})$.*

Proof. S is stable set of G iff S is a clique in $\overline{G}$, same with the other. □

Remark. *In a clique each vertex must have a different colour. Thus $\omega(G) \leq \chi(G)$. With Proposition 13.3 we get $\alpha(G) \leq \overline{\chi}(G)$.*

13.4 Definition. A graph is *perfect* if $\omega(G) = \chi(G)$.

13.5 Theorem. *A graph G is perfect (iff $\overline{G}$ is perfect) iff neither G nor $\overline{G}$ contains an odd cycle of length ≥ 5 as in induced subgraph. Thus the property is decidable in polynomial time.*

13.6 Definition (Lovasz' Θ-function). We put

$$\Theta(G) := \max \{\langle \mathbf{1}, X\rangle : \langle I, X\rangle = 1, \forall \{i,j\} \in E. X_{ij} = 0, X \succeq 0\}$$

where I is the identity and $\mathbf{1}$ is the all 1 matrix, so $\langle \mathbf{1}, X\rangle = \sum_{i,j} X_{ij}$. Note that the above is an SDP.

13.7 Proposition. *$\Theta(G)$ can also be defined by using the dual SDP*

$$\Theta(G) = \min \left\{t : Z \preceq tI, \forall i,j. \{i,j \in \overline{E} \vee i = j \to Z_{ij} = 1\right\}$$

or by writing

$$\begin{aligned} \mathcal{Z} &:= \left\{Z \in \mathbb{S}^n : \forall i \in V. Z_{ii} = 1, \forall \{i,j\} \in \overline{E}. Z_{ij} = 1\right\} \\ \Theta(G) &= \min \{\lambda_{\max}(Z) : Z \in \mathcal{Z}\} \end{aligned}$$

Proof. First we have

$$\Theta(G) = \sup \left\{ \langle \mathbf{1}, X\rangle + + \inf \{t(1 - \langle I, X\rangle : t \in \mathbb{R}\} + \sum_{\{i,j\}\in E} \inf \left\{\mu_{ij}(1 - X_{ij}) : \mu \in \mathbb{R}^{n\times n}\right\} : X \succeq 0 \right\}$$

The dual problem is obtained by switching the order of supremum and infimum.

$$\Theta(G) \leq \Theta'(G) = \inf_{t\in\mathbb{R}, \mu\in\mathbb{R}^{n\times n}} t + \sum \mu_{ij} + \underbrace{\sup \left\{ \langle X, \mathbf{1} - tI - \sum_{\{i,j\}\in E} \mu_{ij} E_{ij}\rangle : X \succeq 0 \right\}}_{\begin{cases} 0 & : \mathbf{1} - tI - \sum \mu_{ij} E_{ij} \preceq 0 \\ \infty & : \text{else} \end{cases}}$$

$$\Theta'(G) = \inf \left\{ t : t \in \mathbb{R}, \mu \in \mathbb{R}^{n\times n}, \mathbf{1} - tI - \sum_{\{i,j\}\in E} \mu_{ij} E_{ij} \preceq 0 \right\}$$

Now we obtain the desired dual SDP with the change of variable $Z = \mathbf{1} - \sum \mu_{ij} E_{ij}$. The change back is given by

$$Z \in \mathcal{Z} \Leftrightarrow \exists \mu. Z = \mathbf{1} - \sum \mu_{ih} E_{ij}$$

□

We are going to show that both SDPs are strictly feasible, so they are bounded. Thus strong duality holds and the optimum is attained in both problems.
$X := \frac{1}{n} I$ is strictly feasible for the primal SDP. For the dual we have $Z \prec tI \Leftrightarrow t > \lambda_{\max}(Z)$. So we take an arbitrary matrix $Z \in \mathcal{Z}$, e.g. $Z = \mathbf{1}$ and then some $t > \lambda_{\max}(Z)$. So (t, Z) is strictly feasible for the dual SDP.

13.8 Theorem (Lovasz' sandwich theorem). *$\alpha(G) \leq \Theta(G) \leq \overline{\chi}(G)$.*

Proof. For the first inequality let S be a maximum independent set. Let e_S be the incidence vector (characteristic function) of S. We claim that $X = \frac{1}{|S|} e_S \cdot e_S^T$ is feasible for the primal.

$$\operatorname{tr}(X) = \langle I, X \rangle = \frac{e_S^T e_S}{|S|} = \frac{|S|}{|S|} = 1$$

Furthermore we have

$$X_{ij} = \begin{cases} \frac{1}{|S|} & : i, j \in S \\ 0 & : \text{else} \end{cases}$$

So for all edges we have $X_{ij} = 0$ because S is stable. Hence we obtain a bound

$$\Theta(G) \geq \langle \mathbf{1}, X \rangle = \langle \mathbf{1}, \frac{e_S e_S^T}{|S|} \rangle = \frac{(e_S^T \mathbf{1})^2}{|S|} = |S| = \alpha(G)$$

To see $\Theta(G) \leq \overline{\chi}(G)$, take a minimal k-clique-cover $C_1, \ldots, C_k$ with incidence vectors e_{C_i}. We claim that $t = k$ and

$$Z = kI - \frac{1}{k} \sum_{j=1}^{k} (ke_{C_j} - \mathbf{1})(ke_{C_j} - \mathbf{1})^T$$

is dual feasible. We have

$$tI - Z = \frac{1}{k} \sum_{j=1}^{k} (ke_{C_j} - \mathbf{1})(ke_{C_j} - \mathbf{1})^T \succeq 0$$

Now observe $\sum e_{C_j} = \mathbf{1}$. So we can write

$$Z = kI - \frac{1}{k} \left(k^2 \sum e_{C_j} e_{C_j^T} - 2k\mathbf{1} + k\mathbf{1} \right) = k \left(I - \sum e_{C_j} e_{C_j}^T \right) + \mathbf{1}$$

After permutation of the vertices we have a block matrix

$$\sum e_{C_j} e_{C_j}^T = \begin{pmatrix} \mathbf{1} & \cdots & \\ & \ddots & \\ & \cdots & \mathbf{1} \end{pmatrix}$$

part missing

□

13.9 Lemma. *We have*

$$\inf_{X \succeq_K 0} \langle X, A\rangle = \begin{cases} 0 & : A \succeq_{K^*} 0 \\ -\infty & : else \end{cases} \quad \sup_{X \succeq_K 0} \langle X, A\rangle = \begin{cases} 0 & : A \preceq_{K^*} 0 \\ \infty & : else \end{cases}$$

Proof. For the first part:

$$A \in K^* \Leftrightarrow \forall x \in (K^*)^* = K. \langle A, X\rangle \geq 0$$

Since we can obtain $\langle A, \mathbf{0}\rangle = 0$, and

$$A \in K^* \implies \inf_{x \succeq_K 0} \langle A, X\rangle \geq 0$$

we get the claim for the first case. For $A \notin K^*$ take X_0 with $\langle A, X_0\rangle < 0$. Scaling X_0 goes to $-\infty$. □

Example. *Fix some i, j. Regard the problem*

$$\textbf{??} \min \{X_{ij} : \operatorname{tr} X = 1, X \succeq 0\} \tag{P}$$

We write **??** *as a saddle point of the Lagrangian*

$$\min_{X \succeq 0} X_{ij} + \max_{\lambda} \lambda(1 - \operatorname{tr} X)$$

The dual problem is obtained by switching the order of min *and* max*:*

$$\max_{\lambda} \lambda + \min_{X \succeq 0} X_{ij} - \lambda \operatorname{tr} X$$

Putting $E_{ij} = \frac{1}{2}(e_i e_j^T + e_j e_i^T)$ we get $X_{ij} = \langle X, E_{ij}\rangle$ we can rewrite this to

$$\max_{\lambda} \lambda + \min_{X \succeq 0} \langle X, E_{ij} - \lambda I\rangle$$

Hence the dual problem is

$$\max \{\lambda : E_{ij} - \lambda I = Z, Z \succeq 0\}$$

Therefore taking the dual does not help, but makes the problem more complicated.

13.2 Max-Cut SDP

13.10 Definition. Let $G = (V, E)$ be a graph with edge weights w_{ij} (put $w_{ij} = 0$ for $\{i, j\} \notin E$). A cut is a partition $V = S \cup \overline{S}$. The value of a cut is

$$\operatorname{cut}\left(S, \overline{S}\right) = \sum_{i \in S, j \notin S} w_{ij}$$

The Max-cut problem is to find some cut with the largest value.

Let $x \in \{-1, 1\}^n$ denote the incidence vector of a cut ($x_i = 1 \Leftrightarrow i \in S$). An edge $\{i, j\}$ contributes to the cut iff $x_i x_j = -1$. So the problem can be written as

$$\max \left\{ \sum_{i,j \in V} w_{ij} \cdot \frac{1}{4}(1 - x_i x_j) : \forall i \in V. x_i^2 - 1 = 0 \right\}$$

13.11 Lemma. *Some matrix $X \in \mathbb{S}^n$ satisfies $X_{ij} = x_i x_j$ for some vector $x \in \{-1,1\}^n$ iff*

$$X \succeq 0 \qquad\qquad \operatorname{diag} X = \mathbf{1} \qquad\qquad \operatorname{rank}(X) = 1$$

Proof. We know that psd-matrices of rank 1 have the form $X = uu^T$ for some $u \in \mathbb{R}^n$. Therefore $X_{ij} = x_i x_j \Leftrightarrow X = xx^T$. Then $x_i \in \{-1,1\} \Leftrightarrow x_i^2 = 1 \Leftrightarrow X_{ii} = 1$. □

Now let $W \in \mathbb{S}^n$ be the matrix of weights. This means $\operatorname{cut}(X) = \frac{1}{4}\langle W, \mathbf{1} - xx^T\rangle$. By Lemma 13.11 we can reformulate the Max-Cut problem:

$$\operatorname{maxcut}(G) := \max\left\{\frac{1}{4}\langle W, \mathbf{1} - X\rangle : X \succeq 0, \operatorname{diag} X = \mathbf{1}, \operatorname{rank} X = 1\right\}$$

Not the rank condition causes trouble, since it is non-convex. So we just drop it and thus get an SDP-relaxation

$$\operatorname{rel}(G) := \frac{1}{4} \cdot \max\left\{\langle W, \mathbf{1} - X\rangle : X \succeq 0, \forall i. X_{ii} = 1\right\}$$

Since we only dropped a constraint, we immediately have $\operatorname{maxcut}(G) \leq \operatorname{rel}(G)$.
The algorithm of Godmans and Williamson uses this SDP to construct an approximation for Max-Cut.

1. Compute a solution X^* of this SDP.
2. Take a decomposition $X^* = HH^T$. Denote by h_i^T the i-th row of H, so $X_{ij}^* = \langle h_i, h_j\rangle$.
3. Choose a vector uniformly at random over the unit sphere. To do this, draw $z \sim \mathcal{N}(0,1)$ and take $r := \frac{z}{\|z\|}$.
4. Define $S = \{i : r^T h_i > 0\}$.

13.12 Theorem (Godmans, Williamson). *Let S be the random cut returned by the above algorithm. Then*

$$\mathbb{E}\left[\operatorname{cut}\left(S, \overline{S}\right)\right] \geq \alpha \operatorname{rel}(G) \geq \alpha \operatorname{maxcut}(G)$$

where $\alpha \approx 0.87856$.

13.13 Lemma. *Let u, v be on the unit sphere of $\mathbb{R}^n$. Let $r \in \mathbb{R}^n$ be drawn uniformly at random on the sphere. Put $H := \left\{x : x^T r \geq 0\right\}$. Then u and v are separated by H with probability $\frac{\arccos(u^T v)}{\pi}$.*

Proof. Note that $\theta := \arccos(u^T v)$ is the angle between those two vectors. We have again angle θ in the opposite direction, so the probability of separating them is $\frac{2\theta}{2\pi}$. □

of Theorem 13.12.

$$\mathbb{E}\left[\operatorname{cut}\left(S, \overline{S}\right)\right] = \sum_{\{i,j\}\in E} w_{ij} \mathbb{P}\left[i \in S, j \notin S\right] = \sum_{\{i,j\}\in E} w_{ij} \frac{\arccos(h_i^T h_j)}{\pi} = \sum_{\{i,j\}\in E} w_{ij} \frac{1 - h < -i^T h_j}{2} \cdot \underbrace{\frac{2\arccos(h_i^T}{\pi \cdot (1 - h_i^T}}_{\geq \alpha}$$

To show this last inequality, fix i, j. Then

$$\frac{2}{\pi} \cdot \frac{\arccos(h_i^T h_j)}{1 - h_i^T h_j} \geq \inf_{u \in [-1,1]} \frac{2}{\pi} \cdot \frac{\arccos(u)}{1 - u} = \inf_{\theta \in [0,\pi]} \frac{2}{\pi} \cdot \frac{\theta}{1 - \cos\theta} =: \alpha$$

□

Alternative construction of the Max-Cut SDP: Recall that

$$\mathrm{maxcut}(G) = \frac{1}{4}\max\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}(1 - x_i x_j) : \forall i. x_i \in \{-1, 1\}\right\}$$

Let $\mathscr{S}_n := \{x \in \mathbb{R}^n : \|x\| = 1\}$. So our constraint is $x_i \in \mathscr{S}_1$. By allowing $x_i \in \mathscr{S}_n$, we get

$$\frac{1}{4}\max\left\{\sum_{i,j\in V} w_{ij}(1 - \langle x_i, x_j\rangle) : \forall i. x_i \in \mathscr{S}_n\right\} = \frac{1}{4}\max\left\{\sum_{i,j\in V} w_{ij}(1 - X_{ij}) : X \succeq 0\right\}$$

which is another SDP-relaxation.

13.3 SDP relaxation of non.convex (in particular binary) quadratic problems

We consider the probem

$$\min\left\{x^T Q_0 x + c_0^T x + q_0 : x \in \mathbb{R}^n, x^T Q_i x + c_i^T x + q_i \gtrless 0\right\} \tag{QCQP}$$

We can model binary constraints via $x_i^2 = x_i$ (if $x_i \in \{0, 1\}$), or $x_i^2 = 1$ (if $x_i = \pm 1$).
Observe that QCQP is not convex in general. QCQP can be rewritten as

$$\min\left\{\langle Q_0, X\rangle + c_0^T x + q_0 : x \in \mathbb{R}^n, X \in \mathbb{S}^n, \langle Q_i, X\rangle + c_i^T x + q_i \gtrless 0, X = xx^T\right\}$$

Now we relax the constraint $X = xx^T$ be replacing it with the weaker (but convex)

$$X \succeq xx^T \Leftrightarrow \begin{pmatrix} X & x \\ x^T & 1 \end{pmatrix} \succeq 0$$

13.14 Proposition. *The SDP*

$$\min\left\{\langle Q_0, X\rangle + c_0^T x + q_0 : x \in \mathbb{R}^n, X \in \mathbb{S}^n, \langle Q_i, X\rangle + c_i^T x + q_i \gtrless 0, X \succeq xx^T\right\}$$

is a relaxation of QCQP. Its optimal value gives a lower bound to the optimal value of QCQP.

13.15 Remark. Binary constraints $x_i \in \{0, 1\}$ in the SDP relaxation become $X_{ii} = x_i$.

For short write $\overline{X} = \begin{pmatrix} X & x \\ x^T & 1 \end{pmatrix}$ and the same for $\overline{X}^*$.

13.16 Proposition. *If (X^*, x^*) is an optimal solution of the SDP, and $\mathrm{rank}(\overline{X}^*) = 1$, then x^* solves the original problem.*

Proof. If $X \succeq xx^T$ and $\mathrm{rank}\, X = 1$, then we can write $\overline{X}^* = (u, \alpha)(u, \alpha)^T$ for some $u \in \mathbb{R}^n$,$\alpha \in \mathbb{R}$. Thus we obtain $X = uu^T$, $\alpha^2 = 1$ and $x = \alpha u = \pm u$. Therefore $X = xx^T$ and we have a solution which is feasible for QCQP (hence optimal, since we regarded a relaxation). □

13.17 Example. We regard k-exact quadratic knapsack:

$$\max\left\{\sum c_{ij} x_i x_j : \sum a_i x_i \leq b, \sum x_i = k, x_i \in \{0, 1\}\right\}$$

This means, we must take exactly k items and any combination of 2 items might have some "synergy" bonus. By taking value function c as diagonal matrix, we obtain the original k-exact knapsack. a_i are the individual weights, b is the capacity.
The SDP relaxation is

$$\max\left\{\langle C, X\rangle : \sum a_i x_i \leq b, \sum x_i = k, \overline{X} \succeq 0, \mathrm{diag}(X) = x\right\}$$

We can add the redundant constraint $\mathbf{1}X\mathbf{1} = k^2$ (which comes from squaring $\mathbf{1}^T x = k$).

13.4 Completely positive formulation of mixed-integer QP

13.18 Definition (Copositive cone). A matrix $X \in \mathbb{S}^n$ is *copositive* if $u^T X u \geq 0$ for all $u \in \mathbb{R}^n_+$ (note that in contrast to psd-matrices, we only demand this for $u \geq 0$). This set is denoted $\mathscr{C}_n$.

13.19 Definition (Completely positive cone). A matrix $X \in \mathbb{S}^n$ is *completely positive* if $X = \sum_{i=1}^q u_k u_k^T$ for some $q \in \mathbb{N}$ and $u_k \in \mathbb{R}^n_+$. (Again only regard $u_k \geq 0$.) The set is denoted $\mathscr{C}_n^*$.

13.20 Proposition. *1. If $X \in \mathscr{C}_n^*$ we can always choose $q \leq \frac{n(n+1)}{2}$.*

2. We have the chain

$$\mathscr{C}_n^* \subseteq \mathbb{S}^n_+ \cap \mathbb{R}^{n \times n}_+ \subseteq \mathbb{S}^n_+ \subseteq (\mathbb{S}^n_+ + \mathbb{R}^{n\times n}_+) \subseteq \mathscr{C}_n$$

3. $\mathscr{C}_n$ and $\mathscr{C}_n^*$ are proper cones and dual to each other (thus the notation $\mathscr{C}_n^*$ is justified).

Proof. 1. $\mathscr{C}_n^* \subseteq \mathbb{S}^n$, and the affine dimension of $\mathbb{S}^n$ is $\frac{n(n+1)}{2}$.

2. trivial. For the last note $\mathbb{S}^n_+, \mathbb{R}^{n \times n}_+ \subseteq \mathscr{C}_n$ and it is a cone.

3. It suffices to show duality and that one of the cones is proper.

   We show that $\mathscr{C}_n$ is proper:

   convex: It is the intersection of infinitely many half-spaces.

   convex: clear

   non-empty interior: We have $\mathbb{S}^n_+ \subseteq \mathscr{C}_n$ and $\mathbb{S}^n_+$ has non-empty interior

   pointed: Let $X \in \mathscr{C}_n \cap -\mathscr{C}_n$. Then we get $u^T X u = 0$ for all $u \in \mathbb{R}^n_+$. By taking $u = e_i$, we get $X_{ii} = 0$. Then we continue with $u = e_i + e_j$ and obtain $0 = 2X_{ij} + X_{ii} + X_{jj} = 2X_{ij}$. Hence $X = \mathbf{0}$, which shows that $\mathcal{C}_n$ is pointed.

   Now let $Y \in \operatorname{dual}(\mathscr{C}_n^*)$. This means

$$\begin{aligned} &\forall X \in \mathscr{C}_n^*. \langle X, Y \rangle \geq 0 \\ \Leftrightarrow &\forall u_1, \ldots, u_q \geq 0. \sum u_k^T Y u_k \geq 0 \\ \Leftrightarrow &\forall u \geq 0. u^T Y u \geq 0 \Leftrightarrow Y \in \mathscr{C}_n \end{aligned}$$

□

Now we consider the following MIQP: $B \subseteq [n]$ denote the indices of binary variables.

$$\min \left\{ x^T Q x + c^T x : Ax = b, X \geq 0, \forall i \in B. x_i \in \{0,1\} \right\}$$

Now in the relaxation the matrix $X = xx^T$ is completely positive, due to the constraint $x \geq 0$. Furthermore, we rewrite $a_i^T x = b_i$ to $a_i^T x x^T a_i = b_i^2$.

13.21 Theorem (Burer). *Under some mild assumptions the completely positive programme* specify

$$\min \left\{ \langle Q, X \rangle + c^T x : Ax = b, \forall i. a_i^T X a_i = b_i^2, \forall i \in B. X_{ii} = x_i, \overline{X} \succeq 0 \right\}$$

is equivalent to the original MIQP.

14 Robust Optimisation

So far we regarded problem of the type

$$\min \{f_0(x) : \forall i \in [n].f_i(x) \leq 0\}$$

where all f_i are convex.
If we have some constraint of the type

$$x_1 + x_2 + x_3 + x_7 \leq 2 \qquad\qquad x \geq 0$$

we can think of this as a precise constraint. But of we get our constraint from real world data, we can have error in the measurement, e.g.

$$1.2037x_1 + 2.3303x_2 - 5.7701x_3 \leq 0$$

Here it might happen that a slight change in the coefficients makes the problem infeasible.
Formally, assume that the objective and constraints depend on a parameter θ.
If we have some estimate θ_0 of θ, one can solve the nominal problem

$$\min \{f_0(x, \theta_0) : \forall i.x_i(x, \theta_0) \leq 0\} \tag{NP}$$

The solution of eq. (NP) is not guaranteed to be feasible for the "true" value of the unknown parameter θ. In Robust Optimisation we protect ourselves by optimising against the worst-case in some uncertainty set $\Theta \subseteq \mathbb{R}^k$.
Formally the robust counterpart is

$$\min \left\{ \sup_{\theta \in \Theta} f_0(x, \theta) : \forall i.\forall \theta \in \Theta.f_i(x, \theta) \leq 0 \right\} \tag{RP}$$

This is an alternative to the stochastic programming approach, where θ is a random variable and one solves

$$\min \{\mathbb{E}_0 f_0(x, \theta) : \forall i.\mathbb{P}\,(f_i(x, \theta) \leq 0) = 1\} \tag{SP}$$

14.1 robust Linear Programming

Consider the robust LP

$$\min \left\{ \sup_{\theta \in \Theta} c(\theta)^T \cdot x : \forall \theta \in \Theta.A(\theta)x \leq b(\theta) \right\} \tag{RLP}$$

Wlog we can assume that c and b are not affected by the uncertainty, by moving it to A. To see this, observe that eq. (RLP) is equivalent to

$$\min \left\{t : \forall \theta \in \Theta.A(\theta)x - b(\theta)z \leq 0, c(\theta)^T x \leq t, z = 1\right\}$$

Let $a_i^T(\theta$ denote the i-th row of $A(\theta)$ and rewrite

$$\min \left\{c^T x : \forall i.\forall \theta \in \Theta.a_i^T(\theta)x \leq b_i\right\}$$

Constraints of this form are called semi-infinite linear constraints, because they involve a finite number of variables but an infinite amount of linear constraints parametrised by θ. It still is a convex problem.

14.2 Robust counterpart of eq. (RLP) for different uncertainty models

An uncertainty model is given by

- the functions $a : \Theta \to \mathbb{R}^n$
- the uncertainty set $\Theta \subseteq \mathbb{R}^k$

Alternatively we can specify the row-uncertainty set

$$\mathcal{A} = \{a(\theta) : \theta \in \Theta\}$$

such that the semi-definite constraint

$$\forall \theta \in \Theta . a(\theta)^T x \leq b \tag{SIC}$$

is equivalent to

$$\forall a \in \mathcal{A} . a^T x \leq b \tag{SIC}$$

14.2.1 Polyhedral uncertainty

$\mathcal{A}$ is a polytope described by its vertices $\mathcal{A} = \mathrm{conv}(\{v_1, \ldots, v_k\})$. Equivalently

$$a(\theta) = \sum_{i=1}^{k} \theta_i v_i \qquad\qquad \Theta = \left\{\theta \geq 0 : \sum \theta_i = 1\right\}$$

14.1 Proposition. *In the polyhedral uncertainty model eq.* (SIC) *is equivalent to* $\forall i \in [k] . v_i^T x \leq b$.

Proof. Assume $\forall i . v_i^T x \leq b$. Then if $a \in \mathcal{A}$, i.e. $a = \sum \theta_i v_i$ for some $\theta \in \Theta$, we have

$$\underbrace{\sum \theta_i (v_i^T}_{a} x) \leq \underbrace{\sum \theta_i}_{=1} b$$

and thus $a^T x \leq b$. Conversely if $\forall a \in \mathcal{A} . a^T x \leq b$, then in particular this holds at $v_i \in \mathcal{A}$. □

14.2.2 Conic uncertainty model

$$\mathcal{A} = \{\underbrace{\overline{a} + P\theta}_{a(\theta)} : \underbrace{F\theta \preceq_K h}_{\theta \in \Theta}\}$$

for some proper cone $K \subseteq \mathbb{R}^l$, i.e. $h \in \mathbb{R}^l$.

14.2 Theorem. *Assume that the conic inequality is essentially strictly feasible. Then in the conic uncertainty model eq.* (SIC) *is equivalent to the following system that involves additional variable* $z \in \mathbb{R}^l$*:*

$$\exists z \in \mathbb{R}^l . \left(\overline{a}^T x + h^T z \leq b, F^T z = P^T x, z \succeq_{K^*} 0\right)$$

Proof. Observe that eq. (SIC) is equivalent to

$$\begin{aligned}
&\sup_{a \in \mathcal{A}} a^T x \leq b \\
\Leftrightarrow &\sup_{\theta : F\theta \preceq_K h} (\overline{a} + P\theta)^T x \leq b \\
\Leftrightarrow &\sup_{\theta : F\theta \preceq_K h} \theta^T P^T x \leq b - \overline{a}^T x
\end{aligned}$$

The problem in the last line is a conic programming problem and string duality holds ($p^* = d^*$), because we assume $F\theta \preceq_K h$ is essentially strictly feasible. So

$$\sup \left\{\theta^T u : Ft \overset{\preceq}{_K} h\right\} = \inf \left\{h^T z : F^T z = u, z \succeq_{K^*} 0\right\}$$

holds for all u. For $u = P^T x$ we obtain

$$\text{eq. (SIC)} \Leftrightarrow \exists z \succeq_{K^*} 0. F^T z = u, h^T z \leq b - \overline{a}^T x$$

□

14.2.3 Budgeted uncertainty set

In this uncertainty model we assume that we know a nominal scenario $\overline{a}$ and for each coordinate i there is a maximal deviation $|a_i - \overline{a}_i| \leq \delta_i$. In addition we give a constant $\Gamma \geq 0$ such that $\sum \frac{|a_i - \overline{a}_i|}{\delta_i} \leq \Gamma$.
Equivalently we can parametrise the budgeted uncertainty set as follows, introducing $\theta_i = \frac{|a_i - \overline{a}_i|}{\delta_i}$:

$$\mathcal{A} = \{\overline{a} + \operatorname{diag}(\delta)\theta : \|\theta\|_\infty = 1, \|\theta\|_1 \leq \Gamma\}$$

Let $K \in \mathbb{R}^l$. Then we regard the problem

$$\forall \Theta. F\Theta \preceq_K h \rightarrow (\overline{a} + P\Theta)^T x \leq b \tag{10}$$

$$\Leftrightarrow \exists z \in \mathbb{R}^l. \overline{a}^T x + h^T z \leq b, F^T z = P^T x, z \succeq_{K^*} 0 \tag{11}$$

14.3 Example (Robust Knapsack). Regard the continuous knapsack problem

$$\max \left\{p^T x : w^T x \leq W, x \geq 0\right\}$$

where p is the profit, w are the weights, W is the capacity.
However, this problem is easy to solve, by just taking the most valuable (by ratio) item.
Now assume that w is uncertain and lives in a Γ-budgeted uncertainty set

$$w \in \mathcal{W} = \{\overline{w} + \operatorname{diag}(\delta)\theta : \|\theta\|_1 \leq \Gamma, \|\theta\|_\infty \leq 1\}$$

For knapsack, we can restrict to positive deviations

$$\widehat{\mathcal{W}} = \left\{\overline{w} + \delta^T\theta : \sum \theta_i \leq \Gamma, \forall i. 0 \leq \theta_i \leq 1\right\}$$

This is a constraint of the same form as eq. (10), with

$$\overline{a} = \overline{w} \qquad P = \operatorname{diag}(\delta) \qquad F = (I, -I, \mathbf{1})^T \qquad h = (\mathbf{1}, \mathbf{0}, \Gamma) \qquad K = \mathbb{R}_+^{2n+1}$$

Hence the semi-infinite constraint

$$\forall w \in \mathcal{W}. w^T x \leq W \Leftrightarrow \exists (z, y, \xi) \geq 0. \overline{w}^T x + \mathbf{1}^T z + \Gamma\xi \leq W, z - y + \Gamma\mathbf{1} = \operatorname{diag}(\delta)x$$

So the robust counterpart of the knapsack problem reduces to this LP.

$$\max \left\{p^T x : \overline{w}^T x + \mathbf{1}^T z + \Gamma\xi \leq W, z + \Gamma\mathbf{1} \geq \operatorname{diag}(\delta)x, x, z, \xi \geq 0\right\}$$

14.2.4 Ellipsoidal uncertainty

Let $\mathcal{E}$ be some ellipsoid.

$$\forall a \in \mathcal{E}.a^T x \leq b \Leftrightarrow \forall \theta.\|\theta\|_2 \leq 1 \rightarrow (\overline{a} + P\theta)^T x \leq b$$

This model is related to normally distributed measures $a \sim \delta(\overline{a}, \Sigma)$. Its dual problem is an SOCP (note that the dual for the polyhedral case is an LP).

14.3 Robust counterpart of SOCP

Here we regard the question whether we can handle the robust counterpart of semi-infinite constraints of the form

$$\forall \theta \in \Theta.\forall \theta \in \|A(\theta)x + b(\theta)\| \leq c(\theta)^T x + d(\theta)$$

In general the answer is "no", but possible is Θ is a Euclidean ball and c, d are constant. Norms are per default 2-norms.

Remark. *If A and b are affine wrt θ, then*

$$\|A(\theta)x + b(\theta)\| \leq t \Leftrightarrow \|y_0(x) + L(x)\theta\| \leq t$$

for some affine functions y_0 and L. Say

$$A(\theta) = A_0 + \sum \theta_i A_i \qquad\qquad b(\theta) = b_0 + \sum \theta_i b_i$$

then we can rewrite

$$A(\theta)x + b(\theta) = \underbrace{A_0 x + b_0}_{=:y_0(x)} + \underbrace{\sum \theta_i (A_i x + b_i)}_{=:L(x)}$$

14.4 Lemma. *For all $s \in \mathbb{R}$ we have*

$$\inf \left\{ s v^T L \theta : \|\theta\| \leq 1 \right\} = \inf \left\{ v^T L z : \|z\| \leq |s| \right\}$$

14.5 Theorem. *Let $A, B \in \mathbb{S}^n$ and assume $\exists x. x^T A x > 0$. Then*

$$\left(\forall x. x^T A x \geq 0 \rightarrow x^T B x >\geq 0 \right) \Leftrightarrow \exists \lambda \geq 0. B \succeq \lambda A$$

Proof. Exercise □

This can be seen as a strong duality result for the problem

$$\min \left\{ x^T B x : x^T A x \geq 0 \right\}$$

14.6 Theorem. *The semi-infinite SOCP-constraint*

$$\forall \|\theta\| \leq t. \|y_0(x) + L(x)\theta\| \leq 1$$

is equivalent to

$$\exists \lambda \geq 0. \begin{pmatrix} t - \lambda & y_0(x)^T & 0 \\ y_0(x) & tI & L(x) \\ 0 & L(x)^T & \lambda I \end{pmatrix} \succeq 0$$

Proof. Fix some x and write $y_0 = y_0(x)$ and $L = L(x)$. By using Schur-complement Lemma 9.8, the semi-infinite SOC-constraint is equivalent to

$$\begin{aligned}
\forall \|\theta\| \leq 1. M := \begin{pmatrix} t & (y_0 + L\theta)^T \\ y_0 + L\theta & tI \end{pmatrix} &\succeq 0 \\
\Leftrightarrow \forall \|\theta\| \leq 1. \forall s \in \mathbb{R}, v \in \mathbb{R}^n. (s,v) \cdot M \cdot (s,v)^T &\geq 0 \\
\Leftrightarrow \forall \|\theta\| \leq 1. \forall s \in \mathbb{R}, v \in \mathbb{R}^n. s^2 t + 2s(y_0 + L\theta)^T v + t\|v\|^2 &\geq 0 \\
\Leftrightarrow \forall s \in \mathbb{R}, v \in \mathbb{R}^n. s^2 t + 2s y_0^T v + t\|v\|^2 + \inf\left\{ s v^T L\theta : \|\theta\| \leq 1 \right\} &\geq 0 \\
\overset{\text{Lemma 14.4}}{\Longleftrightarrow} \forall s \in \mathbb{R}, v \in \mathbb{R}^n. \forall \|z\|^2 \leq s^2. s^2 t + 2s y_0^T v + t\|v\|^2 + 2v^T L z &\geq 0 \\
\Leftrightarrow \left((s,v,z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I \end{pmatrix} \begin{pmatrix} s \\ v \\ z \end{pmatrix} \geq 0 \to (s,v,z) \begin{pmatrix} t & y_0^T & 0 \\ y_0 & tI & L \\ 0 & L^T & 0 \end{pmatrix} \begin{pmatrix} s \\ v \\ z \end{pmatrix} \geq 0 \right)
\end{aligned}$$

and we are done using Theorem 14.5. $\square$

14.4 Adjustable robust counterpart for 2-stage problems

Regard the problem

$$\min\left\{ c^T x + p^T y : Ax \leq 0, Tx + Wy \leq h \right\}$$

In the following table, all functions of θ are affine.

type of constraint	uncertainty set Θ	refo
$\forall \theta \in \Theta. a(\theta)^T x \leq b(\theta)$	conic inequality $F\theta \preceq_K h$, budgeted/ellipsoidal	CP over
$\leq f$ for all $\|A(\theta)x + b(\theta)\| \leq c(\theta)^T x + d(\theta)$	-	hard, even
$\|A(\theta)x + b(\theta)\| \leq cTx + d$	Ball	
$\forall \theta. \exists y. T(\theta)x + W(\theta)y \leq t(\theta)$	-	

14.4.1 Two-stage-problem

$$\min\left\{ c^T x + p^T y : Ax \leq b, Tx + Wy \leq h \right\}$$

The robust counterpart is

$$\sup_{\theta \in \Theta} \inf_{y} c^T x + p^T y : Ax \leq b, \forall \theta \in \Theta. T(\theta)x + W(\theta)y \leq b(\theta)$$

Assuming that $W(\theta) = W$ is a fixed recourse: Then we get the relaxation

$$\min_{x, y(\theta)} \sup_{\theta} c^T x + p^T y(\theta) : Ax \leq b, \forall \theta \in \Theta. T(\theta)y + Wy(\theta) \leq h(\theta), y(\theta) = y_0 + y\theta$$

14.7 Theorem. *Let $\Theta = \{\theta : F\theta \preceq_k f\}$ essentially strictly feasible. Let t_i be the rows of T with $t_i(\theta) = z_i + T_i\theta$ and $h_i(\theta) = \eta_i + h_i^T\theta$. Then a* safe *solution to the robust 2-stage LP can be obtained by solving the conic programme, where the constraint $t_i(\theta)^T x + w_i^T y(\theta) \leq h_i(\theta)$ is replaced by*

$$\begin{aligned}
z_i^T x + w_i^T y_0 + z_i f &\leq \eta_i \\
F^T z_i = T_i^T x + Y^T w_i - h_i, z_i &\succeq_{K^*} 0
\end{aligned}$$

15 Interior Point Methods

We regard the problem

$$\min \{f(x) : x \in \operatorname{int} \chi, Ax = b\}$$

where f is convex and χ is convex.
We have $\chi \subseteq \operatorname{dom} f$. For all $t \geq 0$, we consider the perturbed problem

$$\min \{t \cdot f(x) + F(x) : Ax = b\} \tag{P_t}$$

where F is a "barrier function", i.e.

- F is strongly convex, so $\exists v > 0. \nabla^2 F \succeq v I_n$.
- F goes to ∞ along every sequence $x_i \in \operatorname{int} \chi$ where $x_i \to \partial \chi$.

As F is strongly convex, there is an only minimiser of eq. (P_t).

$$x^*(t) = \operatorname{argmin} \{t \cdot f(x) + F(x) : Ax = b\}$$

This defines the central path $(x^*(t) : t \in \mathbb{N})$. We will show that the limit $x^*(\infty)$ is an optimal solution of eq. (P_t).
The advantages of this method:

- Use techniques from unconstrained optimisation.
- For a certain class of problems we can prove convergence to an ε-suboptimal solution, in time polynomial in input size and $-\log \varepsilon$.
- works well both in theory and in practice

15.1 The Newton Method for Unconstrained Problems

Here we want to solve the problem

$$p^* = \inf \{f(x) : x \in \mathbb{R}^n\}$$

We assume f is strongly convex , so $\nabla^2 f(x) \succ 0$ for all $x \in \operatorname{dom} f$. (This assumption is justified, since $tf + F$ from above is strongly convex.)
Given a current iterate $x \in \operatorname{dom} f$, we take a 2nd order approximation of f around x:

$$f(u+u) \approx f(x) + \nabla f(x)^T u + \frac{1}{2} u^T \nabla^2 f(x) u =: \widehat{f}(u)$$

The minimum of $\widehat{f}(u)$ satisfies

$$u = -\left(\nabla^2 f(x)\right)^{-1} \nabla f(x)$$

This defines the Newton direction $\Delta x = -\left(\nabla^2 f(x)\right)^{-1} \nabla f(x)$. The Newton method moves along Δx to find the next iterate $x^+ = x + \delta \Delta x$ for some $\delta \leq 1$.
The Newton decrement is

$$\lambda(x) = \sqrt{\nabla f(x)^T \cdot \nabla^2 f(x)^{-1} \cdot \nabla f(x)}$$

One can show that $\frac{1}{2}\lambda(x)^2 = f(x) - \inf_u \widehat{f}(u)$.

15.2 Analysis of Convergence of Newton's Method

15.2.1 Analysis for Regular Functions

The analysis depends on 3 regularity parameters

- ν: strong convexity parameter
- L: Lipschitz constant of ∇f
- M: Lipschitz constant of $\nabla^2 f$

There exist 2 numbers γ and η (which depend on ν, L, M) and 2 parameters α, β used to set the step length δ such that:

1st phase: $f(x^{(k)}) - f\left(x^{(k+1)}\right) \geq \gamma > 0$

2nd phase: quadratic convergence

15.2.2 Convergence Analysis of Self-concordant Functions

15.1 Definition. Function $f : \mathbb{R}^n \to \mathbb{R}$ is *self-concordant* if

- $\operatorname{dom} f$ is open
- $\nabla^2 f(x) > 0$ for all $x \in \operatorname{dom} f$, in particular f is convex
- $f(x_i) \to \infty$ for every sequence $x_i \to \partial \operatorname{dom} f$
- $f \in C^3$, and for all line segments $x + th$ in $\operatorname{dom} f$, the restriction $F(t) := f(x + th)$ satisfies $|F'''(t)| \leq 2F''(t)^{\frac{3}{2}}$ for all $t \in \operatorname{dom} F$, i.e.

$$\forall t \in \operatorname{dom} F. \left| \frac{\mathrm{d}}{\mathrm{d}t} F''(t)^{-\frac{1}{2}} \right| \leq 1$$

Recall that the Newton decrement is

$$\lambda(x) = \left(\nabla f(x)^T \cdot \nabla^2 f(x)^{-1} \cdot \nabla f(x)\right)^{\frac{1}{2}} = \left\| \nabla^2 \underbrace{f(x)^{-1} \nabla f(x)}_{Delta_x} \right\|_x$$

where $\| \cdot \|_x$ is the *local norm*, associated with the inner product $\langle u, v \rangle_x = u^T \nabla^2 f(x) v$.

15.2 Proposition. *If f is self-concordant and $\lambda(x) \leq 0.68$, then $f(x) - p^* \leq \lambda(x)^2$.*

Proof. Let $x \in \operatorname{dom} f$, $h \in \mathbb{R}^n$ and consider $F : t \mapsto x + ht$. By self-concordance, we have $F''(t)^{-1/2} \leq F''(0)^{-1/2} + t$ for all $0 \leq t \in \operatorname{dom} F$. So

$$F''(t) \geq \frac{1}{\left(F''(0)^{-1/2} + t\right)^2} = \frac{F''(0)}{\left(1 + tF''(0)^{-1/2}\right)^2}$$

for all $0 \leq t \in \operatorname{dom} F$. Integration yields

$$F'(t) \geq F'(0) + \frac{F'(0)t}{1 + tF''(0))^{1/2}} = F'(0) + F''(0)^{1/2} - \frac{F''(0)^{1/2} t}{1 + tF''(0))^{1/2}}$$

Integrating a second time yields

$$F(t) \geq F(0) + t(F'(0) + F''(0)^{1/2}) + \log\left(1 + tf''(0)^{1/2}\right) \tag{12}$$

The right hand side is minimised over $\mathbb{R}$ at

$$t^* = \frac{-F'(0)}{F''(0) + F''(0)^{1/2}F'(0)}$$

Plugging this value into eq. (12), we obtain

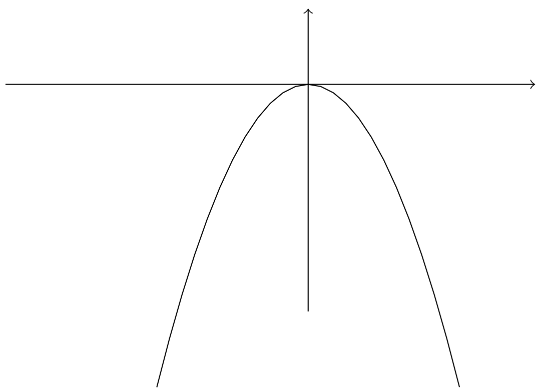
$$F(t) \geq F(0) - F'(0)F''(0)^{-1/2} + \log\left(1 + F'(0)F''(0)^{-1/2}\right) = f(x) + \underbrace{u + \log(1-u)}_{\rho(u)}$$

where $u := -F'(0)F''(0)^{-1/2}$. Now we make the derivatives of F explicit:

$$\begin{aligned} F'(0) &= h^T \nabla f(x) \\ F''(0) &= h^T \nabla^2 f(x) h \end{aligned}$$

$$F'(0)F''(0)^{-1/2} = \frac{h^T\nabla f(x)}{\sqrt{h^T\nabla^2 f(x) h}} = \frac{\langle h, \overbrace{\nabla^2 f(x)^{-1}\nabla f(x)}^{} -\Delta_x}{\|h\|_x}\rangle \leq \|\Delta_x\|_x = \lambda(x)$$

Since the direction h was chosen arbitrary, we obtain a global lower bound $p^* \geq f(x) + \rho(\lambda(x))$. Check $|u| \leq \lambda(x)$ implies $\rho(u) \geq \rho(\lambda(x))$, i.e. ρ is concave. Our bound was chosen such that $\forall 0 \leq u \leq 0.68. - \rho(u) \leqq u^2$. $\square$



15.3 Proposition. *Let $\delta = \frac{1}{1+\lambda(x)}$ and consider the single damped Newton step $x^+ := x + \delta\Delta_x$. Then*

- $x^+ \in \operatorname{dom} f$
- $f(x) - f(x^+) \geq \lambda(x) - \log(1 + \lambda(x))$ *In particular, if $\lambda(x) \geq \frac{1}{4}$, then $f(x) - f(x^+) \geq 0.026$, so a positive constant.*
- *If $\lambda(x) \leq \frac{1}{4}$, then $2\lambda(x^+) \leq (2\lambda(x))^2$.*

15.4 Theorem. *Let f self-concordant. Given an initial point $x^{(0)} \in \operatorname{dom} f$. We consider the iterates $x^{(k+1)} = x^{(k)} - \frac{1}{1+\lambda(x^{(k)})} \cdot \Delta_x$. The number if iterations to obtain an ε-suboptimal solution is in*

$$\underbrace{\mathcal{O}(1)}_{\leq 38} \cdot \left(f\left(x^{(0)}\right) - p^*\right) + \underbrace{\log\log\frac{1}{\varepsilon}}_{\text{“}\leq 6\text{''}}$$

Proof. The number of iterations in the first phase is at most $\frac{1}{0.026}\left(f\left(x^{(0)}\right)-p^*\right)$ and $\frac{1}{0.026}\approx 38$. During the second phase, show by induction, that after k further iterations

$$\left(f\left(x^{(k)}\right)-p^*\right)\leq\lambda(x^{(k)})^2\leq\frac{1}{2^{2^{k+1}}}$$ □

Note that we have ignored equality constraints, since we can express $Ax=b$ by $\exists z.x=x_0+Uz$ for some matrix U, depending on A.

15.3 Path Following Algorithms

15.3.1 Analysis for Regular Functions

We have the problem

$$\min\left\{f_0(x):\forall i\leq m.f_i(x)\leq 0, Ax=b\right\} \tag{P}$$

The central path is defined by

$$x^*(t)=\operatorname{argmin}\left\{tf_0(x)+\sum_{i=1}^{m}-\log(-f_i(x)):Ax=b\right\} \tag{Q_t}$$

Assumptions

- f_i are strongly convex, i.e. particular twice differentiable
- eq. (P) is essentially strictly feasible

15.5 Proposition. *Assume $x^*(t)$ solves eq. (Q_t). Then $x^*(t)$ is feasible for eq.* (P) *and*

$$f\left(x^*(t)\right)-p^*\leq\frac{m}{t}$$

missing lecture

16 Polynomial Optimisation

16.1 Notation. Denote $\mathbb{R}_d[x]=\{p\in\mathbb{R}[x]:\deg p\leq d\}$. Any $p\in\mathbb{R}_d[x]$ is associated with its vector of coefficients, writing $p\in\mathbb{R}^{d+1}$. (I.e. we do not use any sparsity.)

16.2 Definition. A polynomial is non-negative if $\forall x\in\mathbb{R}.p(x)\geq 0$, written $p\geq 0$. Restricting the degree, we get the cone

$$\mathcal{P}_d^{\text{pos}}=\{p\in\mathbb{R}_d[x]:p\geq 0\}$$

When minimising a polynomial, we observe

$$\min\left\{p(x):x\in\mathbb{R}\right\}\sim\max\{\gamma\in\mathbb{R}:p-\gamma\geq 0\}\sim\max\left\{\gamma\in\mathbb{R}:p-\gamma\succeq_{\mathcal{P}_d^{\text{pos}}}\right\}$$

16.3 Definition. A polynomial $p\in\mathbb{R}_{2d}[x]$ is a sum of squares, if there are $p_i\in\mathbb{R}_d[x]$ such that $p=\sum p_i^2$. Their cone is denoted $\mathcal{P}_d^{\text{sos}}$.

16.4 Proposition. *If d is even, then $\mathcal{P}_d^{\text{pos}}$ and $\mathcal{P}_d^{\text{sos}}$ are proper cones.*

16.5 Theorem. $\mathcal{P}_{2d}^{\text{pos}} = \mathcal{P}_{2d}^{\text{sos}}$.

Proof. Squares are trivially non-negative.
Now let $0 \leq p \in \mathcal{P}_{2d}^{\text{pos}}$. We factor p over $\mathbb{C}$ and obtain $p = p_{2d} \cdot \prod_{i=1}^{2d}(x - a_i)$. Recall $p(\overline{z}) = \overline{p(z)}$, so $p(\overline{z}) = 0 \Leftrightarrow p(z) = 0$. Any real root must have even multiplicity. So after reordering the roots, we can write $p = p_{2d} \cdot \prod_{i=1}^{d}(x - a_i)(x - \overline{a_i})$. Hence by putting $q := \sqrt{p_{2d}} \cdot \prod_{i=1}^{d}(x - a_i)$ we have

$$p = q \cdot \overline{q} = |q(x)|^2 = \mathrm{Re}(q)^2 + \mathrm{Im}(q)^2$$

So p can even be written as a sum of 2 squares. $\square$

16.6 Theorem. *$p \in \mathbb{R}_{2d}[x]$ is SOS iff there exists some $M \in \mathbb{S}^{d+1}$ such that*

$$M \succeq 0 \qquad\qquad S_k(M) := \sum_{i+j=k} M_{ij} = p_k$$

i.e. summing over the k-th antidiagonal yields p_k.

Proof. Let $v_x = (1, x, \ldots, x^d)$. Then we have $p = v_x^T M v_x$ iff the second constraint holds (summing up and reordering, to group same powers). Furthermore

$$M \succeq 0 \Leftrightarrow M = P^T P \qquad \text{for some matrix } P$$

$$\Leftrightarrow p(x) = v_x^T P^T P v_x = \|P v_x\|^2 \Leftrightarrow p(x) = \sum_{i=0}^{d} (P_i v_x)^2 \Leftrightarrow p \text{ is SOS}$$

where the P_i are the rows of P. $\square$

16.7 Example. Let $p = x^2 + 4x + 5$. Then the above problem has the solution

$$M = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 21 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 21 & 1 \end{pmatrix}$$

which gives the decomposition $p = (2 + x)^2 + 1^2$.

16.1 Multivariate Polynomials

Now regard $p \in \mathbb{R}[x_1, \ldots, x_n]$, with the notation $\mathbb{R}_d[x]$, only here $x = (x_1, \ldots, x_n)$. A polynomial is written $p = \sum_{\alpha \in \mathbb{N}^n} p_\alpha x^\alpha$. Again, we do not have sparsity, so even zero coefficients are noted. For the cones we use $\mathcal{P}_{n,d}^{\text{pos}}$ and $\mathcal{P}_{n,d}^{\text{sos}}$.

16.8 Proposition. *The Motzkin polynomial $p = x^2y^4 + x^4y^2 + 1 - 3x^2y^2 \in \mathcal{P}^{\text{pos}} \setminus \mathcal{P}^{\text{sos}}$.*

16.9 Theorem (Hilbert, 1888). *$\mathcal{P}_{n,2d}^{\text{pos}} = \mathcal{P}_{n,2d}^{\text{sos}}$ iff $n = 1$, $2d = 2$ or $(n, 2d) = (2, 4)$.*

16.10 Theorem. *$p \in \mathbb{R}_{2d}[x]$ is SOS iff there exists some*

$$M \in \mathbb{S}_+^{\binom{n+d}{n}} \qquad\qquad \sum_{\alpha+\beta=\gamma} M_{\alpha,\beta} = p_\gamma$$

which are $\binom{n+2d}{n}$ linear constraints.

But $(1 + x^2 + y^2)p(x, y) \in \text{SOS}$, where p is Motzkin. So this is a certificate for non-negativity (SOS divided by something positive). This leads to the Lasserre Hierarchy

$$v_r^* = \sup\left\{\gamma : \left(1 + \sum x_i^2\right)^r \cdot (p - \gamma) \in \text{SOS}\right\}$$

The sequence v_r^* converges to the optimum.