Algebraic Geometry

Inofficial lecture notes
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0 introduction

First let us recall the background of linear algebra and compare this to concepts in algebraic geometry.

	Linear Algebra	Algebraic Geometry
motivation	Solve a system of linear equations	solve a system of polynomial equations
concepts	vector space over a field,dimension	algebraic variety over a field, coordinate ring, sheaf of rings, dimension, degree, genus
qualitative statements	$\dim\ker\varphi + \dim\operatorname{im}\varphi = n$	Main focus
algorithms (symbolic)	Gaussian elimination	Gröbner basis
numerical algorithms	inner product, norm, iterative methods	numerical algebraic geometry, freshly emerging; homotopy methods

Algebraic geometry combines many fields:

- Algebra
- Geometry
- Topology
- Number theory $\rightsquigarrow$ arithmetic geometry (but left out in this lecture)

This lecture focuses on varieties over $\mathbb{C}$ (mainly we need "algebraically closed). Further we will put the focus on geometric ideas. We will follow the lecture held by Andreas Gathmann from 2002/03, sometimes their corresponding numbers will be given.

0.1 Example (Exercise 0.1.1). Let $n \geq 1$ an regard

$$C_n = \{(x,y) \in \mathbb{C}^2 : y^2 = \underbrace{(x-1)(x-2)\dots(x-2n)}_{=:f(x)}\}$$

If we want to have some solution, we just choose x and compute y. As a manifold it has dimension 1. To actually show that C_n is a smooth submanifold of $\mathbb{C}^2$ we write

$$C_n = \{(x,y) \in \mathbb{C}^2 : F(x,y) = 0\} \qquad\qquad F(x,y) = y^2 - f(x)$$

When taking the derivative we get

$$\operatorname{grad} F(x,y) = (-f'(x), 2y) \stackrel{!}{=} 0 \implies y = 0, f'(x) = 0, \implies f(x) = 0$$

But $f(x)$ only has single roots. Therefore $\forall (x,y) \in C_n. \operatorname{grad} F(x,y) \neq 0$. So C_n is a smooth submanifold.
To understand this example better, we take a look at the case $n = 1$. Let $x \in \mathbb{C} \setminus \{1,2\}$. Then $y = \pm\sqrt{(x-1)(x-2)}$. Consider some circle around 1 with radius r. $x(\varphi) = r \cdot e^{i\varphi}$ and $y(\varphi) = \sqrt{r} \cdot e^{\frac{1}{2}i\varphi}$, in particular $x(0) = r$, $y(0) = \sqrt{r}$ an $y(2\pi) = -\sqrt{r}$.
Next we want to know, how our solution set looks topologically. What we have is two planes, where in each a circle (the smallest containing 1 and 2) is cut out and both of these circles are identified. In general, we will have n holes in each of the 2 planes.
When we add some point ∞, and go to the Riemann sphere, we get a more compact image. In the case $n = 2$ get a torus (the hole of the torus is the gap between the tubes). After *compactification* we get a surface of "genus" $n-1$ (intuitively $n-1$ holes).
The first one to visualise these concepts was Riemann, although at his time, topology was not developed yet.

0.2 Example (Exercise 0.1.2). In Example 0.1, the function f only had single roots, which was an important condition. Now let

$$C_n = \{(x,y) \in \mathbb{C}^2 : y^2 = f(x)\}$$

and $f(x)$ has a double root. Remember that a cut was a circle between two roots of f. So this circle now degenerates into a single point, so the tube becomes a line. This creates a singularity.

0.3 Example (Exercise 0.1.3). Let $C_d := \{(x,y) \in \mathbb{C}^2 : f(x,y) = 0\}$ and f is some general polynomial of degree d. Then we write

$$C'_d := \{(x,y) \in \mathbb{C}^2 : l_1(x,y) \dots l_d(x,y) = 0\}$$

where the l_i are linear "in general position".
The "Homotopy type" of C_d and C'_d should be the same. Note that C_d is smooth, whereas C'_d is not.
The zero set of C'_d is a union of d complex lines $\cong \mathbb{C}$. The compactification of a line is $\mathbb{C}_\infty$, the Riemann-sphere. The compactification of all the lines are d sphere, who are pairwise connected by lines (their connections form a complete graph).

0.4 Lemma (Degree-genus formula for planar curves). *To compute the genus in Example 0.3, we have*

$$g = \binom{d}{2} - (d-1) = \binom{d-1}{2}$$

Proof. Consider the spheres a nodes from an initially empty graph. We need $d-1$ edges to connect them. Each additional edge creates a loop, which topologically means a hole. $\square$

1 Affine varieties

1.1 Algebraic sets and the Zariski topologiy

Let k be some algebraically closed field.

1.1 Definition. We call $\mathbb{A}^n = k^n$ an affine space. Then $S \subseteq k[X_1, \dots, X_n]$ has a zero set $Z(S) := \{a \in \mathbb{A}^n : \forall f \in S. f(a) = 0\}$. Such sets are called *algebraic sets*, or *affine varieties*. If $S = \{f_1, \dots, f_r\}$, we write $Z(S) = Z(f_1, \dots, f_r)$.

1.2 Example.
1. We always have $\mathbb{A} = Z(0)$ and $Z(k[X_1, \dots, X_n]) = \emptyset$.
2. $Z(X_1 - a, \dots, X_n - a) = \{a\}$. any linear subspan is affine.

Let (S) be the *ideal* in $k[X]$ generated by S. Note $Z(S) = Z((S))$.

1.3 Lemma. *Let R be some commutative ring. Equivalent are*

1. *Every ideal of R is finitely generated.*
2. *Every ascending chain of ideals is stationary.*

In this case, we call R a Noetherian ring.

1.4 Theorem. *In particular, if k is a field, then $k[X_1, \dots, X_n]$ is Noetherian.*

Proof. Assume ideal $I \subseteq R[X]$ is not finitely generated. Take some $0 \neq f_0 \in I$ of minimum degree. Inductively take $f_{i+1} \in I \setminus (f_0, \dots, f_i)$ of minimum degree. Then we have $\deg f \leq \deg f_{i+1}$. Let a_i be the leading coefficient of f_i. Consider $I_i := (a_0, \dots, a_i) \subseteq I_{i+1}$. If R is Noetherian, then at some point we will have $a_{m+1} \in (a_0, \dots, a_m)$. Let $r_0, \dots, r_m \in R$ such that $a_{m+1} = \sum r_i a_i$. Define the new polynomial

$$f := f_{m+1} - \sum_{i=0}^{m} r_i X^{\deg f_{m+1} - \deg f_i} f_i$$

Therefore $\deg f < \deg f_{m+1}$, which is a contradiction. $\square$

1.5 Lemma.
1. *If $S_1 \subseteq S_2 \subseteq k[X]$, then* stuff
2. *$Z(\bigcup_i S_i) = \bigcap Z(S_i)$.*
3. *$Z(S_1 \cdot S_2) = Z(S_1) \cup Z(S_2)$.*

1.6 Definition (Zariski topology). The closed sets of $\mathbb{A}^n$ are the algebraic sets. Define the Zariski topology on a subset $X \subseteq \mathbb{A}^n$ as the induced topology.

1.7 Example. about $\mathbb{A}^1$

1.2 Hilbert's Nullstellensatz

1.8 Definition. For a subset $X \subseteq \mathbb{A}^n$ define *vanishing ideal* of X as

$$I(X) := \{f \in k[x] : \forall a \in X. f(a) = 0\}$$

$$\text{algebraic sets in } \mathbb{A}^n \xleftrightarrow[Z]{I} \text{ideals in } k[x]$$

Figure 1: bijection between algebraic sets ans ideal

So we have

1.9 Proposition (Hilbert's Nullstellensatz, weak form). *Assume k is algebraically closed. Then the maximal ideals in $k[x]$ are all of the form $(x_1 - a_1, \dots, x_n - a_n)$.*

Before we prove this, let us recall some things from Algebra.

1.10 Remark.
- First note that $m_a := (x_1 - a_1, \dots, x_n - a_n)$ is maximal. To obtain this result, recall that an ideal is maximal iff its factor is a field.
- We also need the fact that there is an infinite sequence $a_0, a_1, \dots \in \mathbb{C}$, which are algebraically independent.

1.11 Lemma. *Let $K \subset L = K(a_1, \dots, a_n)$ some finitely generated field extension. Take $N \subseteq \{1, \dots, n\}$ maximal such that $(a_i)_{i \in N}$ are algebraically independent over K. Wlog we have $N = \{1, \dots, m\}$. This yields an intermediate field $K \leq K(a_1, \dots, a_m) \leq L$. The first extension is purely transcendental, the second one is a finitely generated algebraic extension.*

Proof of Proposition 1.9. For the full proof, see Algebra II. We will provide a shorter proof for the case $k = \mathbb{C}$.
Let $m \subset \mathbb{C}[x]$ be some maximal ideal. By Hilbert's basis theorem, there are $f_1, \dots, f_r \in \mathbb{C}[x]$ such that $m = (f_1, \dots, f_r)$. Let K be the subfield of $\mathbb{C}$ generated by the coefficients of $(f_1, \dots, f_r)$. Next we go down to a smaller gin of coefficients and put $m_0 := m \cap K[x]$.

m_0 is maximal ideal of $K[x]$: Assume we had an intermediate ideal $m_0 \subset m_0' \subset K[x]$. Then over $\mathbb{C}$ we also get an intermediate ideal $m = m_0\mathbb{C}[x] \subset m_0'\mathbb{C}[x] \subset \mathbb{C}[x]$, which is a contradiction.

There is a ring morphism $L := K[x]/m_0 \to \mathbb{C}$: By Lemma 1.11 we can split $\mathbb{Q} \leq L$ into transcendental and algebraic part

$$\mathbb{Q} \leq E := \mathbb{Q}(y_1, \dots, y_r) \leq L$$

Now we define the map $\varphi : \mathbb{Q}(y_1, \dots, y_r) \to \mathbb{C}$ via $y_i \mapsto a_i$, where the a_i are some algebraically independent numbers (which exist by Remark 1.10).

Now let $E \leq E(b)$ be some algebraic extension. Then we can extend $\varphi : E \to \mathbb{C}$ to some map from $E(b)$. But from E to L we have a sequence of simple extensions, so we can extend φ to a map $\varphi : L \to \mathbb{C}$.

Put $\alpha_i := \varphi(x_i)$. Then we find the common root

$$0 = \varphi(f_i(x_1, \dots, x_n)) = f_i(\alpha_1, \dots, \alpha_n)$$ □

Recall, that we assume our field k to be algebraically closed, and put $\mathbb{A}^n(k) := k^n$. We have a map φ from $\mathbb{A}^n$ to the maximal ideals in $k[x_1, \dots, x_n]$ via $a \mapsto (x_1 - a, \dots, x_n - a)$. This is a special case of figure 1. The Nullstellensatz says that φ is a bijection.

1.12 Remark.

1. $X_1 \subseteq X_2 \implies I(X_1) \supseteq I(X_2)$ and $I_1 \subseteq I_2 \implies Z(I_1) \supseteq Z(I_2)$.
2. Let X be algebraic. Then $Z(I(X)) = X$.

of item 2. Direction $\supseteq$ is clear by definition.
Let $X = Z(J)$, J exists since X is algebraic. Now we have $I(X) = I(Z(J)) \supseteq J$. Using item 1 we get $Z(I(X)) \subseteq Z(J) = X$. □

1.13 Example. Assume $X = \{a\} \subseteq \mathbb{A}^n$, so $I(X) = (x-a)$. Let $m \in \mathbb{N}_{>0}$. Then $Z((x-a)^m) = Z(x-a)$. Put $J := (x-a)^m$. Then $Z(J) = \{a\}$ but $I(X) = (x-a) \supset J$.
So we see, multiplicities create problems.

1.14 Definition. Let J be an ideal in $k[x]$. The *radical* of J is

$$\sqrt{J} := \{f \in k[x] : \exists m \in \mathbb{N}. f^m \in J\}$$

The ideal J is called *radical ideal* if $\sqrt{J} = J$.

1.15 Remark.

1. $\sqrt{J}$ is an ideal.
2. $J \subseteq \sqrt{J}$.
3. If X is algebraic, then $I(X)$ is radical.
   Let $f \in \sqrt{I(X)}$. Then we have $f^m \in I(X)$ for some $m \in \mathbb{N}$. Thus $\forall p \in X. f(p)^m = 0$, so $f(p) = 0$. So we already have $f \in I(X)$.
4. Taking the radical is monotone.

1.16 Theorem. *Let J be an ideal in $k[x]$. Then $I(Z(J)) = \sqrt{J}$.*

Proof. **$\supseteq$:** The radical is monotone and we have $I(Z(J)) \supseteq J$. But the LHS already is its own radical, so $I(Z(J)) = \sqrt{(I(Z(J))} \supseteq \sqrt{J}$.

$\subseteq$: Rabonivich's trick: add a new variable t and define the ideal $\widetilde{J} := J + (tf - 1) \subseteq k[x,t]$. Now assume $(p, \tau) \in Z(\widetilde{J})$. Since $J \subseteq \widetilde{J}$, this in particular means $p \in Z(J)$. But then we need $0 = \tau \cdot f(p) - 1 = -1$, which cannot be. So $Z(\widetilde{J}) = \emptyset$.

Using the weak Nullstellensatz (Proposition 1.9), we get $\widetilde{J} = (1)$. Hence we have a representation

$$1 = (ft-1)g_0 + \sum_{i=1}^{r} f_i g_i$$

where $f_i \in J$ and $g_u \in k[x,t]$. Substitute t by f^{-1}. This yields an identity of rational functions

$$1 = \sum_{i=1}^{r} f_i(x) \cdot g\left(x, f^{-1}\right))$$

Let N be the largest degree of the g_i in t. Then we can rewrite this as a polynomial equation

$$f^N = \sum_{i=1}^{r} f_i(x) f^N g_i(x, f^{-1})$$

The RHS are multiples of the f_i, so it lies in $(f_1, \dots, f_r) = J$. But this shows $f^N \in J$, hence $f \in \sqrt{J}$.

$\square$

1.17 Example. Assume we have one variable and $f \in k[x]$, f monic. Then we can factor it

$$f = (x - a_1)^{m_1} \dots (x - a_r)^{m_r}$$

Then $\sqrt{(f)} = ((x - a_1) \dots (x - a_r))$. This means the ideal of f is radical iff f is square-free. Since we have just one variable, the question can be decided efficiently.
However, for many variables, the question whether an ideal, given by some generators, is radical has a best know double-exponential-time algorithm.

1.3 Irreducibility

Example. *Regard the following set $Z = (x_1, x_2) = Z(x_1) \cup Z(x_2)$. This can be written as a union of proper subsets $z(x_i) \subset Z(x_1, x_2)$.*

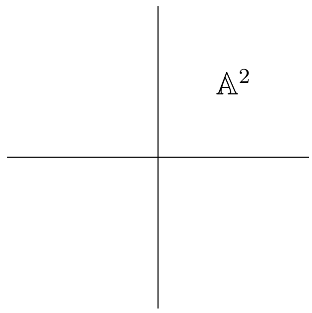


Figure 2: A set which is the union of two of its subsets.

1.18 Definition.

1. A topological space X is called *reducible* if there are closed sets $X_1, X_2 \subset X$ such that $X = X_1 \cup X_2$. Otherwise we call X *irreducible*
2. A topological set X is called *disconnected* if there are closed sets $X_1, X_2 \subset X$ such that $X_1 \cap X_2 = \emptyset$ and $X = X_1 \cup X_2$. Otherwise call X *connected.*

1.19 Example. These concepts strongly depend on the topology.

1. Let $X\mathbb{R}$ with euclidean topology. Then $X = (-\infty, 0] \cup [0, \infty)$. So $\mathbb{R}$ is reducible.
2. $\mathbb{A}^1$ is irreducible.
3. If X is Hausdorff and irreducible, then X consists of a single point.
4. Let X be irreducible, then X is connected. The converse implication does not hold.

1.20 Lemma. *An algebraic set X is irreducible iff $I(X)$ is a prime ideal.*

Proof. $\Rightarrow$ Suppose X is irreducible. First note $I(X) \neq k[x]$, since $X \neq \emptyset$. To show that $I(X)$ is prime, assume $fg \in I(X)$. Then $X \subseteq Z(f) \cup Z(g)$. To get an equality, we write $X = (Z(f) \cap X) \cup (Z(g) \cap X)$. Since X is irreducible, wlog $X = Z(f) \cap X$, which means $Z(f) \subseteq X$, so $f \in I(X)$.

$\Leftarrow$ Now suppose $I(X)$ is a prime ideal. Let $X = X_1 \cup X_2$, with X_i closed. Then $I(X) = I(X_1) \cap I(X_2)$ (exercise).

Since $I(X)$ is prime, this implies wlog $I(X) = I(X_1)$, so $X = X_1$. Hence X is irreducible. $\square$

1.21 Corollary. *$\mathbb{A}^n$ is irreducible.*

Proof. $I(\mathbb{A}^n) = \{0\}$ is a prime ideal, since $k[x]$ is an integral domain. Then apply Lemma 1.20. $\square$

1.22 Lemma. *Let $f \in k[x_1, \dots, x_n]$ be a square-free polynomial. Then $Z(f)$ is an irreducible algebraic set iff f is irreducible.*

Proof. Assume f is irreducible, then $ZI(Z(f)) = (f)$ is prime.
Alternatively: Let $Z(f)$ irreducible and $f = gh$. Wlog $Z(g) = Z(f)$. Then

$$\sqrt{(g)} = I(Z(g)) = I(Z(f)) = \sqrt{(f)} \overset{\text{sq.free}}{=} (f)$$

But g is a divisor of f, so $(g) = (f)$.
On the other hand, if $f = gh$ is a proper factorisation, then $Z(f) = Z(g) \cup Z(h)$. $\square$

More generally, let $f \in k[x_1, \dots, x_n]$ square-free with decomposition $f = f_1 \cdot \ldots \cdot f_r$ and each f_i irreducible. Then $Z(f) = \bigcup Z(f_i)$.
Let X be a topological space and $S \subseteq X$. Then we have the closure of S defined as

$$\overline{S} := \bigcap_{\substack{S \subseteq A \subseteq X \\ A \text{ closed}}} A$$

1.23 Remark. If $X \subseteq \mathbb{A}^n$, $S \subseteq X$, then $\overline{S} = Z(I(S))$.

1.24 Lemma. *Let X be a topological space. Then X is irreducible iff every non-empty open subset $U \subseteq X$ is dense, i.e. $\overline{U} = X$.*

Proof. **$\Rightarrow$:** Assume $U \subseteq X$ is open. Take any open set $\emptyset \neq W \subseteq X$. Assume $U \cap W = \emptyset$. Then we go over to the complements and get $X = (X \setminus U) \cup (X \setminus W)$. So X would be reducible.

$\Leftarrow$: analogous $\square$

1.25 Definition. A topological space X is called *Notherian* if every descending chain $X \supseteq X_1 \supseteq \dots$ of closed sets is stationary.

1.26 Proposition. *If X is algebraic, then X is Notherian.*

Proof. Let $(X_i)_{i \in \mathbb{N}}$ be our descending chain of closed sets. This corresponds to a chain of ideals in $k[x]$

$$I(X_1) \subseteq I(X_2) \subseteq \dots$$

However, since $k[x]$ is Noetherian, we know that this chain is stationary, i.e. $\exists m. \forall i > m. I(X_m) = I(X_i)$. Going back to the zero sets, we get

$$X_i = Z(I(X_i)) = Z(I(X_m)) = X_m$$ $\square$

1.27 Proposition. *1. Every Noetherian topological space X is finite union of irreducible closed subsets $X = X_1 \cup \ldots \cup X_r$.*

2. *Assume $X_i \not\subseteq X_j$ for $i \neq j$. Then the above decomposition is unique up to permutation.*

Proof. 1. Assume the statement were false for X. Then X is reducible (otherwise it were its own decomposition), say $X = X_1 \cup X_1'$ with $X_1, X_1' \neq X$. But then the statement must fail for one of these, say X_1. Then we continue the argument and we get an infinite descending chain $X_1 \supseteq X_2 \supseteq \ldots$, which contradicts Noetherian.

2. Assume we have two decompositions

$$X = X_1 \cup \ldots \cup X_t = X_1' \cup \ldots \cup X_s'$$

where all X_i, X_j' are closed and irreducible.

$$X_1 = \bigcup_{j=1}^{s} (X_1 \cap X_j')$$

Since X_i is irreducible, there must be some j with $X_1 = X_1 \cap X_j'$. Wlog we say $X_1 \subseteq X_1'$. But the other way round we get $X_1' \subseteq X_j$ for some j. This yields $X_1 \subseteq X_j$, so $j = 1$ and we have $X_1 = X_1'$. Now we just subtract X_1 from everything. Note $\overline{X_i \setminus X_1} = X_i$ (Due to our assumption, it can't be empty and $\emptyset \neq U = X_i \setminus X_1$ is an open subset of X_i, so $\overline{U} = X_i$). So we just continue with the equality

$$\bigcup_{i=2}^{t} \overline{X_i \setminus X_1} = \bigcup_{j=2}^{s} \overline{X_j' \setminus X_1}$$

□

1.28 Example. Let $X = Z(f) \subseteq \mathbb{A}^n$ square-free. Write $f = f_1 \cdot \ldots \cdot f_r$ as factorisation into irreducible polynomials. Then $Z(f) = Z(f_1) \cup \ldots Z(f_r)$ is a decomposition into irreducible components.

1.29 Definition. An *affine variety* is an irreducible algebraic set.

1.30 Definition. Let X be an irreducible topological space. The *dimension* of X is the largest $n \in \mathbb{N}$ such that there exists a chain

$$X_0 \subset X_1 \subset \ldots \subset X_n = X$$

where all X_i are irreducible and closed.
If X is Notherian with irreducible components $X_1, \ldots, X_r$, then we define $\dim X := \max \dim X_i$.

1.31 Example.
- We have the obvious cases $\dim\{p\} = 0$ and $\dim \mathbb{A}^1 = 1$.
- For higher dimension the definition just yields $\dim \mathbb{A}^n \geq n$. Equality holds, but it requires further proof.

2 Functions, Morphisms and Varieties

2.1 Functions on affine varieties

2.1 Definition. Let $X \subseteq \mathbb{A}^n$ an affine variety. Then we put

$$A(X) := k[x_1, \ldots, x_n]/I(X)$$

the *coordinate ring* of X, also written $k[X]$.
Any $f \in A(X)$ define a function $X \to k$ via $p \mapsto F(p)$, where $f = F \mod I(X)$ and $F \in k[x]$. This is well defined, because if $\widetilde{F} - F \in I(X)$, then $\widetilde{F}(p) = F(p)$ for all $p \in X$.

2.2 Remark. $I(X)$ is a prime ideal, so $A(X)$ is an integral domain.

2.3 Definition. Let $X \subseteq \mathbb{A}^n$ an affine variety. Then *quotient field* of $A(X)$ is called *field of rational functions in* X.

2.4 Definition. Let $X \subseteq \mathbb{A}^n$ an affine variety and $p \in X$. Then the *local ring* of X defined at p

$$\mathcal{O}_{X,p} := \left\{ \frac{f}{g} : f, g \in A(X), g(p) \neq 0 \right\}$$

2.5 Remark.

1. For $\varphi \in \mathcal{O}_{X,p}$ the expression $\varphi(p) := \frac{f(p)}{g(p)}$ is well-defined.
2. $\mathcal{O}_{X,p}$ is a local ring in the sense of Algebra II.

$$m_{X,p} := \left\{ \frac{f}{g} \in \mathcal{O}_{X,p} : f(p) = 0 \right\}$$

is an ideal. The evaluation map $\mathrm{ev} : \mathcal{O}_{X,p} \twoheadrightarrow k$ via $\frac{f}{g} \mapsto \frac{f(p)}{g(p)}$ is a ring morphism. $\ker(\mathrm{ev}) = m_{X,p}$, which means $\mathcal{O}_{X,p}/m_{X,p} \cong k$, so $m_{X,p}$ is a maximal ideal. Furthermore let $I \subseteq \mathcal{O}_{X,p}$ be another ideal with $I \nsubseteq m_{X,p}$. Then there exists $\frac{f}{g} \in I$ such that $f(p) \neq 0$. But in this case $1 = \frac{f}{g} \cdot \frac{g}{f} \in I$, so $I = \mathcal{O}_{X,p}$. Therefore $m_{X,p}$ is the only maximal ideal.

2.6 Example. Let $X := Z(x_1x_4 - x_2x_3) \subseteq \mathbb{A}^4$. (The polynomial is irreducible.) We have

$$\begin{tikzcd}
\mathbb{A}^4 \arrow[r] & k \\
X \arrow[u, "\subseteq"] \arrow[ur, dotted, "\overline{x_1}"']
\end{tikzcd}$$

$$A(X) = k[x_1, \dots, x_n]/I(X) = k[\overline{x_1}, \dots, \overline{x_n}]$$

where $\overline{x_i} = x_i \mod I(X)$.
Now let

$$K(X) \ni \varphi \in \frac{\overline{x_1}}{\overline{x_2}} = \frac{\overline{x_3}}{\overline{x_4}}$$

Note that $A(X)$ is not factorial.

$$U := \{p \in X : p_2 \neq 0 \vee p_4 \neq 0\}$$

$\varphi \in \mathcal{O}_X(U)$. Take $p \in U$. Verify $\varphi \in \mathcal{O}_{X,p}$. If $p_2 \neq 0$, write $\varphi = \frac{\overline{x_1}}{\overline{x_2}}$. If $p_4 \neq 0$, write $\varphi = \frac{\overline{x_3}}{\overline{x_4}}$.
What we want (but is impossible): $f, g \in A(X)$ such that

$$\forall p \in U. g(p) \neq 0 \wedge \varphi(p) = \frac{f(p)}{g(p)}$$

2.2 Relative Nullstellensatz

Let X be an affine variety,$Y \subseteq X$ closed, $I \subseteq A(X)$ some ideal.

$$I_X(Y) := \{f \in A(X) : \forall p \in Y. f(p) = 0\} \qquad \text{vanishing ideal}$$
$$Z_X(I) := \{p \in X : \forall f \in I. f(p) = 0\} \qquad \cdot\text{zero set}$$

2.7 Theorem. *We have*

$$Z_X(I_X(Y)) = Y \qquad I_X(Z_X(I)) = \sqrt{I}$$

Proof. Exercise, derive from the case $X = k^n$. □

Let X be an affine variety, $p \in X$ and $A(X)$ an integral domain. Then $K(X)$ is defined as the field of fractions of $A(X)$.

$$\varphi \in \mathcal{O}_{X,p} := \left\{ \varphi \in K(X) : \exists f, g \in A(X). g(p) \neq 0, \varphi = \frac{f}{g} \right\}$$

Then $\varphi(p) := \frac{f(p)}{g(p)}$ is well-defined. Furthermore we put $m_{X,p} := \{\varphi \in \mathcal{O}_{X,p} : \varphi(p) = 0\}$, which is the unique maximal ideal.

2.8 Definition. Let $U \subseteq X$ open. Then we define $\mathcal{O}_X(U) := \bigcap_{p \in U} \mathcal{O}_{X,p}$ the *ring of regular fractions* on U.

This is a well-defined function. Let $\psi : U \to k$, via $p \mapsto \psi(p)$. For all $p \in U$ there exists some $V \subseteq U$, a neighbourhood of p and there exist $f, g \in A(X)$ such that

$$\forall q \in V. g(q) \neq 0 \wedge \psi(q) = \frac{f(q)}{g(q)}$$

Remark. *Any $\psi : U \to k$ satisfying the properties arise from a $\varphi \in \mathcal{O}_{X,p}$.*

Proof. Suppose we have ??subsets $V, V' \subseteq U$ and $f, f', g, g' \in A(X)$ with f, g ??on V, f', g' ??on V' where

?

?

?

$$\forall p \in V. \psi(p) = \frac{f(p)}{g(p)} \qquad \forall q \in V. \psi(q) = \frac{f(q)}{g(q)}$$

Claim. $f/g = f'/g'$

Proof. X is irreducible. Therefore $\emptyset \neq V \cap V'$ is dense in X. Hence

$$\forall p \in V \cap V'. f(p) g'(p) = f'(p) g(p)$$

so $fg' - f'g \in A(X)$. □

□

2.9 Definition. Let $f \in A(X)$. Then $X_f := \{p \in X : f(p) \neq 0\}$ is the *distinguished open subset.*

2.10 Proposition. *We have*

$$\mathcal{O}_X(X_f) = \left\{ \frac{g}{f^r} : g \in A(X), r \in \mathbb{N} \right\} =: A(X)_f$$

In particular (using $f = 1$) we have $\mathcal{O}_X(X) = A(X)$.

Proof. The inclusion $A(X)_f \subseteq \mathcal{O}_X(X_f)$ is trivial.
Take $\varphi \in \mathcal{O}_X(X_f)$.

$$\forall p \in X_f . \exists g_p, h_p \in A(X) . g_p(p) \neq 0 \land \varphi = \frac{h_p}{g_p}$$

Define the ideal $I := \langle g_p : p \in X_f \rangle \subseteq A(X)$. Then $Z_X(I) \subseteq Z_X(f)$. By the relative Nullstellensatz we get

$$\sqrt{I} = I_X(Z_X(I)) \supseteq I_X(Z_X(f)) \ni f$$

Therefore there is some $r \in \mathbb{N}$ with $f^r \in I$. Hence there exist some $u_p \in A(X)$ such that $f^r = sum_p u_p g_p$ as a finite sum. Multiplying with φ yields

$$f^r \varphi = \sum_p u_p \underbrace{g_p \varphi}_{=h_p} = \underbrace{\sum_p u_p h_p}_{=:H} \in A(X)$$

Hence $\forall p \in X_f . \varphi(p) = \frac{H(p)}{f(p)^r}$, which shows $\varphi \in A(X)_f$. □

2.11 Remark. Let $g \in k[x, y]$ non-constant. Then $|Z(g)| > 1$, i.e. the zero set is not a single point.

Proof. Regard g as polynomial in y and write

$$g(x, y) = f_0(x) + f_1(x)y + \ldots + f_n(x)y^n$$

for some $f_i \in k[x]$. Since g is non-constant, it has a root and wlog we say $0 = g(0, 0) = f_0(0)$. If $f = 0 = 0$ we are done, because any pair $(x, 0)$ is a root.
So assume $f_0 \neq 0$. If f_0 has another root, we are done as well. Otherwise (after scaling) we have $f_0 = x^m$.
So we have the form

$$g(x, y) = x^m + f_1(x)y + \ldots + f_n(x)y^n$$

Then find some $\varepsilon \neq 0$ such that $g(x, \varepsilon) \neq 0$. Since k is algebraically closed, this has another root. Over $\mathbb{C}$ use some sufficiently "small" ε. Then we cannot have cancellation. □

Take $U := k^2 \setminus \{(0, 0)\}$. What is $\varphi \in \mathcal{O}_{k^2}(U)$?
Not only locally, but globally we have $\varphi = \frac{f}{g}$ for $f, g \in k[x_1, x_2]$ and $\gcd(f, g) = 1$. For all $p \in U$ with $g(p) \neq 0$ we have $\varphi(p) = \frac{f(p)}{g(p)}$. But by Remark 2.11 g must be constant, i.e. $g \in k$. Therefore $\varphi = f$.

2.12 Remark. The above argument shows $\mathcal{O}_{k^2} = k[x_1, x_2]$.

2.13 Corollary. *$U = k^2 \setminus \{(0, 0)\}$ is not a distinguished open set.*

2.3 Sheaves

Let X be some topological space.

2.14 Definition (Presheaf). A *presheaf* F of rings:

- assign to each open subset $U \subseteq X$ a ring $F(U)$
- For every inclusion $U \subseteq V$ of open sets we have a ring morphism $\rho_{V,U} : F(V) \to F(U)$
- $F(\emptyset) = 0$
- $\rho_{U,U} = \mathrm{id}$
- for $U \subseteq V \subseteq W$ we have $\rho_{W,U} = \rho_{V,U} \circ \rho_{W,V}$ (note we have weird order of composition)

The elements of $F(U)$ are called the *sections* of F on U. For $V \subseteq U$ we have $\rho_{U,V}(f) = f_{|V}$.

2.15 Definition. A presheaf is called a *sheaf* if F satisfies the *glueing property*/sheaf-axiom:
Let $U \subseteq X$ open and $\{U_i : i\}$ an open cover of U and $f_i \in F(U_i)$ such that

$$\forall i, j. f_{i|U_i \cap U_j} = f_{j|U_i \cap U_j}$$

Then there exists some $f \in F(U)$ such that $f_{|U_i} = f_i$.

2.16 Example. Let X be some affine variety. Let $V \subseteq U \subseteq X$ open sets. We have the restriction $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

Claim. *The restriction $u \mapsto \mathcal{O}_X(U)$ defines a sheaf.*

We call $\mathcal{O}_X$ the structure sheaf of the affine variety X.

The above definitions can equivalently be given in terms of categories.

2.17 Definition. Let $\mathscr{C}_X$ be the category of open sets in X, where we have a morphisms for each containment $U \subseteq V$. A presheaf is a contravariant functor F from $\mathscr{C}_X$ to the category of sets. A presheaf of rings/k-algebas/... is a contravariant functor F from $\mathscr{C}_X$ to the category of rings/k-algebras/....

2.18 Example. Let $X = \mathbb{R}$. For $U \subseteq \mathbb{R}$ open let$F(U)$ be the ring of constant real-valued functions on U. For $U \subseteq V \subseteq \mathbb{R}$ open, $\rho_{V,U}$ is the restriction of functions.
This is a presheaf but not a sheaf. To see this let $U_1 = (0, 1)$, $U_2 = (2, 3)$ and $U = U_1 \cup U_2$. Choose functions $f_i : U_i \to \mathbb{R}$ with $f_i : x \mapsto i$. Then there is no constant function that restricts to both f_1 and f_2.

2.19 Example. Let $X = \mathbb{R}^n$ and let $F(U)$ be the ring of continuous functions on U. Further let $\rho_{V,U}$ be the restriction of functions from V to U.
This is a sheaf. For an open set $U \subseteq X$ with cover $U = \bigcup_i U_i$ and continuous functions $f_i : U_i \to \mathbb{R}$ we define $f(x) = f_i(x)$ for $x \in U_i$. This is well-defined due to the given conditions that the functions agree on their intersection.

The difference between Example 2.18 and Example 2.19 is *global* versus *local* property. To make Example 2.18 work, we have to weaken the condition to locally constant functions.

2.20 Example. If $X \subseteq \mathbb{A}^n$ is an affine variety, then the ring $\mathcal{O}_X(U)$ of regular functions on open subsets of X form a sheaf of rings $\mathcal{O}_X$, called the *sheaf of regular functions on X*.

2.21 Definition. A *ringed space* is a pair $(X, \mathcal{F})$ where X is a topological space and $\mathcal{F}$ is a sheaf of rings on X. In this case $\mathcal{F}$ is called the *structure sheaf* of the ringed space, usually written $\mathcal{O}_X$.

2.22 Example. If $\mathcal{F}$ is a sheaf on X, $U \subseteq X$ open, then the restriction $\mathcal{F}_{|U}$ with $F(V) = (F_{|U})(V)$ for all $V \subseteq U$ is again a sheaf.

2.23 Definition. Let $P \in X$ be some fixed point, $\mathcal{F}$ a presheaf on X. Define the equivalence on pairs (U, φ) where $P \in U$ open, $\varphi \in \mathcal{F}(U)$ via:

$$(U, \varphi) \sim_P (U', \varphi') :\Leftrightarrow \exists V \text{ open}. P \in V \subseteq U \cap U' \wedge \varphi_{|V} = \varphi'_{|V}$$

The set of all such equivalence classes is called the *stalk* $\mathcal{F}_P$ at P. It elements are called *germs*.

2.24 Remark. If $\mathcal{F}$ is a presheaf of rings/..., then it stalks are rings/.... The addition works via

$$[(U, \varphi)]_P + [(U', \varphi')]_P := \left[(U \cap U', \varphi_{|U \cap U'} + \varphi'_{|U \cap U'})\right]_P$$

The multiplication works likewise. Due to our assumptions this is well-defined.

2.25 Lemma. *Let X be an affine variety and $P \in X$. The stalk of $\mathcal{O}_X$ at P is isomorphic to $\mathcal{P}_{X,P}$.*

Proof. By definition we have

$$\mathcal{O}_X(U) = \bigcap_{P' \in U} \mathcal{O}_{X,P'} \subseteq \mathcal{O}_{X,P} \subseteq K(X)$$

$\subseteq$: For $[(U, \varphi)]_P$ we have $\varphi \in \mathcal{O}_X(U) \subseteq \mathcal{O}_{X,P}$. Assume $[(U, \varphi)]_P \sim [(U', \varphi')]_P$. we need to show that φ and φ' are equal in $\mathcal{O}_{X,P}$. First take some open V with $P \in V \subseteq U \cap U'$ and $\varphi_{|V} = \varphi'_{|V} \in \mathcal{O}_X(V) \subseteq \mathcal{O}_{X,P}$.

$\supseteq$: Let $\psi \in \mathcal{O}_{X,P}$. Then $\psi = \frac{f}{g}$ for some $f, g \in A(X)$ and $g(P) \neq 0$. Let $U = X \setminus Z(g)$. Hence $\psi \in \mathcal{O}_X(U)$ and $[(U, \psi)]_P \in (\mathcal{O}_X)_P$ (the stalk of $\mathcal{O}_X$ at P). □

2.4 Morphisms between affine varieties

2.26 Definition. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces, where $\mathcal{O}_X, \mathcal{O}_Y$ are sheaves of k-value functions. Let $f : X \to Y$ be some map.

1. If $U \subseteq Y$ open and $\varphi : U \to k$, then the *pullback* is $f^*\varphi = \varphi \circ f : f^{-1}(U) \to k$.
2. f is called a *morphism* if it is continuous and it pulls back regular functions to regular functions, i.e.

$$\forall U \subseteq Y \text{ open}. f^*\mathcal{O}_Y(U) \subseteq \mathcal{O}_X(f^{-1}(U))$$

Recall that for an affine variety $X \subseteq \mathbb{A}^n$ we have $\mathcal{O}_X(X) = A(X)$.

2.27 Lemma. *Let $f : X \to Y$ be a continuous map between affine varieties. TFAE*

1. *f is a morpism*
2. *$\forall \varphi \in A(Y). f^*\varphi \in A(X)$*
3. *$\forall p \in X. \forall \varphi \in \mathcal{O}_{Y,f(p)}. f^*\varphi \in \mathcal{O}_{X,p}$*

Proof. **item 1⇒item 2** trivial

item 2⇒item 3 Take $\varphi \in \mathcal{O}_{Y,f(p)}$. Write $\varphi = \frac{g}{h}$ with $g, h \in A(Y)$ and $h(f(p)) \neq 0$. Then $f^*g, f^*h \in A(X)$ by item 2. Therefore

$$f^*\varphi = \frac{f^*g}{f^*h} \implies f^*\varphi \in \mathcal{O}_{X,p}$$

item 3⇒item 1 Use the fact $\mathcal{O}_Y(U) = \bigcap_{q \in U} \mathcal{O}_{Y,q}$, otherwise clear □

2.28 Example. 1. Let $f : \mathbb{A}^1 \to \mathbb{A}^1$ via $x \mapsto x^2$. This is continuous. For any polynomial $\varphi : \mathbb{A}^1 \to k$ the composition $\varphi \circ f$ is a polynomial, so f is a morphism by Lemma 2.27.

2. Let $f : \mathbb{A}^n \to \mathbb{A}^m$ given by $x \mapsto (f_1(x), \dots, f_m(x))$ where $f_i \in k[x]$ are polynomials. Then f is a morphisms of affine varieties.

   Furthermore every morphisms has this shape. To see this, let $y_i : \mathbb{A}^m \to k$ be the projection on the i-th component. Then $f^*y_i = y_i \circ f = f_i$ is a polynomial.

3. More generally let $X \subseteq A^n, Y \subseteq \mathbb{A}^m$ affine varieties. Take a polynomial map $f : X \to Y$. Then f is a morphism of affine varieties and every morphism has this form. The proof is the same as in the global case before.

We have the following correspondence:

affine varieties X	$A(X)$ k-algebra finitely generated integral domain

2.29 Lemma. *Let A be some finitely generated k-algebra which is an integral domain. Then there is some affine variety $X \subseteq k^n$ such that $A(X) \cong A$.*

Proof. Let $A = \langle a_1, \dots, a_n \rangle$. Consider the (surjective) k-algebra morphism

$$\pi : k[x_1, \dots, x_n] \twoheadrightarrow A \qquad\qquad x_i \mapsto a_i$$

Then $J := \ker \pi$ is a prime ideal. We put $X := Z(J) \subseteq \mathbb{A}^n$. Then

$$I(X) = I(Z(J)) = \sqrt{J} = J \implies A \cong k[x_1, \dots, x_n]/J = A(X)$$ □

2.30 Remark. Technically $X \mapsto A(X)$ is a contravariant functor. A map $X \xrightarrow{f} Y$ becomes $A(Y) \xrightarrow{f^*} A(X)$ and we have $(g \circ f)^* = f^* \circ g^*$. This functor is full and faithful.

2.31 Lemma. *Let $X \subseteq A^n, Y \subseteq \mathbb{A}^m$ affine varieties and $\psi : A(Y) \to A(X)$ an algebra morphism. Then there is a unique morphism $f : X \to Y$ of affine varieties such that $f^* = \psi$.*

Proof. Write $A(\mathbb{A}^m) = k[x_1, \dots, x_m]$. Then we have the projection $A(Y) = k[\overline{y_1}, \dots, \overline{y_m}]$. Define $f_i : \psi(\overline{y_i}) \in A(X)$. This defines a function $f : X \to k^m$ via $f = (f_1, \dots, f_m)$.
Take $G \in I(Y)$, where $G(y_1, \dots, y_m)$ is a polynomial. Then we can write

$$\begin{aligned} G(f_1, \dots, f_m) &= G(\psi(\overline{y_1}), \dots, \psi(\overline{y_m})) \\ &= \psi(G(\overline{y_1}, \dots, \overline{y_m})) && \psi \text{ morphism} \\ &= \psi(0) = 0 && G \in I(Y) \end{aligned}$$

Hence for $p \in X$ we have $G(f_1(p), \dots, f_m(p)) = 0$. Therefore $(f_1(p), \dots, f_m(p)) \in Z(I(Y)) = Y$. Then f is a morphism. Furthermore $f^*(\overline{y_i}) = f_i = \psi(\overline{y_i})$, so $f^* = \psi$. □

2.32 Definition. A morphism $f : X \to Y$ is called *isomorphism* is there is some morphism $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

2.33 Remark. Not every bijective morphism is an isomorphism.
Let $X = Z(y^2 - x^3) \subseteq \mathbb{A}^2$. Then $f : \mathbb{A}^1 \to X$ via $t \mapsto (t^2, t^3)$ is a bijective morphism. For the inverse we have

$$f^{-1}(x,y) = \begin{cases} \frac{y}{x} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$$

But the pullback of an isomorphism is again an isomorphism.

$$\begin{aligned} A(X) = k[x,y]/(y^2 - x^3) &\xrightarrow{f^*} k[t] \\ x &\mapsto t^2 \\ y &\mapsto t^3 \end{aligned}$$

Then $\mathrm{im}(f^*) = k[t^2, t^3]$ (all without the monomial t^1).

2.5 Products and Tensors

We regard the embeddings

$$a \in A := k[x_1, \dots, x_n] \to k[x_1, \dots, x_n, y_1, \dots, y_m] =: C \leftarrow k[y_1, \dots, y_m] =: B \ni b$$

We define products via $a \cdot b = a(x) \cdot b(y)$.

2.34 Lemma. *1.* $C = \mathrm{span}\{a \cdot b : a \in A, b \in b\}$

2. Let a_i and b_j linear independent. Then the $(a_i b_j)$ are linearly independent.

We have the following universal property If this universal property holds, we call $C := A \otimes B$ the

$$\begin{tikzcd}
& X & \\
A \arrow[ur, "f"] \arrow[dr, "\iota_A"'] & & B \arrow[ul, "g"'] \arrow[dl, "\iota_B"] \\
& C \arrow[uu, dashed, "\gamma"'] &
\end{tikzcd}$$

Figure 3: Universal property for binary coproduct

tensor product. In terms of categories this is the coproduct.

Remark. *For the coproduct we have thw following properties:*

- $k[x] \otimes k[y] = k[x,y]$
- *Because of commutativity we have*

$$\gamma(a_i a_{i'} b_j b_{j'}) = \alpha(a_i a_{i'}) \beta(b_j b_{j'}) = \alpha(a_i)\alpha(a_{i'})\beta(b_j)\beta(b_{j'}) = \gamma(a_i b_j)\gamma(a_{i'} b_{j'})$$

Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties. We define a product $X \times Y \subseteq \mathbb{A}^n \times \mathbb{A}^m$. Then $X \times Y$ is Zariski-closed.

2.35 Lemma. $I(X \times Y) = I(X \times \mathbb{A}^m) + I(\mathbb{A}^n \times Y)$

Proof. $\subseteq$ clear

$\supseteq$ Let $A = I(X) \oplus L$, $B = I(Y) \oplus M$. Then we use the following vector space decomposition

$$\begin{aligned} A \otimes B &= I(X) \otimes I(Y) + I(X) \otimes M + L \otimes I(Y) + L \otimes M \\ &= I(X) \otimes B + A \otimes I(Y) + L \otimes M \subseteq I(X \times \mathbb{A}^m) + I(\mathbb{A}^n \times Y) + l \otimes M \\ &\subseteq I(X \times Y) + L \otimes M \subseteq A \otimes B \end{aligned}$$

So we have all equalities. Hence we now have to show $I(X \times Y) \cap M \otimes L = 0$. To this end let l_i be basis of L, m_i a basis of M. For $f \in L \otimes M$

continue

$\square$

2.36 Proposition. *Let X, Y be affine varieties. Then $X \times Y$ is irreducible, hence an affine variety.*

Proof. The elegant proof consists of the following steps

1. $A(X \times Y) = A(X) \otimes A(Y)$
2. $A(X), A(Y)$ are integral domains
3. Kathlén

$\square$

Pedestrian proof of Proposition 2.36. Show that $I(X \times Y)$ is prime. Let $f \cdot g \in I(X \times Y)$. Assume $f \notin I(X \times Y)$. We have to show $g \in I(X \times Y)$. Let $p' \in X$ be arbitrary. Then the function $q \mapsto f(p', q)g(p', q) \in I(Y)$. But $I(Y)$ is prime, so $g(p', \cdot) = 0$ or $f(p', \cdot) = 0$ as a function on Y. Therefore

$$\forall p' \in X . Y \subseteq Z(f(p', \cdot)) \cup Z(g(p', \cdot))$$

Since Y is irreducible, we get $Y \subseteq Z(g(p', \cdot))$ if there is q with $f(p', q) \neq 0$. We get $(Y \setminus Z(f(\cdot, q)) \times Y \subseteq Z(g)$, where the first part is an non-empty open subset of X. Since X is irreducible, $X \setminus Z(f(\cdot, q))$ is dense in X. Hence $X \times Y \subseteq Z(g)$. $\square$

For varieties we have the following diagram for products. This yields the conversee picture for the coordinate ring, defining a coproduct (inversed arrows).

2.37 Lemma. *Let X be an affine variety and $f \in A(X)$. Let $X_f := \{x \in X : f(x) \neq 0\} \neq \emptyset$ be a distinguished open set. Then the ringed space $(X_f, (\mathcal{O}_X)_{|X_f})$ is isomorphic to an affine variety with coordinate ring $A(X)_f$.*

Proof. We define the algebraic set $Z := \{(x, \lambda) \in A \times \mathbb{A}^1 : \lambda f(x) = 1\}$. We have a bijection $\pi : Z \to X_f$ via $(x, \lambda) \mapsto x$; it inverse is $\varphi : x \mapsto (x, \frac{1}{f(x)})$.

φ is a morphism of ringed spaces unfold the definitions

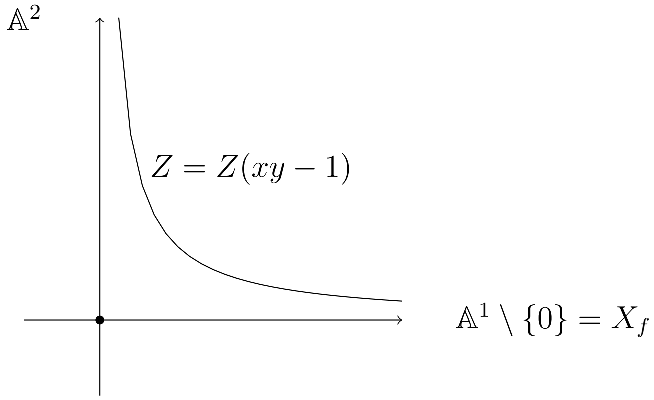
projection π is a morphism of ringed spaces same

$$
\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, dashed, "\exists! h"] & \\
X & & Y \\
 & X \times Y \arrow[ul, "\pi_X"] \arrow[ur, "\pi_Y"'] &
\end{tikzcd}
$$

Figure 4: Universal property for product of varieties

$$
\begin{tikzcd}
 & A(Z) & \\
A(X) \arrow[ur, "f^*"] \arrow[dr, "\pi_X^*"'] & & A(Y) \arrow[ul, "g^*"'] \arrow[dl, "\pi_Y^*"] \\
 & A(X \times Y) \arrow[uu, dashed, "\exists! h^*"'] &
\end{tikzcd}
$$

Figure 5: Universal property for coproduct of coordinate rings



□

As an example, how the proof works, take the following picture.

2.38 Definition. A ringed space $(X, \mathcal{O}_X)$ is called an *affine variety* over k if

1. X is irreducible
2. $\mathcal{O}_X$ is a sheaf over k-value functions
3. X is isomorphic to an affine variety in the old sense.

Example. *The general linear group* $\mathrm{GL}_n(k)$ *is an affine variety but only in the new sense. The regular functions of* $\mathrm{GL}_n(k)$ *are of the form*

$$A \mapsto \frac{F(A)}{\det(A)^e}$$

for polynomials F *and* $e \in \mathbb{N}$.

2.39 Remark. Not every open subset of an affine variety is an affine variety. For example $U := \mathbb{A}^2 \setminus \{0\}$ is not an affine variety. We know $\mathcal{O}(U) = k[x, y]$. There we have the maps

$$U \hookrightarrow \mathbb{A}^2$$
$$\iota^* : k[x, y] \xrightarrow{\sim} \mathcal{O}(U) = k[x, y]$$

If U we an affine variety, then we would have $U \cong \mathbb{A}^2$ as varieties. But $\iota : U \hookrightarrow \mathbb{A}^2$ is not surjective. However, every open set is a finite union of affine varieties.

one lecture missing

2.40 Lemma. *Let* X, Y *be prevarieties and* $f : X \to Y$ *some map. Let* $U_1, \dots, U_r$ *be an open cover of* X *and* $V_1, \dots, V_r$ *an open cover of* Y *where* V_i *are affine such that* $f(U_i) \subseteq V_i$. *Assume*

$$\forall i. \left(f_{|U_i}\right)^* \mathcal{O}_Y(V_i) \subseteq \mathcal{O}_X(U_i) \tag{1}$$

Then f *is a morphism.*

Proof. Wlog we may assume the U_i are affine. First show f is continuous: Use that $f_{|U_i} : U_i \to V_i$ is continuous due to eq. (1). Taka any open $V \subseteq Y$ and decompose it as $V = \bigcup_i (V \cap V_i)$. Then $f^{-1}(V) = \bigcup_i (f^{-1}(V) \cap U_i)$. The remaining part is an exercise. □

Remark. *Take the isomorphism* $\varphi : \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 \setminus \{0\}$ *via* $x \mapsto \frac{1}{x}$. *Then glueing yields* $\mathbb{P}^1$. *Assume we have a non-constant polynomial* $f : \mathbb{A}^1 \to \mathbb{A}^1$. *Then we can extend this to a function* $\widetilde{f} : \mathbb{P}^1 \to \mathbb{P}^1$ *by mapping* $f : \infty \mapsto \infty$. *Check that* $\widetilde{f}$ *is a morphism. Put* $V_1 = \mathbb{P}^1 \setminus \{\infty\}$ *and* $V_2 = \mathbb{P}^1 \setminus \{0\}$ *as affine open cover of* $\mathbb{P}$. *Now we have to check the conditinos of Lemma 2.40. For* $i = 1$, *eq.* (1) *is clear. For* $i = 2$ *take new coordinates* $\widetilde{x} = \frac{1}{x}$ *and* $\widetilde{y} = \frac{1}{y}$. *Write* $f(x) = \sum_{i=0}^n a_i x^i$. *Then*

$$\widetilde{y} = \frac{1}{y} = \frac{1}{f\left(\frac{1}{\widetilde{x}}\right)} = \frac{\widetilde{x}^n}{\widetilde{x}^n f\left(\frac{1}{\widetilde{x}}\right)}$$

which is a rational function. Moreover, since we are close to zero, this denominator is non-zero, so the function is regular.

2.6 Varieties

Let x be some topological space and regard $X \times X$ with product topology.

2.41 Claim. *Equivalent are*

1. *X is Hausdorff, i.e. for any $x_1 \neq x_2$ there exist open $U_i \ni x_i$ such that $U_1 \cap U_2 = \emptyset$.*
2. *$\Delta := \{(x,x) : x \in X\}$ is a closed subset of $X \times X$.*
3. *for all continuous $f_1, f_2 : Y \to X$ the set $\{y \in Y : f_1(y) = f_2(y)\}$ is closed in Y.*

Proof. **item 1⇔item 2:** Show that $X \times X \setminus \Delta$ is open: Take $(x_1, x_2) \in X \times X \setminus \Delta$, so $x_1 \neq x_2$. Take U_i according to Hausdorff. Then $U_1 \cap U_2 = \emptyset$, so $(x_1, x_2) \in U_1 \times U_2 \subseteq X \times X \setminus \Delta$.

The converse direction works the same way.

item 2⇒item 3: Let $f := (f_1, f_2) : Y \to X \times X$ continuous. Then $\{y : f_1(y) = f_2(y) = f^{-1}(\Delta)$ is closed.

item 3⇒item 2: Take the projection $\pi_i : X \times X \to X$. Then $\Delta = \{(x_1, x_2) : \pi_1(X_1, x_2) = \pi_2(x_1, x_2)\}$ is closed. □

2.42 Definition. Let X be a prevariety. We call X a *variety* if for every prevariety Y and every morphisms $f_1, f_2 : Y \to X$ the set $\{y : f_1(y) = f_2(y)\}$ is closed in Y.

Remark. *Let $X, Y \subseteq \mathbb{A}^n$ concrete affine varieties, $X \times Y \subseteq \mathbb{A}^n \times \mathbb{A}^n$.*
Let X, Y prevarieties (Zariski). How to define a topology on $X \times Y$? Take affine open covers

$$X = \bigcup_{i=1}^{m} U_i \qquad\qquad Y = \bigcup_{j=1}^{m} V_j$$

Then we have

$$X \times Y = \bigcup_{i,j} U_i \times V_j$$

Then $W \subseteq X \times Y$ is called open if for all i, j the set $W \cap (U_i \times V_j)$ is open in the Zariski-topology of $U_i \times V_j$. However, this characterisation depends on the covering.
We can turn $X \times Y$ into a prevariety.

Recall the universal property of the product.

2.43 Lemma. *A prevariety X is a variety iff $\Delta(X)$ is closed in $X \times X$. The proof is as before with topological spaces using the categorical characterisation of the product.*

Example. *This prevariety X is not a variety. We have embeddings*

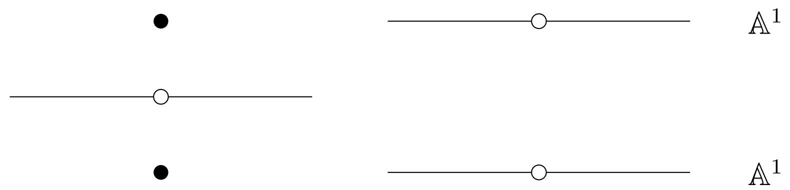
$$j_1 : \mathbb{A}^1 \hookrightarrow X \hookleftarrow \mathbb{A}^1 : j_2$$

but the set $\{y : j_1(y) = j_2(y)\} = \mathbb{A}^1 \setminus \{0\}$ is not closed.

2.44 Lemma. *Every concrete affine variety is a variety.*

$$
\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, dashed, "\gamma"] & \\
X & & Y \\
& X \times Y \arrow[ul, "\pi_X"] \arrow[ur, "\pi_Y"'] &
\end{tikzcd}
$$

Figure 6: Universal property for binary product, π_i are morphisms, for any Z, f, g there exists a unique γ such that the diagram commutes



Proof. Let $X \subseteq \mathbb{A}^n$ closed and $I(X) = (f_1, \dots, f_r)$.

$$\Delta = \{(x, y) : x = y, f_1(x) = 0, \dots, f_r(x) = 0\} \subseteq \mathbb{A}^n \times \mathbb{A}^n$$

is Zariski-closed. $\square$

2.45 Lemma. *Any open or closed irreducible subset of a variety is a variety.*

Proof. Let $\emptyset \neq U \subseteq X$ open. We already know U is a prevariety. Hence $\Delta_U = \Delta_X \cap (U \times U)$ is closed in $U \times U$ since Δ_X is closed in $X \times X$.
Similar for $Y \subseteq X$ closed. $\square$

3 Projective Varieties

3.1 Projective spaces and projective varieties

3.1 Definition. A *projective n-space*over k, denoted $\mathbb{P}^n$ is the set of 1-dimensional linear subspaces of k^{n+1}.
Alternatively wwe may define an equivalence on $k^{n+1} \setminus \{0\}$, saying $a \sim b :\Leftrightarrow \exists t \in k^*.a = tb$. Then $\mathbb{P}^n = (k^{n+1} \setminus \{0\}) / \sim$. We denote the classes by $[a_0 : \ldots : a_n] := [(a_0, \dots, a_n)]_\sim$, called homogeneous coordinates of $\mathbb{P}$.

3.2 Example. Take $[a_0 : a_1] \in \mathbb{P}^1$. If $a_0 \neq 0$, we can write this as $[1 : \alpha_1]$, where $\alpha_1 = \frac{a_1}{a_0}$ is uniquely determined. If $a_0 = 0$, then $[0 : a_1] = [0 : 1] =: \infty$. Again $\mathbb{P}^1 = \mathbb{A}^1 \dot{\cup} \{\infty\}$.

3.3 Example. Take $p := [a_0 : \ldots : a_n] \in \mathbb{P}^n$. If $a_0 \neq 0$, we can write $p = [1 : \alpha_1 : \ldots : \alpha_n]$. These α_i are called affine coordinates and are uniquely determined. For $a_0 = 0$, point p corresponds to a point in $\mathbb{P}^{n-1}$. So again $\mathbb{P}^n \cong \mathbb{A}^n \dot{\cup} \mathbb{P}^{n-1}$.

We have the surjection

$$k^{n+1} \setminus \{0\} \twoheadrightarrow \mathbb{P}^n(k)$$
$$(x_0, \dots, x_n) \mapsto [x_0 : \ldots : x_n]$$

Regarding the case $k = \mathbb{C}$, we can turn $\mathbb{P}^n(\mathbb{C})$ into a topological space via quotient topology of euclidean topology on $\mathbb{C}^{n+1} \setminus \{0\}$.
If we take the sphere $S^{2n+1} = \{x \in \mathbb{C}^{n+1} : \|x\| = 1\}$, which is compact, then we have the surjection

image

Let $f \in k[x_0, \dots, x_n]$ homogeneous of degree d. Then in this space

$$Z(f) = \{[a_0 : \dots : a_n] \in \mathbb{P}^n : f(a_0, \dots, a_n) = 0\}$$

We can decompose $k[x]$ into homogeneous subspaces

$$k[x] = \bigoplus_{d \in \mathbb{N}} k[x]_d$$

This yields a decomposition for each polynomial $f = \sum_d f^{(d)}$.

3.4 Lemma. *Let $I \subseteq k[x]$ an ideal. Equivalent are*

1. *I is generated by homogeneous polynomials.*
2. *For all $f \in I, d \in \mathbb{N}$ we have $f^{(d)} \in I$.*

Such ideal are called homogeneous.

Proof. **item 1⇒item 2** Assume $I = (f_1, \dots, f_r)$ with f_i homogeneous. Take $f \in I$, written as $f = \sum a_i f_i$ with $\deg f_i = d_i$. Then

$$f^{(d)} = sum(a_i f_i)^{(d)} = \sum a_i^{d-d_i} \cdot f_i \implies f^{(d)} \in I$$

item 2⇒item 1 We simply have

$$I = (f_1, \dots, f_r) = \left(f_1^{(0)}, f_1^{(1)}, \dots\right)$$

□

3.5 Definition. Let $I \subseteq k[x]$ a homogeneous ideal. Define the *zero set*

$$Z(I) = \{[a_0 : \dots : a_n] : \forall f \in I. f(a) = 0\}$$

We call the sets $Z(I)$ the algebraic sets of $\mathbb{P}^n$.

3.6 Lemma.

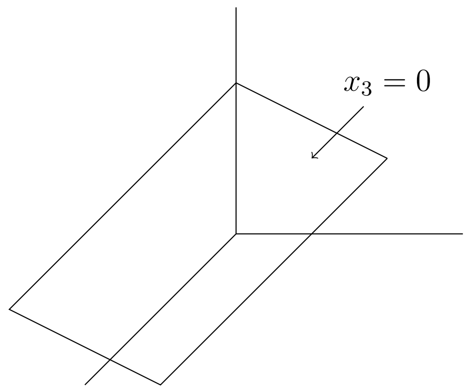
1. *Let $I_1 \subseteq I_2$ homogeneous ideals. Then $Z(I_1) \supseteq Z(I_2)$.*
2. *Let $(I - I)$ be a family of homogeneous ideals. Then $\bigcap Z(I_i) = Z\left(\bigcup_i I_i\right)$.*
3. *Let I_1, I_2 homogeneous ideals. Then $Z(I_1) \cup Z(I_2) = Z(I_1 \cdot I_2)$.*

Proof. Same as for $\mathbb{A}^n$. □

We define the Zariski topology on $\mathbb{P}^n$ to be the topology where the algebraic sets are the closed sets.
Let $L \subseteq \mathbb{A}^{n+1}$ be some linear subspace of dimension $d + 1$.

$$\begin{aligned} \pi : \mathbb{A}^{n+1} \setminus \{0\} &\twoheadrightarrow \mathbb{P}^n \\ \setminus \{0\} &\mapsto \pi(L \setminus \{0\}) \end{aligned}$$

where the latter is a projective linear subspace of $\mathbb{P}^n$ of dimension d.

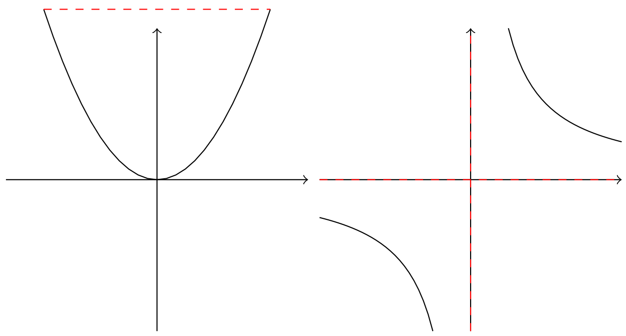


Example. *Describe a line in* $\mathbb{P}^3$. *E.g. take* $x_0 = x_3 = 0$.

$$\{x : x_0 = x_3 = 0\} = \{[0 : a_1 : a_2 : 0] : 0 \neq (a_1, a_2) \in k^2\}$$

This is isomorphic to $\mathbb{P}^1$.

Example. *Two conics in* $\mathbb{A}^n$, *let* $X_1 = \{x : x_2 = x_1^2\}$ *and* $X_2 = \{x : x_1 x_2 = 1\}$. *However, in the*



projective setting, we add a point at infinity and then both curves are isomorphic.

Regard the embedding

$$\begin{aligned} \mathbb{A}^2 &\xrightarrow{\sim} \mathbb{P}^2_{x_0} \subseteq \mathbb{P}^2 \\ (x_1, x_2) &\mapsto [1 : x_1 : x_2] \\ \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}\right) &\leftarrow\!\shortmid [x_0 : x_1 : x_2], x_0 \neq 0 \end{aligned}$$

Let $f(x) = x_2 - x_1^2 = 0$. Then we have the homogenisation $f' = x_0 x_2 - x_1^2 = 0$, or more general $f' = x_0^d \cdot f\left(\frac{x}{x_0}\right)$.
The closure in $\mathbb{P}^2$ is given by the equation $x_0 x_2 - x_1^2 = 0$. If $x_0 = 0$, then $x_1 = 0$, so the only point added was $[0 : 0 : 1]$, which is the point at infinity. For an intuition, that latter two entries $(0, 1)$ show in which direction this infinity lies. This additional point turn the parabola into a circle.
However, for the hyperbola $x_1 x_2 - 1 = 0$, we get the same homogenisation $x_1 x_2 - x_0^2 = 0$ (up to permutation of variables). Here the points added are $[0 : 0 : 1]$ and $[0 : 1 : 0]$. Again this turn the hyperbola into a circle.

3.7 Definition. Let $X \subset \mathbb{P}^n$. Then the *vanishing ideal* $I(X)$ is the ideal generated by the homogeneous polynomials vanishing on X.

Remark. *Also for the Zariski topology on* $\mathbb{P}^n$ *we want the following as for* $\mathbb{A}^n$

- *irreducible sets*
- *decomposition into irreducible sets*

For a homogeneous ideal I in $k[x]$ we have

$$Z_A(I) \subseteq \mathbb{A}^{n+1} \qquad\qquad Z_{\mathbb{P}}(I) \subseteq \mathbb{P}^n$$

3.8 Lemma. *Let $X \subseteq \mathbb{P}^n$ a non-empty algebraic set. Then X is irreducible iff $I(X)$ is a prime ideal.*

Proof. as for $\mathbb{A}^n$ □

Remark (Warning). *The ideal $\mathbf{m} = (x_0, \dots, x_n)$ is a homogeneous ideal in $k[x]$. However, $Z_{\mathbb{A}}(\mathbf{m}) = \{0\}$, but $Z_{\mathbb{P}}(\mathbf{m}) = \emptyset$.*

Remark. *Let $f_1, \dots, f_n$ be homogeneous polynomials.*

Explanation why we can assume same degree in zero set

3.2 Cones and Projective Nullstellensatz

3.9 Definition. A affine algebraic set $X \subseteq \mathbb{A}^{n+1}$ is called a *cone* of $X \neq \emptyset$ and $\forall \lambda \in k. \lambda X \subseteq X$. If $X \subseteq \mathbb{P}^n$ is projective algebraic, then

$$C(X) := \{(x_0, \dots, x_n) : [x_0 : \dots : x_n] \in X\} \cup \{0\}$$

is called the *cone over* X.

3.10 Lemma. 1. *Let $I \subset k[x]$ be a homogeneous ideal. If $X = Z_{\mathbb{P}}(I) \subseteq \mathbb{P}^n$, then $C(X) = Z_{\mathbb{A}}(I)$.*

2. *Conversely if $X \subseteq \mathbb{P}^n$ is projective algebraic, then $I(C(X)) = I(X)$*

of the pitfall in item 2. Let $f \in k[x_0, \dots, x_n]$. f vanishes on a cone $C \ni p \neq 0$. Hence for all d, $f^{(d)}$ vanishes on C. This means

$$\forall \lambda. 0 = f(\lambda p) = \sum_d f^{(d)}(\lambda p) = \sum_d \lambda^d f^{(d)}(p)$$

and therefore $\forall d. f^{(d)}(p) = 0$. □

Definition. *Let A be some k-algebra. If we have a decomposition*

$$A = \bigoplus_{d \in \mathbb{N}} A_d$$

where $A_d \cdot A_e \subseteq A_{d+e}$, then A is a graded *k-algebra.*

We are now interested in the grading of our polynomial ring. Therefore we partition

$$k[x] = \bigoplus_{d \in \mathbb{N}} k[x]_d$$

into homogeneous subsets. Let $I \subseteq k[x]$ is an ideal, then I is called *homogeneous* if $I = \bigoplus_{d\in\mathbb{N}} I_d$ where $I_d = I \cap k[x]_d$. In this case

$$k[x]/I = \bigoplus_{d\in\mathbb{N}} k[x]_d/I_d$$

is again a graded k-algebra.
Now let $I \subseteq k[x]$ be some homogeneous ideal. Then $Z_{\mathbb{A}}(I) \subseteq \mathbb{A}^{n+1}$ is an algebraic set and a *cone.* On the other hand $Z_{\mathbb{P}}(I) \subseteq \mathbb{P}^n$ is a (projective) algebraic set. $Z_{\mathbb{A}}(I)$ is the cone over $Z_{\mathbb{P}}(I)$.
We have a bijection between projective algebraic sets in $\mathbb{P}^n$ and affine cones in $\mathbb{A}^{n+1}$. As special case we get $\emptyset \subseteq \mathbb{P}^n \leftrightarrow \{0\} \subseteq \mathbb{A}^{n+1}$. Recall for an algebraic cone $C(X)$ we have the bijection to homogeneous ideals, where for the corresponding ideal we have $I \subseteq (x_0, \ldots, x_n)$.

3.11 Proposition (Projective Nullstellensatz).

1. *Let $X \subseteq \mathbb{P}^n$ an algebraic set. Then $Z_{\mathbb{P}}(I(X)) = X$.*
2. *Let $I \subseteq k[x]$ be some homogeneous ideal such that $I \subseteq (x_0, \ldots, x_n)$. Then $I(Z_{\mathbb{P}}(I)) = \sqrt{I}$.*
3. *$\mathbb{Z}_{\mathbb{P}}(I) = \emptyset \Leftrightarrow \sqrt{I} = (x_0, \ldots, x_n)$.*

Proof.

1. As in the affine case.
2. Let $X = \mathbb{Z}_{\mathbb{P}}(I)$. Then

$$I(\mathbb{Z}_{\mathbb{P}}(I)) = I(X) = I(C(X)) = I(Z_{\mathbb{A}}(I) = \sqrt{I}$$

where the last part follows from the affine Nullstellensatz.

□

Now the question is, what the regular functions $\mathcal{P}_{\mathbb{P}^n}(U)$ are? They are rational functions. However, they also have to be well-defined,i.e. independent of the representative. Therefore for a regular function $\frac{p(x)}{q(x)}$ we need $\deg p = \deg q$, both homogeneous, and finally for regularity $q(x) \neq 0$.

3.3 Projective varieties as ringed spaces

Let $X \subset \mathbb{P}^n$ be a projective variety. Define

$$S(X) := k[x_0, \ldots, x_n)/I(X)$$

as the homogeneous coordinate ring. $S(X)$ is a graded algebra and an integral domain.

3.12 Definition. The field of rational functions in X is defined as

$$K(X) := \left\{ \frac{f}{g} : \exists d \in \mathbb{N}. f, g \in S(X)_d, g \neq 0 \right\}$$

This is a subfield of the field of fractions of $S(X)$.
For $p \in X$ define

$$\mathcal{O}_{X,p} := \left\{ \frac{f}{g} \in K(X) : g(p) \neq 0 \right\} \qquad \text{local ring}$$

For open $U \subseteq X$ define

$$\mathcal{O}_X(U) := \bigcap_{p\in U} \mathcal{O}_{X,p} \qquad \text{ring of regular functions on } U$$

We obtain a sheaf of k-valued functions.

3.13 Example. For $X = \mathbb{P}^n$ we get $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) \cong k$, i.e. only the constant functions. More generally $\mathcal{O}_X(X) \cong k$.

3.14 Proposition. *Let $X \subseteq \mathbb{P}^n$ be a projective variety. Then $(X, \mathcal{O}_X)$ is a prevariety.*

Proof. Putting $\mathbb{P}^n_i := \{x \in \mathbb{P}^n : x_i \neq 0\}$, we can write $\mathbb{P}^n = \bigcup_{i \in \mathbb{N}} \mathbb{P}^n_i$. These sets are open. Wlog we focus on $i = 0$. We have bijections

$$\mathbb{A}^n \to \mathbb{P}^n_0 (a_1, \dots, a_n) \qquad \mapsto (1 : a_1 : \dots : a_n)(a_0 : \dots : a_n) \mapsto \left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right)$$

and these are inverse to each other.
Have $X = \bigcup_{i \in \mathbb{N}} (X \cap \mathbb{P}^n_i)$, where $X \cap \mathbb{P}^n_i$ are open subsets of X. If $X = Z(f_1, \dots, f_r)$ where f_j are homogeneous polynomials, define

$$g_j(x_1, \dots, x_n) := f_j(1, x_1, \dots, x_n) \qquad \text{dehomogenisation}$$

Define $Y := Z(g_1, \dots, g_r) \subseteq \mathbb{A}^n$ an algebraic set. We get bijections inverse to each other $Y \leftrightarrows \mathbb{P}^n_0 \cap X$. We have to check that these bijections are morphisms of k-ringed spaces.
Let $U \subseteq Y$ open and $\varphi \in \mathcal{O}_Y(U)$. Locally φ is given by $\varphi = \frac{p}{q}$, where p, q are polynomials. The pullback with respect to the map $Y \leftarrow \mathbb{P}^n_0 \cap X$ looks like

$$\begin{tikzcd} k & \\ Y \arrow[u, "\varphi"] & \mathbb{P}^n_0 \cap X \arrow[l] \arrow[ul, dashed, "\psi"'] \end{tikzcd}$$

$$\left(\tfrac{a_1}{a_0}, \dots, \tfrac{a_n}{a_0}\right) \longleftarrow\!\!\shortmid (a_0 : \dots : a_n)$$

$$\psi(a_0 : \dots : a_n) = \frac{p\left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right) a_0^D}{q\left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right) a_0^D}$$

If $D \geq \deg p$, then

$$F(a_0 : \dots : a_n) := p\left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right) a_0^D$$

is a polynomial of degree D and

$$F(ta_0, \dots, ta_n) = p\left(\frac{ta_1}{a_0}, \dots, \frac{ta_n}{a_0}\right) a_0^D = p\left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}\right) a_0^D t^D \qquad \square$$

3.15 Lemma. *Let $X \subseteq \mathbb{P}^n$ a projective variety. Let $f_1, \dots, f_m \in k[x]$ homogeneous polynomials of the same degree without a common root, i.e. $\forall p \in X. \exists i. f_i(p) \neq 0$. then the f_i define a morphism*

$$X \mapsto \mathbb{P}^m \qquad p \mapsto (f_0(p), \dots, f_m(p))$$

$$\{x \in X : f_0(x) \neq 0\} = \varphi^{-1}(\mathbb{P}^m_0) \xrightarrow{\varphi} \mathbb{P}^m_0 \cong \mathbb{A}^m$$

$$\begin{tikzcd}
 & (y_0 : \ldots : y_m) \arrow[r] \arrow[dl, "\tau"'] \arrow[d, "\text{regular}"] & \left(\frac{y_1}{y_0}, \ldots, \frac{y_m}{y_0}\right) \\
F\left(\frac{y_1}{y_0}, \ldots, \frac{y_m}{y_0}\right) & k &
\end{tikzcd}$$

Proof. First note that the above map is well-defined.
Again we partition $\mathbb{P}^m = \bigcup_{i \in \mathbb{N}} \mathbb{P}^m_i$ and focus on $i = 0$. Then

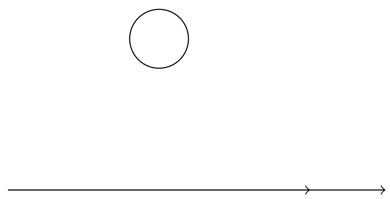
$$(\tau \circ \varphi)(p) = F\left(\frac{y_1}{y_0}, \ldots, \frac{y_m}{y_0}\right)$$

so $\tau \circ \varphi$ is a regular function. □

3.16 Example. Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^2$ define by $(s : t) \mapsto (s^2 : st : t^2) = (x : y : z)$. Then φ is a morphism by the above Lemma 3.15.

Claim. *$\varphi : \mathbb{P}^2 \to Z_{\mathbb{P}}(xz - y^2)$ is a bijection.*

we have to check



example incomplete

Our next goal is to describe $\mathbb{P}^n \times \mathbb{P}^m$. Consider the map

$$f : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$$
$$((x_0 : \ldots : x_n), (y_0 : \ldots : y_m)) \mapsto [(x_i y_j)_{i \leq n, j \leq m}]$$

First note that the right hand side is non-zero and well-defined. In homogeneous coordinates z_{ij} we have $z_{ij} z_{i'j'} = z_{ij'} z_{i'j}$, because we just have products (which commute).

3.17 Proposition. *1. The image $X := f(\mathbb{P}^n \times \mathbb{P}^m)$ is a projective variety in $\mathbb{P}^{(n+1)(m+1)-1}$ which is the zero set of the quadratic polynomials $z_{ij} z_{i'j'} = z_{ij'} z_{i'j}$.*

2. The map $f : \mathbb{P}^n \times \mathbb{P}m \to X$ is an isomorphism.

3. The closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$ are exactly the subsets that are zero sets of polynomials in $k[x, y]$which are bihomogeneous in x and y respectively.

Proof. 1. We already noticed that $f(x, y)$ satisfy the quadratic equations. The equations can be rewritten as

$$0 = \det \begin{pmatrix} z_{ij} & z_{ij'} \\ z_{i'j} & z_{i'j'} \end{pmatrix}$$

and this is the condition of having a rank 1 matrix. But the rank-1-matrices are exactly those, which arise as product of two vectors and these vectors are unqiue up to scaling. Hence this map is injective.

2. By Example 3.16 f is a morphism since it is given by quadratic polynomials. We have to check that the inverse

$$f^{-1} : X \to \mathbb{P}^n \times \mathbb{P}^m = \bigcup_{i=0}^{n} \bigcup_{j=0}^{n} (\mathbb{P}^n)_{x_i} \times (\mathbb{P}^m)_{y_j} \cong \mathbb{A}^n \times \mathbb{A}^m$$

is a morphism. Regarding the pullback, we have where wlog we have $z_{00} = 1$.

$$\begin{tikzcd}
\mathbb{P}^{(n+1)(m+1)-1} \arrow[dr] & f\left((\mathbb{P}^n)_{x_i} \times (\mathbb{P}^m)_{y_j}\right) \arrow[r, phantom, "f^{-1}"'] & (\mathbb{P}^n)_{x_i} \times (\mathbb{P}^m)_{y_j} \cong \mathbb{A}^n \times \mathbb{A}^m \\
 & z_{ij} \arrow[r, mapsto] & ((z_{i0})_i, (z_{0j}))
\end{tikzcd}$$

some more confusing stuff

3. Let $Y \subseteq O\mathbb{P}^n \times \mathbb{P}^m$ be closed. Then $f(Y)$ is closed in $X \subseteq \mathbb{P}^{(n+1)(m+1)-1}$. So it is the zero set of homogeneous polynomials $\varphi_k(z)$ of degree d_k. Then Y is the zero set of $\varphi_k(x \otimes y$ and $\varphi_k(x \otimes y)$ is homogeneous of degree d_k in x and homogeneous of degree d_k in y. Conversely the zero set of bihomogeneous polynomials can be written as the zero set of the bihomogeneous polynomials of the same degree (in x and y) by replacing a single polynomial f with many polynomials $x_i^d f$. Such polynomials are homogeneous polynomials in the $x_i y_j$. □

3.18 Example. $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to $X \subseteq \mathbb{P}^3$ where X is the zero set of $z_{00}z_{11} - z_{01}z_{10}$. This will be a quadric surface.
First we fix a subvariety $\{\xi\} \times \mathbb{P}^1 \subseteq \mathbb{P}^\times\mathbb{P}^1$ for all $\xi \in \mathbb{P}^1$. All of these are lines. Likewise we have lines $\mathbb{P}^1 \times \{\eta\} \subseteq \mathbb{P}^\times\mathbb{P}^1$ for $\eta \in \mathbb{P}^1$. We get two partitions

$$\mathbb{P}^2 \times \mathbb{P}^1 = \bigcup_{\xi \in \mathbb{P}^1} \{\xi\} \times \mathbb{P}^1 = \bigcup_{\eta \in \mathbb{P}^1} \mathbb{P}^1 \times \{\eta\} = \{(\xi, \eta) : \xi, \eta \in \mathbb{P}^1\}$$

And this is a double-ruled surface.

We already know that $\mathbb{P}^n$ is a prevariety, now we want to show that it is a variety. To this end we want to show that $\Delta(\mathbb{P}^n)$ is closed in $\mathbb{P}^n \times \mathbb{P}^n$.

$$\Delta(\mathbb{P}^n) = \left\{(x,y) : \operatorname{rk}\begin{pmatrix} x_0 & \dots & x_n \\ y_0 & \dots & y_n \end{pmatrix} \leq 1\right\} = \left\{(x,y) : \forall i < j.\det\begin{pmatrix} x_i & x_j \\ y_i & y_j \end{pmatrix} = 0\right\}$$

So we have characterised it by a set of bihomogeneous polynomials. Thus $\Delta(\mathbb{P}^n)$ is closed in $\mathbb{P}^n \times \mathbb{P}^n$.

3.19 Corollary. *Every projective variety X is a variety.*

Proof. Let $X \subseteq \mathbb{P}^n$ irreducible closed. Then $\Delta(X) = \Delta(\mathbb{P}^n) \cap (X \times X)$. Since $\Delta(\mathbb{P}^n)$ is closed in $\mathbb{P}^n \times \mathbb{P}^n$, $\Delta(X)$ is closed in $X \times X$. □

3.4 The Main Theorem in Projective Varieties

3.20 Lemma. *Let $f_1, \ldots, f_s \in k[x] \setminus k$ homogeneous with $Z_{\mathbb{P}^n}(f_1, \ldots, f_s) = \emptyset$. Then there exists some $d \in \mathbb{N}$ such that $k[x]_d \subseteq (f_1, \ldots, f_s)$.*

Proof. By the Nullstellensatz $\sqrt{(f_1, \ldots, f_s)} = (X_0, \ldots, X_n)$ because $\mathbb{Z}_{\mathbb{A}^{n+1}}(f_1, \ldots, f_s) = \{0\}$. Thus there exist some $d_i \in \mathbb{N}$ with $x_i^{d_i} \in (f_1, \ldots, f_s)$. Put $d := \sum d_i - n$. Then for all $\alpha \in \mathbb{N}^{n+1}$ with $|\alpha|_1 = d$ there is some i with $\alpha_i \geq d_i$ (otherwise $|\alpha| \leq \sum_{i=0}^{n}(d_i) - 1$). Hence $x^\alpha \in (f_1, \ldots, f_s)$. $\square$

However, this is just the existence. To get some quantitative statement, we can use the Effective Nullstellensatz. Let $D := \max \deg f_i$. Then $d \leq D^n$ is sufficient (shown only around 1990, while the existence was around 1900).

3.21 Theorem. *The projection $\pi_2 : \mathbb{P}^m \times \mathbb{A}^n \to \mathbb{A}^n$ is closed, i.e. it maps closed sets to closed sets.*

Remark (Illustration). *Let $Z \subseteq \mathbb{P}^m \times \mathbb{A}^n$ closed, zero set of equations $f_i(x, y) = 0$, where f_i are bihomogeneous. Then there exist polynomials $g_1, \ldots, g_M$ such that*

$$y \in \pi_2(Z) \Leftrightarrow \exists x. f_1(x, y) = \ldots = f_s(x, y) = 0 \Leftrightarrow g_1(y) = \ldots = g_M(y) = 0$$

So this basically is quantifier elimination.

Remark. *If we use resultants, then we basically do quantifier elimination as well. However, there we demand that the polynomials are monic. This corresponds to the condition that we take the projective variety.*

Theorem 3.21 even holds in a more general setting. Let X, Y be topological spaces, X compact and $Z \subseteq X \times Y$ closed. Then the projection $\pi_2 : Z \to X$ is closed.

Example. *Theorem 3.21 does not hold if we take an affine variety instead of a projective one. Take $\pi_2 : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$. The set $Z := Z(xy - 1)$ is closed. But $\pi_2(Z) = k \setminus \{0\}$ is not closed, so π_2 is not closed.*

Proof of Theorem 3.21. Let $Z \subseteq \mathbb{P}^m \times \mathbb{A}^n$ closed. Then Z is the zero set of polynomials $f_1(X, Y)$, $\ldots$, $f_s(X, Y)$, where f_i are homogeneous in X, say of degree d_i. Then $y \notin \pi_2(Z)$ iff $f_1(X, y)$, $\ldots$, $f_s(X, y)$ have no common root in $\mathbb{P}^m$. By Lemma 3.20 this happens iff

$$\exists d \in \mathbb{N}. \mathscr{H}_d := k[x]_d \leq (f_1(X, y), \ldots, f_s(X, y))$$

We have to show $\{y \in \mathbb{A}^n : y \notin \pi_2(Z)\}$ is open. To this end it is enough to show for all $d \in \mathbb{N}$ that

$$U_d := \{y \in \mathbb{A}^n : \mathscr{H}_d \leq (f_1(X, y), \ldots, f_s(X, y))\}$$

is an open subset of $\mathbb{A}^n$.
Fix some $y \in \mathbb{A}^n$. We have $y \in U_d$ iff the following linear map is surjective

$$T_y : \mathscr{H}_{d-d_1} \times \ldots \times \mathscr{H}_{d-d_s} \to \mathscr{H}_d$$
$$(g_1, \ldots, g_s) \mapsto \sum_{i=1}^{s} g_i f_i(X, y)$$

This means $\operatorname{rk}(T_y) \geq \dim \mathscr{H}_d =: N$. Let M_y be the representation matrix with respect to the bases of monomials of the spaces $\mathscr{H}_{d-d_1}, \ldots, \mathscr{H}_{d-d_s}$. The entries of M_y are polynomials in y. Then $\operatorname{rk}(T_y) \geq N$ iff there is some non-vanishing $N \times N$-subdeterminant of M_y. And this is an open condition in the Zarisky topology. $\square$

Put $\delta := \max \deg f_i$. Then (by the Effective Nullstellensatz) we may take $d \leq \delta^m$, but then $N \approx \frac{\delta^{m^2}}{m!}$, so the size grows doubly exponential.

3.22 Corollary. *Let X be a projective variety and Y any variety. Then the projection $\pi_2 : X \times Y \to Y$ is a closed map.*

Proof. Closedness is a local property. So we may assume Y is affine, so $Y \subseteq \mathbb{A}^n$ closed. Let $X \subseteq \mathbb{P}^m$ be closed. Let $Z \subseteq \mathbb{P}^m \times \mathbb{A}^n$ be closed. Then by Theorem 3.21 $\pi_2(X \times Y)$ is closed in $\mathbb{A}^n$, so $\pi_2(X \times Y)$ is closed in Y. $\square$

3.23 Definition. A variety X is called *complete* if the projection $\pi : X \times Y \to Y$ is a closed map for all varieties Y.

Corollary 3.22 shows that all projective varieties are complete.

3.24 Corollary. *Let $f : X \to Y$ be a morphism of varieties and assume X is complete. Then $f(X)$ is closed in Y.*

Proof. Define $\Gamma : X \to (X \times Y)$ via $x \mapsto (x, f(x))$. Hence $f = \pi_2 \circ \Gamma$. Furthermore $\Gamma(X)$ is the inverse image of Δ under $f \times \mathrm{id}$. So $\Gamma(X) = (f \times \mathrm{id})^{-1}(\Delta(Y))$ is closed. Therefore $f(X) = \pi_2(\Gamma(X))$ is closed by the Main Theorem (Theorem 3.21). $\square$

3.25 Corollary. *Let $X \subseteq \mathbb{P}^n$ be some projective variety consisting of more than one point. Let $f \in k[x] \setminus k$ homogeneous. Then $Z(f) \cap X \neq \emptyset$.*

Proof. Assume otherwise. Then f is non-zero on X. Let $P, Q \in X$ be two distinct points and choose $g := (x - P)^{\deg f}$, so $g(P) = 0$ and $g(Q) \neq 0$. Define the morphism $F : X \to \mathbb{P}^1$ via $R \mapsto (f(R), g(R))$. Note that this is well-defined, since f is non-zero (so we cannot get 0) and both are homogeneous of the same degree. Then $F(X) \subseteq \mathbb{P}^1$ is closed, due to Corollary 3.24 and irreducible (because X is irreducible). So $F(X) = \mathbb{P}^1$ or $F(X)$ is a single point. Since $(0 : 1) \notin F(X)$, so $F(X)$ is a single point. But $F(P) = (1 : 0)$ and $F(Q) \neq (1 : 0)$ so we have to distinct points in $F(X)$. $\square$

Remark. *Corollary 3.25 fails for affine varieties. We could take parallel lines in the plane. $X = Z(y) \subseteq \mathbb{A}^2$ and $f = y - 1$.*

3.26 Corollary. *Every regular function on a complete variety is constant.*

Proof. Let $f : X \to \mathbb{A}^1$ be a regular function. Then we can regard it as a morphism $f : X \to \mathbb{P}^1$. By Corollary 3.24 $f(X) \subset \mathbb{P}^1$ is closed, irreducible. So $f(X)$ is a single point. $\square$

3.27 Example (Veronese Embedding). Let

$$\left\{ f_i(x_0, \ldots, x_n) : 0 \leq i \leq N := \binom{n+d}{n} - 1 \right\}$$

be the set of all monomials in $k[x]_d$. The map $F_d : \mathbb{P}^n \to \mathbb{P}^N$ via $(x_0 : \ldots : x_n) \mapsto (f_0(x) : \ldots : f_N(x))$ is a morphism. $F_d(\mathbb{P}^n) \subseteq \mathbb{P}^N$ is closed by Theorem 3.21, so $F_d(\mathbb{P}^n)$ is a projective variety (called *Veronese variety*).

Claim. *$F_p : \mathbb{P}^n \to F(\mathbb{P}^n)$ is an isomorphism.*

injectivity: Let $x_i \neq 0$, then the image will contain $x_i^{d-1}x_0, \ldots, x_i^{d-1}x_n$, from which we can uniquely recover $x_0, \ldots, x_n$. Hence the map is injective.

inverse morphism: The inverse can be expressed via $\frac{x_j}{x_i} = \frac{x_i^{d-1}x_j}{x_i^d}$. So we recover $\left(\frac{x_0}{x_i} : \ldots : \frac{x_n}{x_i}\right) = (x_0 : \ldots : x_n)$ by rational functions.

Let $g_1(x) = \ldots = g_n(x) = 0$, and g_i some arbitrary homogeneous polynomials of degree d. Then we may write

$$g_i(x) = \sum_{|\mu|=d} a_{i\mu} X^\mu$$

Introducing new variables y_μ we may rewrite these as linear forms

$$l_i = \sum_{|\mu|=d} a_{i\mu} y_\mu$$

Then the Veronese map restricts to a bijection

$$Z_{\mathbb{P}^n}(g_1, \ldots, g_n) \xrightarrow{\sim} Z_{\mathbb{P}^N}(l_1, \ldots, l_N) \cap F(\mathbb{P}^n)$$

3.28 Corollary. *Let $X \subseteq \mathbb{P}^n$ be a projective variety. Let $f \in k[x]_d$, $d > 0$. Then $X \setminus Z_{\mathbb{P}}(f)$ is an affine variety.*

Proof. This is known already if f is a linear form. If $f = x_0$, recall For the general case we use the

$$\begin{tikzcd}
(\mathbb{P}^n)_{x_0} \arrow[r, no head] & \{x : x_0 \neq 0\} \arrow[r, "\sim"] & \mathbb{A}^n \\
X \cap (\mathbb{P}^n)_{x_0} \arrow[u, no head, "\subseteq"] & & \text{image} \arrow[u, no head, "{\subseteq, \text{closed}}"]
\end{tikzcd}$$

veronese map of degree d. □

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, no head, "\subseteq"] & F(\mathbb{P}^n) \arrow[r, no head, "\subseteq"] & \mathbb{P}^N \\
X \arrow[u, no head, "\subseteq"] \arrow[r, "\sim"] & F(X) \arrow[u, no head, "\subseteq"] & \\
X \setminus Z_{\mathbb{P}^n}(f) \arrow[u, no head, "\subseteq"] \arrow[r, "\sim"] & F(X) \setminus Z_{\mathbb{P}^N}(l) \text{ affine variety} &
\end{tikzcd}$$

4 Dimension

4.1 The Dimension of Projective Varieties

4.1 Definition. Let X be a Noetherian topological space. The *dimension* $\dim X$ is the maximal $n \in \mathbb{N}$ such that there exists a chain

$$X_0 \subset X_1 \ldots \subset X_n \subseteq X$$

where the X_i are irreducible, closed subsets.

4.2 Lemma. *1. If $X_0 \subset \ldots \subset X_n$ is a chain of maximal length, then $\dim X_i = i$.*

2. If $Y \subset X$ is a closed subvariety, then $\dim Y < \dim X$.

3. Let $f : X \to Y$ be a surjective morphism of projective varieties Then every chain $Y_0 \subset \ldots \subset Y_n$ can be lifted to a chain X_i such that $f(X_i) = Y_i$. In particular $\dim X \geq \dim Y$.

Proof. 1. If otherwise, then we could switch the chain up to X_i to obtain a longer chain for X.

2. Any chain for Y can be properly extended to a chain for X.

3. Induction on $\dim Y$.

IB: For $\dim Y = 0$, Y is just a single point. then everything is clear.

IS: Take a chain $Y_0 \subset \ldots \subset Y_n = Y$. Let $Z_1, \ldots, Z_r$ be the irreducible components of $f^{-1}(Y_{n-1})$. Then we have

$$f(Z_1) \cup \ldots \cup f(Z_r) = Y_{n-1}$$

By Theorem 3.21 all $f(Z_i)$ are closed. Since Y_{n-1} is irreducible, there must be some i with $f(Z_i) = Y_{n-1}$. Next we apply the induction hypothesis to $f : Z_i \to Y_{n-1}$ (note $\dim Y_{n-1} < \dim Y$). So there is a chain $X_0 \subset \ldots \subset X_{n-1} = Z_i$ such that $f(X_j) = Y_j$. Then w extend this chain by X and we are done. ☐

In the general setting we have some projective variety $X \subset \mathbb{P}^n$. Let $p \in \mathbb{P}^n \setminus X$. Wlog $p(0 : \ldots : 0 : 1)$ by change of coordinates. Let $H \subseteq \mathbb{P}^n$ be a projective linear subspace of codimension 1 with $p \notin H$. Wlog $H = \{x : x_n = 0\}$. Define a projection map $\pi : \mathbb{P}^n \setminus \{p\} \to H$ via

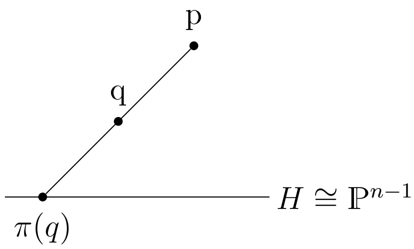


Figure 7: Projection from point p onto hyperplane H

$(a_0 : \ldots : a_n) \mapsto (a_0 : \ldots : a_{n-1} : 0)$. Now we can restrict this map to X, so we get the projection $X \to \pi(X) \subseteq H$ and we know $\pi(X)$ is closed by Theorem 3.21. Note X is a zero set of polynomials and be intersection with a line, the intersection becomes the zero set of a univariate polynomials. Hence it only has finitely many points. This means the fibres of π are finite.

missing lecture

4.3 Proposition. *Let $\subseteq \mathbb{P}^N$ be a projective variety with $\dim X \geq 1$, $f \in k[x] \setminus k$ homogeneous such that $f \notin I(X)$. Then $\dim(X \cap Z(f)) = \dim X - 1$.*

Remark. *We already know $Z_X(f) := X \cap Z(f) \neq \emptyset$.*

Proof. Using the Veronese-embedding we can assume f is linear. Put $n := \dim X$ and $X_1 := Z_X(f)$. Then $\dim X_1 < \dim X$. By the Lemma there exists a linear form f_1, vanishing on none of the ref components of X_1. Put $X_2 := Z_{X_1}(f_1)$. Suppose $X = C_1 \cup \ldots \cup C_s$ is the decomposition into irreducible components. Hence

$$Z_{X_1}(f_1) = Z_{C_1}(f_1) \cup \ldots \cup Z_{C_s}(f_1)$$

We have $Z_{C_i}(f_1) \subset C_i$ for all i. Therefore

$$\dim Z_{C_i}(f_1) < \dim C_1 \leq \dim X_1$$

Thus we have $\dim X_2 = \dim Z_{X_1}(f_1) < \dim X_1$.
If $X_2 \neq \emptyset$, proceed inductively: Find a linear form f_2 such that $X_3 := Z_{X_2}(f_2)$ satisfies $\dim X_3 < \dim X_2$. Note that f_2 is linearly independent of f_0, f_1.
Continuing this way, we obtain linear forms $f_0, \ldots, f_m$ such that $Z(f_0, \ldots, f_m) = \emptyset$ and $m \leq n$. Consider a morphism $\varphi : X \to \mathbb{P}^m$ given by $x \mapsto (f_0(x) : \ldots : f_m(x))$ (which is well-defined, since started with a homogeneous form). After change of coordinates assume $f_i(x) = x_i$. So φ is a composition of projections as considered before. Thus $\varphi(X) \subseteq \mathbb{P}^m$ is closed. By the corollary $\dim \varphi(X) = \dim X$. Therefore ref

$$n = \dim X = \dim \varphi(X) \leq m \leq n$$

So $n = m$, and this is only possible if the dimension goes down by 1 in each step. Hence $\dim X_{i+1} = \dim X_i - 1$, which in particular means $\dim Z_X(f) = n - 1$. $\square$

4.2 Noether Normalisation

4.4 Theorem. *Let $X \subseteq \mathbb{P}^n$ be a projective variety. Then there is some $m \leq n$ and there are linear forms $\varphi_1, \ldots, \varphi_m$ such that $Z_X(\varphi_0, \ldots, \varphi_m) = \emptyset$ an*

$$\varphi : X \to \mathbb{P}^m \qquad x \mapsto (\varphi_0(x) : \ldots : \varphi_m(x))$$

is surjective with finite fibres. Furthermore for all homogeneous $f \in k[x]$ there is a representation

$$f^D + a_1 f^{D-1} + \ldots + a_D = 0 \tag{2}$$

in $S(X) = k[x]/I(X)$ where $s_i \in k[x]$ homogeneous of degree $D - i$.

sketch. We obtain φ as a composition of projections $X \to \mathbb{P}^{n-1}$ with centre $p \in \mathbb{P}^n \backslash X$ as considered before. The property eq. (2) is shown as in **??** using the morphism lemma missing

$$X \to \mathbb{P}^m \qquad (x_0 : \ldots : x_n) \mapsto (x_0^d : \ldots : x_{m-1}^d : f(x))$$

where $d := \deg f$. After a change of coordinates $Z(x_0, \ldots, x_m) = \emptyset$, so this morphism is well-defined. $\square$

4.5 Remark. 1. $m = \dim x$.

2. One can shown that Zariski-almost-all choices $(\varphi_0, \ldots, \varphi_m)$ satisfy the theorem. (There is some open set of possible choices, which satisfy the conditions, and open implies dense.) So when construction these forms, we may choose them at random.

4.6 Theorem. *Let $X \subseteq \mathbb{A}$ be an affine variety, not a point. Then there is some $m \leq n$ and there are affine linear forms $\psi_1, \ldots, \psi_m$ such that*

$$\psi : X \to \mathbb{A}^m \qquad x \mapsto (\psi_1(x), \ldots, \psi_m(x))$$

is surjective with finite fibres and for all $g \in k[x]$ there is a representation

$$g^D + b_1 g^{D-1} + \ldots + b_D = 0$$

in $A(X)$ where $b_i \in k[x]$.

Remark.
- *The comorphism*

$$\psi^* : k[y] \hookrightarrow A(X)$$

  is regular and provides an integral ring extension. *Note that $y_1, \ldots, y_m$ are algebraically independent.*

- *Recall the transcendence degree of $k(X)$ over k is the maximal number of algebraically independent elements from X. We will show $\dim X = \operatorname{trdeg}_k k(X)$.*

- *One can show that almost all choices of $\psi_1, \ldots, \psi_m$ are good for Theorem 4.6.*

4.3 Algebraic Intermezzo

Let $K \subseteq L$ be some finite field extension. We defined $[L : K] = \dim_K L$. For $a \in L^*$ consider the map $\mu_a : L \to L$ via $b \mapsto ab$. μ_a is a linear isomorphism of k-vector spaces.

Definition. *The* norm *of a is $N_{L/K}(a) := \det(\mu_a) \in K^*$.*

4.7 Proposition.
1. *$N_{L/K} : L^* \to K^*$ is a group homomorphism (group structure given by $\mu_{a_1 a_2} = \mu_{a_1} \circ \mu_{a_2}$.)*
2. *If $a \in K$, then $N_{L/K}(a) = a$.*
3. *If $K \leq L \leq E$ are finite field extensions, $a \in L$, then*

$$N_{E/K}(a) = \left(N_{L/K}\right)^{[E:L]}$$

4. *Take minimal polynomial*

$$\min_K(\alpha) = X^d + \lambda_1 X^{d-1} + \ldots + \lambda_d \in K[X]$$

   then $N_{L/K}(\alpha) = \pm \lambda_d^{[L:K(\alpha)]}$.

Proof.
1. When did we do this?
2.
3. Take some basis $e_1, \ldots, e_m$ of L as a K-vector space, and $f_1, \ldots, f_n$ basis of L-vector space E. For $a \in L$ with $ae_j = \sum \lambda_{ij} e_i$ we get $ae_j f_l = \sum \lambda_{ij} e_i f_l$. The representation matrix of the multiplication with a in E has the block diagonal form

$$\begin{pmatrix} \Lambda & & \\ & \ddots & \\ & & \Lambda \end{pmatrix} \qquad \Lambda = [\lambda_{ij}] \in K^{m \times m}$$

   or written with tensors $\mu_{E/K}(a) = \mu_{L/K}(a) \otimes \mathrm{id}$. Therefore $N_{E/K}(a) = (\det \Lambda)^n = \left(N_{L/K}(a)\right)^n$.

4. $K(\alpha)$ has basis $1, \alpha, \dots, \alpha^{d-1}$. The representation matrix of μ_α in this basis is

$$\begin{pmatrix} 0 & \dots & 0 & -\lambda_d \\ 1 & & & -\lambda_{d-1} \\ & \ddots & & \vdots \\ & & 1 & -\lambda_1 \end{pmatrix} \qquad N_{K(\alpha)/K}(\alpha) = (-1)^{d-1}(-\lambda_d) = \pm\lambda_d$$

By item 3 we get $N_{L/K}(\alpha) = \pm\lambda_d^{[L:K(\alpha)]}$. □

4.4 Integral Ring extensions

4.8 Definition. Let $S \subseteq R$ be an extension of rings. An element $a \in R$ is called *integral* (algebraic) over S if there is a *monic* polynomial $f \in S[x]$ such that $f(a) = 0$. The ring extension is called integral if any element $a \in R$ is integral over S.

4.9 Theorem. *1. Let $S \subseteq R$ be a ring extension. Then $\{a \in R : a \text{integral over } S\}$ is a subring, called the integral closure of S in R.*

2. If $S \subseteq R$ and $R \subseteq T$ are integral ring extension, then $S \subseteq T$ is integral.

Remark. • *The first part can be shown constructively. For computations with algebraic numbers, there are corresponding operations with their minimal polynomials, so we just have to check that the results are monic again.*

• *A crucial observation is: a integral over K iff $S[a]$ is a finitely generated S-module.*

• *To show the hard direction, we need $M \cdot \mathrm{ad}(M) = \det M \cdot I$.*

4.10 Definition. Let S be an integral domain and K its field of fractions. We call S *integrally closed* if S is the integral closure of S in K, i.e. If $a \in K$ integral over S, then $a \in S$.

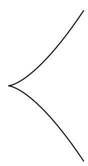
4.11 Proposition. *Every factorial ring S is integrally closed.*

Proof. Suppose $\frac{a}{b} \in K$ is integral over S, where $a, b \in S$, $b \neq 0$ and $\gcd(a, b) = 1$. Take the minimal polynomial

$$\left(\frac{a}{b}\right)^d + \lambda_1 \left(\frac{a}{b}\right)^{d-1} + \dots + \lambda_d = 0 \implies a^d + \lambda_1(a^{d-1}b + \dots + \lambda_d b^d) = 0$$

Hence $b \mid a^d$, so b is a unit. Thus $\frac{a}{b} \in S$. □

4.12 Example. Regard polynomial ring $k[t]$ with subring $S := k[t^2, t^3]$. The field of fractions is $K = k(t)$. t is integral over S since it is a zero of $X^2 - t^2 \in S[X]$. But $t \notin S$, so S is not integrally closed. The corresponding picture is Neil's parabola which has a singularity at the origin.



4.13 Lemma. *Let $S \subseteq R$ be an extension of integral domains and $K \subseteq L$ their corresponding fields of fractions. If $a \in R$ is integral over S, then $N_{L/K}(a)$ is integral over S.*
In particular $N_{L/K}(a) \in S$ if S is integrally closed.

Proof. Let $f = \sum \lambda_i X^{d-i} \in K[X]$ be the minimal polynomial of a over K with $\lambda_0 = 1$. Let $a := a_1, \ldots, a_r \in \overline{L}$ be the zeros of f in the algebraic closure. Let $g \in S[X]$ be monic with $g(a) = 0$. Since $S \subseteq K$ and f is minimal, we get $f \mid g$, so write $g = fh$ for some $h \in K[x]$. Furthermore $g(a_i) = 0$ as well. Hence $a_1, \ldots, a_r$ are integral over S. The λ_i are elementary symmetric polynomials in $a_1, \ldots, a_r$, so the λ_i are integral over S. Therefore $N_{L/K}(a) = \pm\lambda_d^{[L:K(a)]}$ is integral over S. □

Now again regard affine Noether normalisation of an affine variety $X \subseteq \mathbb{A}^n$. Let ψ_i be affine linear, sufficiently general. Put

$$\psi : X \to \mathbb{A}^m \qquad\qquad x \mapsto (\psi_1(x), \ldots, \psi_m(x))$$

Then ψ is surjective with finite fibres and $\psi^*(k[y]) \subseteq A(X)$ is an integral extension.

- ψ is a closed map (see the proof) For a closed subvariety $Z \subseteq X$ we have $\dim Z \geq \dim \psi(Z)$, proof as for Lemma 4.2
- As for **??** we see that $\dim Z = \dim \psi(Z)$.

As a conclusion we have $\dim X = m$.

4.14 Proposition. *Let X be a variety an $U \subseteq X$ non-empty open. Then $\dim U = \dim X$. So dimension is a local property.*

Proof. Regard a longest chain of irreducible closed subsets

$$\emptyset \neq U_0 \subset \ldots \subset U_n = U$$

So $\dim U = n$. Consider the chain of their closures in X

$$\overline{U}_0 \subseteq \ldots \subseteq \overline{U}_n = X$$

We have $\overline{U}_i \subset \overline{U}_{i+1}$ since $U_i = \overline{U}_i \cap U$ (note that U_i is closed in U). So $\dim U \leq \dim X$.

Step 1 Let $X_0 \subset \ldots \subset X_n = X$ be a longest chain of irreducible subsets. Assume $X_0 \cap U \neq \emptyset$. Consider

$$\emptyset \neq X_0 \cap U \subseteq X_1 \cap U \subseteq \ldots \subseteq X_n \cap U$$

which is a chain of irreducible closed subsets. Suppose $X_i \cap U = X_{i+1} \cap U$. Then $X_{i+1} = X_i \cup (X_{i+1} \setminus U)$ which is a proper union of closed sets. This contradicts irreducibility. So in this case $\dim U \geq \dim X$.

Step 2 Let X be a projective variety. We will, construct a chain $X_0 \subset \ldots \subset X_n = X$ such that $X_0 \cap U \neq \emptyset$, so we can use Step 1. Use descending recursion starting with $X_n = X$.

Assume we already constructed $X_i \subset \ldots \subset X_n =$ such that $X_i \cap U \neq \emptyset$ and $\dim X_i = i$. Pick a non.constant homogeneous polynomial f that does not vanish on any of the components of $X_i \setminus U$ (it just intersects them). By the dimension theoremthere is a component of $Z_X(f)$ ref of dimension $\dim X_i - 1 = i - 1$. Call this component X_i.

We have to show $X_{i-1} \cap U \neq \emptyset$. Otherwise $X_{i-1} \subseteq X_i \setminus U$. By construction X_{i-1} is a proper subset of any of the components of $X_i \setminus U$. Hence $\dim X_{i-1} < \dim(X_i \setminus U) < \dim X_i$, which would mean $\dim X_{i-1} \leq i - 2$. ↯

This shows that statement for projective varieties.

Step 3 Let $X \subseteq \mathbb{A}^n$ an affine variety. Take $\overline{X} \subseteq \mathbb{P}^n$ the projective closure. Then $\emptyset U \subseteq X \subseteq \overline{X}$. Apply Step 2 twice, which yields $\dim X = \dim \overline{X}$ and $\dim U = \dim \overline{X}$.

Step 4 Let X be any variety. Take a longest chain $X_0 \subset \ldots \subset X_n = X$. Let V be an affine open subset of X containing the point X_0. By Step 1 $\dim V = \dim X$. As before, find a non-empty affine open subset $W \subseteq U$. Then $\dim W = \dim U$. Since W and V are both open $W \cap V \neq \emptyset$. Then

$$\dim X = \dim V = \dim(V \cap W) = \dim W = \dim U \qquad \square$$

4.15 Corollary. *1.* $\dim \mathbb{A}^n = n$ *as* $\mathbb{A}^n$ *is an open subset of* $\mathbb{P}^n$.

2. $\dim(\mathbb{P}^n \times \mathbb{P}^m) = n + m$ *as* $\mathbb{A}^n \times \mathbb{A}^m$ *is an open subset of* $\mathbb{P}^n \times \mathbb{P}^m$.

Next we show a special case of the Affine Dimension Theorem.

4.16 Proposition. *Let* $f \in k[x]$ *non-constant. Then any component of* $Z(f) \subseteq \mathbb{A}^n$ *has dimension* $n-1$.

Proof. Factor f into irreducibles and write $f = \prod f_i^{e_i}$, with $f_i \nsim f_j$. Hence

$$Z(f) = Z(f_1) \cup \ldots \cup Z(f_r)$$

and the $Z(f_i)$ are the irreducible components of $Z(f)$. So we may assume $X := Z(f)$ is irreducible. The projective closure $\overline{X} \subseteq \mathbb{P}^n$ is the zero-set of the homogenisation $\widetilde{f} \in k[x_0, x]$ of f. By **??** $\dim Z_{\mathbb{P}^n}\left(\widetilde{f}\right) = \dim \overline{X} - 1$. Now $Z_X(f) = Z_{\mathbb{P}^n}\left(\widetilde{f}\right) \cap \mathbb{A}^n$ is a non-empty open subset of $\overline{X}$. By Proposition 4.14 we get $\dim Z_{\mathbb{A}^n}(f) = \dim_{\mathbb{P}^n}(\widetilde{f}) = n-1$. $\square$

For the converse let $X \subseteq \mathbb{A}^n$ be closed irreducible with $\dim X = n-1$. Take $f \in I(X) \setminus \{0\}$. Then $\overline{X} \subseteq Z(f)$. Then one of the irreducible components $Z(f_i)$ must correspond to X.

4.17 Remark. We have a bijection correlation between closed subvarieties of $\mathbb{A}^n$ of dimension $n-1$ and non-constant irreducible polynomials in $k[x]$. Varieties of dimension $n-1$ are called hyper-surfaces of $\mathbb{A}^n$.

4.18 Theorem. *Let* $X \subseteq \mathbb{A}^n$ *be an affine variety with* $\dim X \geq 1$ *and* $f \in k[x]$ *such that* $f \notin I(X)$. *Then every irreducible component of* $Z_X(f)$ *has dimension* $\dim X - 1$.

Proof. First argue that it is sufficient to prove $\dim Z_X(f) \geq \dim X - 1$ if $Z_X(f) \neq \emptyset$.
Let $Z_X(f) = Z_1 \cup \ldots \cup Z_r$ be the decomposition into irreducible components. We want to show $\dim Z_1 = \dim X - 1$. Take $g \in k[x]$ such that

$$g \in I(Z_2 \cup \ldots \cup Z_r) \setminus I(Z_1)$$

If such g would not exist, then

$$I(Z_2 \cup \ldots \cup Z_r) \subseteq I(Z_1) \implies Z_2 \cup \ldots \cup \supseteq Z_1 \quad \lightning$$

Put $X_g := X \setminus Z_X(g)$. Then X_g is an affine variety. Furthermore $X_g \cap Z_X(f) = X_g \cap Z_1$, which is irreducible and $\emptyset \neq Z_1 \cap X_g \subseteq Z_1$ is open. Our reduced theorem yields

$$\dim Z_1 = \dim(U \cap Z(f)) \geq \dim U - 1 = \dim X - 1$$

Now we show $\dim Z_X(f) \geq \dim X - 1$. We already know this statement for $X = \mathbb{A}^n$. Idea: Use Noether normalisation Let $\psi : X \to \mathbb{A}^m$ be some surjective morphism such that

$$S := k[y_1, \dots, y_m] \overset{\psi^*}{\hookrightarrow} A(X) =: R$$

is an integral extension and the map is injective. Let K, L be the fields of fractions of S, R. Then we have a group morphism $N_{L/K} : L^* \to K^*$. We put $f_0 := N_{L/K}(f) \in K^*$. f is integral over S, so f_0 is integral over S (since S is integrally closed, namely factorial). But $f_0 \in K$, so $f_0 \in S$.

Claim. $\sqrt{fR} \cap S = \sqrt{f_0 S}$

Proof. **$\supseteq$:** We already know $f_0 \in S$. Recall $f_0 = N_{L/K}(f)$. The minimal polynomial $\min_K(f) = X^d + \lambda_1 X^{d-1} + \ldots + \lambda_d$ has coefficients $\lambda_i \in S$ (again integrality of S).

$$\lambda_d = f^d + \lambda_1 f^{d-1} + \ldots + \lambda_{d-1} f = \underbrace{(f^{d-1} + \lambda_1 f^{-2} + \ldots + \lambda_{d-1})}_{\in R} \cdot f \in fR$$

By some Lemma $f_0 = N_{L/K}(f) = \pm\lambda_d^{[L:K(f)]} \in fR$. So $f_0 \in fR \cap S \subseteq \sqrt{fR} \cap S$, which shows $\sqrt{f_0 S} \subseteq \sqrt{fR} \cap S$.

$\subseteq$: Let $g \in \sqrt{fR} \cap S$. Then there exists some $N \geq 1$ such that $g^N \in fR$, so there is some $h \in R$ with $g^N = fh$.

$$\left(N_{L/K}(g)\right)^N = N_{L/K}\left(g^N\right) = \underbrace{N_{L/K}(f)}_{f_0} \cdot \underbrace{N_{L/K}(h)}_{\in S}$$

By using $N_{L/K}(g) = g^{[L:K]}$ we get

$$g^{N[L:K]} \in f_0 S \implies g \in \sqrt{f_0 S}$$ ☐

Claim. *Geometrically the claim states $\psi(Z_X(f)) = Z_{\mathbb{A}^m}(f_0)$. So we transfer the setting to $\mathbb{A}^m$.*

Proof. Put $Z := Z_X(f)$. Then the claim is equivalent to

$$\sqrt{fR} \cap S = (\psi^*)^{-1}\left(\sqrt{fR}\right) \overset{\text{NST-Satz}}{=} (\psi^*)^{-1}(I_X(Z)) = I_{\mathbb{A}^m}(\psi(Z)) = I(Z_{\mathbb{A}^m}(f_0)) \overset{\text{NST-Satz}}{=} \sqrt{f_0 S}$$

☐

By the theorem for $\mathbb{A}^m$ we already know $\dim Z_{\mathbb{A}^m}(f_0) \geq m - 1$. So

$$\dim Z_X(f) \geq \dim \psi\left(Z_X(f)\right) \geq \dim Z_{\mathbb{A}^n}(f_0) = n - 1 = \dim X - 1$$ ☐

Remark. *An earlier sketch for the proof was*

$$\begin{tikzcd} X \arrow[r, dash, "\supseteq"] \arrow[d, "\psi"'] & Z_X(f) \arrow[d] \\ \mathbb{A}^n \arrow[r, dash, "\supseteq"] & Z_{\mathbb{A}^n}(f_0) \end{tikzcd}$$

$$\psi(Z_X(f)) \supseteq Z_{\mathbb{A}^n}(f_0)$$
$$\dim Z_X(f) \geq \dim \psi\left(Z_X(f)\right) \geq \dim Z_{\mathbb{A}^n}(f_0) = n - 1$$

Now we can refine **??**.

4.19 Corollary. *Let $X \subseteq \mathbb{P}^n$ be a projective variety with dimension $\dim X \geq 1$ and $f \in k[x_0, \dots, x_n] \setminus k$ homogeneous such that $f \notin I(Z)$. Then $Z_X(f) \neq \emptyset$ and* every *irreducible component of $Z_X(f)$ has dimension $\dim X - 1$.*

Proof. Go to the affine charts and use the affine dimension theorem. □

4.20 Corollary. *Let $X \subseteq \mathbb{P}^n$ be a projective variety of dimension $\dim X = n \geq r \geq 1$ and let $f_1, \dots, f_r \in k[x] \setminus k$ be homogeneous. Then $Z_X(f_1, \dots, f_r) \neq \emptyset$ and every irreducible component of $Z_X(f_1, \dots, f_r)$ has dimension $\geq n - r$.*

Proof. Induction on r.

Base case: This is Corollary 4.19.

Step: Let $Z_X(f_1) = X_1 \cup \ldots \cup X_s$ irreducible components. By the dimension theorem $\dim X_i \geq n - 1$. By the induction hypothesis for every i we have $Z_{X_i}(f_2, \dots, f_r) \neq \emptyset$ and every irreducible component of it has dimension $\geq (n-1) - (r-1) = n - r$. For every irreducible component of $Z_X(f_1, \dots, f_r)$ there is some i such that it occurs as a component of $Z_{X_i}(f_2, \dots, f_r)$. □

4.21 Corollary.
1. *If $f_1, \dots, f_n \in k[x] \setminus k$ are homogeneous, then $Z_{\mathbb{P}^n}(f_1, \dots, f_n) \neq \emptyset$.*
2. *Special case $n = 2$: Any two projective curves in $\mathbb{P}^2$ intersect. (This is clearly false in $\mathbb{A}^2$.)*

4.22 Corollary. *Let $X \subseteq \mathbb{A}^N$ be an affine variety with $\dim X = n \geq r \geq 1$ and let $f_1, \dots, f_r \in kx9 \setminus k$. Then either $Z_X(f_1, \dots, f_r) = \emptyset$ or every irreducible component of $Z_X(f_1, \dots, f_r)$ has dimension $\geq n - r$.*

Proof. As for $\mathbb{P}^n$. □

For some affine variety X we have a topological definition of $\dim X$, so we have the concept of $\dim A(X)$. However, the latter is an algebraic object. So we want to have a connection with the transcendence degree. Via Noether normalisation we have $\psi : X \to \mathbb{A}^m$, with $\dim X = m$. So we have to check $m = \operatorname{trdeg}_K A(X)$. One key observation is

$$\psi^* \left(k[y_1, \dots, y_m]\right) \subseteq A(X) \quad \text{integral}$$

But we want to use a different approach via products.
Let X and Y be varieties and

$$\begin{aligned} \emptyset \subset X_0 \subset \ldots \subset X_n = X \\ \emptyset \subset Y_0 \subset \ldots \subset Y_m = Y \end{aligned}$$

be longest chains of irreducible closed subsets, so $\dim X = n$ and $\dim Y = m$. For their product we get the chain

$$x_0 \times Y_0 \subset \ldots X_0 \times Y_m \subset X_1 \times Y_m \subset \ldots \subset X_n \times Y_m$$

of length $n + m$. So we have $\dim(X \times Y) \geq m + n = \dim X + \dim Y$.

4.23 Proposition. *If X and Y are varieties, then* $\dim(X \times Y) = \dim X + \dim Y$.

Proof. We already know $\dim(\mathbb{A}^m \times \mathbb{A}^n) = \dim\left(\mathbb{A}^{m+n}\right) = n + m$ and the same for $\mathbb{P}$.

Sketch Let X and Y be projective varieties. Use Noether Normalisation. Let $\varphi : X \to \mathbb{P}^m$ and $\psi : Y \to \mathbb{P}^n$ be surjective, finite fibres and integral extensions of the rings. (φ and ψ are finite surjective morphisms). Consider

$$\pi = \varphi \times \psi : X \times Y \to \mathbb{P}^m \times \mathbb{P}^n$$

Now π is surjective, finite fibres,.... As before π is a closed map and the dimension is the same, so $\dim(X \times Y) = m + n$.

Essential: If $Z \subseteq X \times Y$ is closed with $\pi(Z) = \pi(X \times Y)$, then $Z = X \times Y$, use proof of **??**.

Let $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$ affine varieties. Take their projective closures $\overline{X} \subseteq \mathbb{P}^m$ and $\overline{Y} \subseteq \mathbb{P}^n$. Then

$$X = \overline{X} \cap \mathbb{A}^m \qquad\qquad Y = \overline{Y} \cap \mathbb{A}^n$$

$$X \times Y = (\overline{X} \times \overline{Y}) \cap (\mathbb{A}^m \times \mathbb{A}^n)$$

$$\dim(X \times Y) = \dim(\overline{X} \times \overline{Y}) = \dim \overline{X} + \dim \overline{Y} = \dim X + \dim Y$$

If X and Y are any varieties, use affine covers and the previous part. □

From Linear Algebra recall: Let $X, Y \leq k^N$. Then $\dim(X \cap Y) \geq \dim X + \dim Y - N$. Now we show a similar statement for varieties.

4.24 Theorem. *Let $X, Y \subseteq \mathbb{A}^N$ be affine varieties. Then any component of $X \cap Y$ has dimension at least $\dim X + \dim Y - N$.*

Proof. Consider the diagonal $\Delta := \{(x, x) : x \in \mathbb{A}^N\}$. Then we have an isomorpism

$$X \cap Y \xrightarrow{\sim} (X \times Y) \cap \Delta \qquad\qquad x \mapsto (x, x)$$

Moreover Δ is the zero set of the N polynomials $x_i - y_i$. We know $\dim(X \times Y) = \dim X + \dim Y$, and each equation can reduce the dimension by at most 1. Thus

$$\dim(X \cap Y) = \dim((X \times Y) \cap \Delta) \geq \dim X + \dim Y - N$$ □

4.25 Remark. The theorem also holds for projective varieties.

4.5 The structure of morphisms

First we discuss finite morphisms.

4.26 Definition. Let X and Y be affine varieties and $\varphi : X \to Y$ be a morphisms. We call φ *finite* if $\varphi^*(A(Y)) \subseteq A(X)$ is an integral ring extension.

Example. *The morphism $\psi : X \to \mathbb{A}^m$ in the affine Noether Normalisation is finite.*

4.27 Lemma. *A finite morphism φ has finite fibres. (This might be a reason for the name.) But the converse is not true.*

Proof. Let $X \subseteq \mathbb{A}^m$ closed and $x_1, \ldots, x_m$ are the coordinate functions on X. Fix some i. Since x_i is integral over $\varphi^*(A(Y))$, there exist degree $d \in \mathbb{N}$ and coefficients $a_1, \ldots, a_d \in A(Y)$ such that

$$x_i^d + \varphi^*(a_1)x_i^{d-1} + \ldots + \varphi^*(a_d) = 0$$

Let $\eta \in Y$. For all $\xi \in \varphi^{-1}(\eta)$ we have

$$\varphi^*(a_i)(\xi) = a_i(\varphi(\xi)) = a_i(\eta) \qquad\qquad \xi_i^d + a_i(\eta)\xi_i^{d-1} + \ldots + a_d(\eta) = 0$$

This equation only has finitely many solutions, but i was arbitrary, so $\varphi^{-1}(\eta)$ is finite. □

Recall Nakayama's Lemma from Algebra 2.

4.28 Proposition. *Let $A \subseteq B$ be an integral extension of rings. Take some proper ideal $I \subset A$. Then $I \cdot B \subset B$ (the ideal in B generated by I) is a proper ideal.*

4.29 Definition. A morphism $\iota : X \to Y$ of varieties is called *closed embedding* if $\iota(X)$ is a closed subset and $\iota : X \to \iota(X)$ is an isomorphism.

Example. *If Y is a variety, $X \subseteq Y$ closed subvariety. Then the inclusion $X \hookrightarrow Y$ is a closed embedding.*

4.30 Remark.
1. A close embedding $\iota : X \to Y$ of affine varieties is finite.
2. The composition of two finite morphisms is finite.

Proof.
1. First $\iota^* : A(Y) \to A(X)$ is surjective. Thus $\iota^*(A(Y)) = A(X)$, so in particular this extension is integral.
2. Let $A \subseteq B, B \subseteq C$ integral ring extensions. Then $A \subseteq C$ is an integral ring extension (see Algebra 2).

□

4.31 Theorem. *A finite morphism is a closed map.*

Proof. Let $\varphi : X \to Y$ be a finite morphism. We can assume that φ is dominant, i.e. $\overline{(\varphi(X)} = Y$. add def
We have to show that φ is surjective. Assume $Y \subseteq \mathbb{A}^m$ is closed with coordinate functions $y_1, \dots, y_m$. Let $\eta \in Y$. The fibre of $\varphi^{-1}(\eta)$ is given by the equations

$$\forall i . \varphi^*(y_i)(\xi) = y_i(\varphi(\xi) = y_i(\eta)$$

which means

$$\varphi^*(y_1) = \eta_1, \dots, \varphi^*(y_n) = \eta_n$$

This generates an ideal

$$(\varphi^*(y_1) - \eta_1, \dots, \varphi^*(y_n) - \eta_n) \subset A(X)$$

Let $m_\eta := (y_1 - \eta_1, \dots, y_n - \eta_n)$ be the maximal ideal of η in $A(Y)$.

$$m_\eta \subset A(Y) \xRightarrow{\varphi^* \text{ inj.}} \underbrace{\varphi^*(m_\eta)}_{=:I} \subset \varphi^*(A(Y)) = A$$

By assumption $A = \varphi^*(A(Y)) \subseteq A(X) = B$ is an integral extension. By Nakayama $\varphi^*(m_\eta)A(X) = I \cdot B \subset B$, where $\varphi^*(m_\eta)$ is the ideal generated by $\varphi^*(y_i) - \eta_i$. Hence indeed

$$(\varphi^*(y_1) - \eta_1, \dots, \varphi^*(y_n) - \eta_n) \subset A(X)$$

which shows $\varphi^{-1}(y) \neq \emptyset$. □

4.32 Proposition. *If $\varphi : X \to Y$ is a surjective finite morphism, then* $\dim X = \dim Y$.

Proof. Since φ is surjective, φ^* is injective. So $\varphi^*(A(Y)) \subseteq A(X)$ is an integral ring extension and $\varphi^*(K(Y)) \subseteq K(X)$ is an algebraic extension. Hence

$$\dim Y = \operatorname{trdeg}_K K(Y) = \operatorname{trdeg}_K \varphi^*(K(Y)) = \operatorname{trdeg}_K K(X) = \dim X$$ □

But we also want to present an alternative proof.

Proof. For every chain

$$\emptyset \subset Y_0 \subset Y_1 \subset \ldots \subset Y_n = Y$$

of irreducible closed subsets there is a chain

$$\emptyset \subset X_0 \subset X_1 \subset \ldots \subset X_n = X$$

of irreducible closed subsets such that $\varphi(X_i) = Y_i$. (Proof as for **??**, using that φ is closed.)
If $Z \subseteq X$ is a closed subvariety such that $\varphi(Z) = \varphi(X)$ then $Z = X$. (Proof as for **??**.) □

4.33 Remark.

1. The notion of a finite morphism can be extended to general varieties X, Y. A morphism $\varphi : X \to Y$ is called *finite* if any $y \in Y$ has an affine open neighbourhood V such that $\varphi^{-1}(V) =: U$ is affine and $\varphi : U \to V$ is finite.

   Both definitions are consistent. "Finiteness of morphisms" is a local notion.

2. It follows easily that a finite morphism $\varphi : X \to Y$ is a closed map with finite fibres and $\dim X = \dim Y$ if φ is surjective.

3. Let $X \subset \mathbb{P}^n$ some projective variety such that $p := (0 : \ldots : 0 : 1) \notin X$ and $\mathbb{P}^{n-1} \cong H \subset \mathbb{P}^n$ with $p \notin H$. Then the projection

   $$\pi : X \to \mathbb{P}^{n-1} \qquad\qquad (a_0 : \ldots : a_n) \mapsto (a_0 : \ldots : a_{n-1})$$

   (from p we draw a line through $x \in X$ and see which point in H we hit) is finite, as follows from **??**.

Now look at any morphism of varieties $\varphi : X \to Y$, and assume that it is dominant.

1. $\dim X \geq \dim Y$
2. $\dim \varphi^{-1}(y) \geq \dim X - \dim Y$ if $\varphi^{-1}(y) \neq \emptyset$. Moreover, equality holds for almost al y.

Now we regard general morphisms (instead of just finite ones).

4.34 Proposition. *If $\varphi : X \to Y$ is a dominant map of varieties, then $\dim Y \leq dimX$.*

Proof. Wlog X, Y are affine. Then we have

$$\varphi^*(A(Y)) \subseteq A(X) \qquad\qquad \varphi^*(K(Y)) \subseteq K(X)$$

Looking at the transcendence degree we get

$$\dim Y = \operatorname{trdeg}_k K(Y) = \operatorname{trdeg}_k \varphi^*(K(Y)) \leq \operatorname{trdeg}_k K(X) = \dim X$$ □

4.35 Proposition. *Let $\varphi : X \to Y$ be a dominant morphism of varieties. For all $y \in \varphi(X)$ and all components F of the fibre $\varphi^{-1}(y)$ we have $\dim F \geq \dim X - \dim Y$. (From Proposition 4.34 we know this is non-negative.)*

Proof. Wlog Y affine (because it is a local statement), say $Y \subseteq \mathbb{A}^N$ and $\dim Y = m$. By the exercise, there are $g_1, \dots, g_m \in A(\mathbb{C}^N)$ and some open $V \subseteq Y$ such that $Z(g_1, \dots, g_m) \cap V = \{y\}$. (This means locally y is the intersection of m curves. Globally this need not be true, since those curves can have other intersection points.)
Wlog V is affine. Let $U \subseteq \varphi^{-1}(V)$ an affine open set such that $U \cap F \neq \emptyset$. The fibre $\varphi^{-1}(y) \cap U$ is given by the m equalities $\varphi^*(y_i) = \eta_i$ for $i = 1, \dots, m$ if we restrict on U.
As a corollary of the Dimension Theorem, every component of this fibre has dimension $\geq \dim U - m = \dim X - m$. In particular $\dim F = \dim(F \cap U) \geq \dim X - \dim Y$. $\square$

4.36 Example. Consider the map

$$\varphi : \mathbb{A}^2 \to \mathbb{A}^2 \qquad\qquad (x, y) \mapsto (x, xy) = (\xi, \eta)$$

If $\xi \neq 0$, we can find a preimage. However, if $\xi = 0$, then we must have $\eta = 0$ as well in the image. Thus $\varphi(\mathbb{A}^2) = (\mathbb{A}^* \times \mathbb{A}^1) \cup \{(0, 0)\}$. For the fibres we have $\varphi^{-1}(\xi, \eta) = (\xi, \xi^{-1}\eta)$ and $\varphi^{-1}(0, 0) = \{0\} \times \mathbb{A}^1$.

Take some affine variety Y and some $0 \neq s \in A(Y)$. Put

$$Y_s := \{y \in Y : s(y) \neq 0\}$$

which is an affine variety and

$$A(Y_s) = A(Y)_s = \left\{ \frac{g}{s^e} : g \in A(Y), e \in \mathbb{N} \right\}$$

4.37 Theorem (Structure Theorem). *Let $\varphi : X \to Y$ be a dominant morphism of affine varieties. Then $\dim X \geq \dim Y$ and there exists $s \in A(Y) \setminus \{0\}$ and a finite surjective morphism $\psi : X_{\varphi^*(s)} \to A_y \times \mathbb{A}^d$ where $d := \dim X - \dim Y$ such that the following diagram commutes*

$$\begin{tikzcd}
X_{\varphi^*(s)} \arrow[r, two heads, "\psi"] \arrow[dr, "\varphi"'] & Y_s \times \mathbb{A}^d \arrow[d, "\mathrm{proj}"] \\
& Y_s
\end{tikzcd}$$

Proof. Suppose $X \subseteq \mathbb{A}^N$ closed and let x_i be the coordinate functions on X.

$$\varphi * (A(Y)) \subseteq A(X) = k[x_1, \dots, x_N] \implies \varphi^*(K(Y)) \subseteq \underbrace{\varphi^*(K(Y))[x_1, \dots, x_N]}_{\text{fin.gen. } \varphi^*(K(Y))\text{-algebra}} \subseteq K(X)$$

Noether normalisation with base field $\varphi^*(K(Y))$ yields: After sufficiently general linear transformation of coordinates, we can assume that $x_1, \dots, x_d$ are algebraically independent over $\varphi^*(K(Y))$ and

$$\varphi^*(K(Y))[x_1, \dots, x_d] \subseteq \varphi^*(K(Y))[x_1, \dots, x_N] \tag{3}$$

is an integral ring extension. Moreover,

$$\underbrace{\operatorname{trdeg}_k \varphi^*(K(Y))}_{=\dim Y} + \underbrace{\operatorname{trdeg}_{\varphi^*(K(Y))} K(X)}_{\Rightarrow = d} = \operatorname{trdeg}_k K(X) = \dim X$$

$$\begin{tikzcd}
\varphi^*(A(Y)_s) \arrow[r, tail] \arrow[d, dash] & \varphi^*(A(Y)_s)[x_1, \dots, x_N] \arrow[d, dash, "(**)"] \\
\varphi^*(A(Y_s)) & A(X_{\varphi^*(s)})
\end{tikzcd}$$

Each $x_{d+1}, \dots, x_N$ is a zero of a monic polynomial with coefficients in $\varphi^*(K(Y))[x_1, \dots, x_d]$, because eq. (3) is an integral ring extension. Now we can choose $0 \neq s \in A(Y)$ ("common denominator") such that all these polynomials have their coefficients in the ring $\varphi^*(A(Y)_s)[x_1, \dots, x_d]$. We obtain an *integral* extension. Justification for (**): $\varphi^*(A(Y_s)) \subseteq A(X_{\varphi^*(s)}$, so

$$\varphi^*(A(Y_s))[x_1, \dots, x_d] \subseteq A(X_{\varphi^*(s)})$$

We have equality, since (the restrictions of) $x_1, \dots, x_N$ generate $A(X_{\varphi^*(s)})$.
We write

$$\varphi^*(A(Y_s))[x_1, \dots, x_d] = \varphi^*(A(Y_s)) \otimes_k k[x_1, \dots, x_d] \cong A(X_s) \otimes_k A(\mathbb{A}^d) = A\left(Y_s \times \mathbb{A}^d\right)$$

and obtain This corresponds to the commutative diagram of affine varieties where ψ is finite (and

$$\begin{tikzcd}
a \otimes 1 \in A(Y_s) \otimes_k k[x_1, \dots, x_d] \arrow[r, "\sim"] & \varphi^*(A(Y_s))[x_1, \dots, x_d] \arrow[r, tail, "\text{integral}"] & A\left(X_{\varphi^*(s)}\right) \\
a \in A(Y_s) \arrow[u] \arrow[urr, dashed, "\varphi^*"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
X_{\varphi^*(s)} \arrow[r, two heads, "\psi"] \arrow[dr, "\varphi \text{ surjective}"'] & \{y\} \times \mathbb{A}^d \subseteq Y_s \times \mathbb{A}^d \arrow[d, two heads, "\text{proj}"] \\
& y \in Y_s
\end{tikzcd}$$

surjective). $\square$

4.38 Corollary. *Let*$\varphi : X \to Y$ *be a dominant morphism of varieties. Then there exists a non-empty open subset* $V \subseteq Y$ *such that* $V \subseteq \varphi(X)$. *Moreover, for all* $y \in V$ *we have* $\dim \varphi^{-1}(y) = \dim X - \dim Y$.

Proof. Wlog X, Y affine and put $d := \dim X - \dim Y$. By the Structure Theorem after restricting to affine pen subsets, we can assume that section 4.5 commutes. Hence φ is surjective. Let

$$\begin{tikzcd}
X_{\varphi^*(s)} \arrow[r, two heads, "\psi"] \arrow[dr, "\varphi"'] & Y_s \times \mathbb{A}^d \arrow[d, "\text{proj}"] \\
& Y_s
\end{tikzcd}$$

$y \in Y$, then $\varphi^{-1}(y) = \psi^{-1}\left(\{y\} \times \mathbb{A}^d\right)$. Let Z be a component of $\varphi^{-1}(y)$. Then $\psi(Z)$ is closed and the restriction $Z \to \psi(Z)$ is a finite morphism (check). Hence $\dim Z = \dim \psi(Z) \leq d$, since $\psi(Z) \subseteq \{y\} \times \mathbb{A}^d$. On the other hand we already know $\dim Z \geq d$. $\square$

Consider the Boolean algebra generated by the closed subsets of a variety X. These are the sets that can be obtained form closed sets by finitely many unions, intersections and complements of X. Such sets are called *constructible.*

4.39 Theorem (Chevalley). *Let $\varphi : X \to Y$ be morphism of varieties. Then $\varphi(X)$ is constructible.*

Proof. Induction on $\dim \overline{\varphi(X)}$. For $\dim \overline{\varphi(X)}$, $\varphi(X)$ is a point and thus constructible.
Now assume $\dim \overline{\varphi(X)} > 0$. By the corollary there exists some open $V \subseteq \overline{\varphi(X)}$ with $\emptyset \neq V \subseteq$ ref $\varphi(X)$. Put $Y' := \overline{\varphi(X)} \setminus V$. If $Y' = \emptyset$, then $V = \varphi(X)$ is open and we are done (has the form $p(x) \neq 0$ for some polynomial p).
Let $Y' \neq \emptyset$. Then $\dim Y' < \dim \overline{\varphi(X)}$. Let X_i be a component of $\varphi^{-1}(Y')$. Then

$$\varphi(X) = V \cup \varphi\left(\varphi^{-1}(Y')\right) = V \bigcup_i \varphi(X_i)$$

We have $\dim \overline{\varphi(X_i)} \leq \dim Y' < \dim \overline{\varphi(X)}$. By induction hypothesis all $\varphi(X_i)$ are constructible and so $\varphi(X)$ is. $\square$

4.40 Remark (Outlook). Let $\varphi : X \to Y$ be a dominant morphism of varieties. Assume $\dim X = \dim Y$. By the corollary $\varphi^{-1}(y)$ is finite for almost all $y \in Y$. The field extension ref $\varphi^*(K(X)) \subseteq K(X)$ is a finite algebraic extension (same dimension, so same transcendence degree). Let $d := [K(X) : \varphi^*(K(Y))]$ be the degree of this extension. One can show that $\#\varphi^{-1}(y) = d$for almost all $y \in Y$ if the extension is separable.

4.41 Example. Take $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ via $x \mapsto x^2$. Then $\#\varphi^{-1}(0) = 1$ but $\#\varphi^{-1}(y) = 2$ for $y \neq 0$. But it makes sense, to introduce multiplicities. Then we will have the same value for all, compare to the Fundamental Theorem of Algebra. This will lead to the concept of schemes.

4.6 Birational equivalence

4.42 Definition. Let X, Y be varieties.

1. A *rational map* φ from X to Y written $\varphi : X \rightharpoonup Y$ is a morphism $\varphi : U \to Y$ where U is a non.empty open subset of X.
2. We say that two rational maps $\varphi : U \to Y$, $\psi : V \to Y$ are equal if $\forall p \in U \cap V. \varphi(p) = \psi(p)$.
3. We call φ *dominant* if $\overline{\varphi(U)} = Y$.

4.43 Remark.
1. A rational map $\varphi : U \to Y$ is dominant iff $\varphi(U)$ contains a non-empty open subset.
2. If $\varphi : X \rightharpoonup Y$ and $\psi : Y \rightharpoonup Z$ are rational maps and φ is dominant, then $\psi \circ \varphi$ is a well-defined rational map.
3. If both φ and ψ are dominant, then $\psi \circ \varphi$ is dominant. Hence varieties with dominant birational maps form a category.

4.44 Definition. Let X, Y be varieties.

1. A *birational map* $\varphi :\rightharpoonup Y$ is a dominant rational map that has an inverse. I.e.there is a rational map $\psi : Y \rightharpoonup X$ such that $\psi \circ \varphi = \mathrm{id}_X$ and $\varphi \circ \psi = \mathrm{id}_Y$.

2. X and Y are called *birational equivalent* if there is a birational map between them.

Exercise. *1. Varieties X and Y are birational equivalent iff there exist non-empty open subsets $U \subseteq X$ and $V \subseteq Y$ such that U is isomorphic to V.*

2. Varieties X and Y are birational equivalent iff $K(X) \cong K(Y)$ as fields.

Example. *$\mathbb{A}^n$ and $\mathbb{P}^n$ are birational equivalent but not isomorphic.*

4.45 Example (Blow-up of a point in $\mathbb{A}^2$). Consider the morphism $\mathbb{A}^2 \backslash \{0\} \to \mathbb{P}^1$ via $(x_0, x_1) \mapsto (x_0 : x_1)$ and its graph

$$\Gamma := \big\{((x_0, x_1), y_0 : y_1)) \in \big(\mathbb{A}^2 \setminus \{0\}\big) \times \mathbb{P}^1 : x_0 y_1 - x_1 y_0 = 0\big\}$$

The closure $\widetilde{X}$ of Γ in $\mathbb{A}^2 \times \mathbb{P}^1$ is given by

$$\widetilde{X} := \big\{((x_0, x_1), y_0 : y_1)) \in \mathbb{A}^2 \times \mathbb{P}^1 : x_0 y_1 - x_1 y_0 = 0\big\}$$

The projection $\pi : \widetilde{X} \to \mathbb{A}^2$ satisfies:

- π is surjective
- $\pi^{-1}(x_0, x_1) = \{((x_0, x_1), (x_0 : x_1))\}$ (single point) if $(x_0, x_1) \neq \mathbf{0}$.
- $\pi^{-1}(0, 0) = (0, 0) \times \mathbb{P}^1$. Geometrically, we can approach $\mathbf{0}$ from all directions. So the fibre is the union of all points in $\mathbb{P}^1$. This is the blow-up.

The origin has been replace by $\mathbb{P}^1$. In other words it has been "blown up" to a $\mathbb{P}^1$.
π induces an isomorphism

$$\widetilde{X} \setminus \pi^{-1}(0,0) \to \mathbb{A}^2 \setminus \{(0,0)\} \qquad\qquad ((x_0, x_1), (x_0 : x_1)) \leftarrow\!\shortmid (x_0, x_1)$$

In particular $\widetilde{X}$ is birational equivalent to $\mathbb{A}^2$. But we can show that $\widetilde{X}$ and $\mathbb{A}^2$ are not isomorphic, because $\widetilde{X}$ contains $\mathbb{P}^1$ as subvariety, and thus cannot be embedded into $\mathbb{A}^2$.

Remark. *If $\widetilde{X}$ is a self-intersection curve (has a singularity), then we can use the blow-up to make the curve smooth.*

Our goal is the following theorem.

4.46 Theorem. *Assume* $\operatorname{char} k = 0$. *Any variety Z over k is birational equivalent to an affine irreducible hypersurface in some $\mathbb{A}^m$.*

For the proof we will need the Theorem of the primitive element from Algebra II.

Proof. Wlog $Z \subseteq \mathbb{A}^n$ is a closed subvariety. Let $x_1, \dots, x_n$ be the coordinate functions on Z. Then $A(Z) = k[x_1, \dots, x_n]$ and $K(Z) = k(x_1, \dots, x_n)$. Let $d := \dim Z = \operatorname{trdeg}_k K(Z)$. After some permutation of the x_i we can assume $x_1, \dots, x_d$ are algebraically independent. Then $k(x_1, \dots, x_d) \subseteq k(x_1, \dots, x_n)$ is a finite algebraic field extension. Since $\operatorname{char} k = 0$, any field extension is separable, so a finite extension is generated by a single element. Therefore there is some $y \in K(Z)$ such that $K(Z) = k(x_1, \dots, x_d)(y)$. Let $f = T^N + q_1 T^{N-1} + \ldots + q_N$ be the minimal polynomial of y over $k(x_1, \dots, x_d)$. Each coefficient is a quotient, and give all of them the same denominator $q_i = \frac{a_i}{b}$ with $a_i, b \in k[x_1, \dots, x_d]$ and $b \neq 0$. Consider the polynomial

$$F := b \cdot f = bT^N + a_1 T^{N-1} + \ldots + a_N \in k[x_1, \dots, x_d][T]$$

We may assume $\gcd(b, a_1, \ldots, a_N) = 1$, thus F is primitive and hence irreducible (Gauss Lemma). The zero set $\mathscr{H} \subseteq \mathbb{A}^{d+1}$ of F is an irreducible hypersurface. Consider the rational map

$$\varphi : \mathbb{A}^n \supseteq Z \rightharpoonup \mathscr{H} \subseteq \mathbb{A}^{d+1} p \qquad \qquad \mapsto (x_1(p), \ldots, x_d(p), y(p))$$

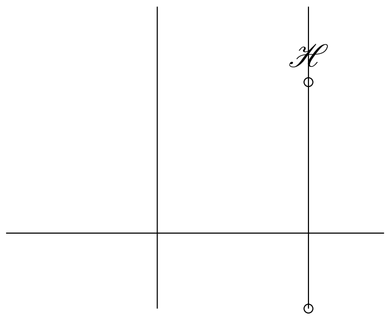
It is defined, where y is defined.

Claim. *φ is dominant.*

Proof. Suppose $G \in k[x_1, \ldots, x_d, T]$ vanishes on $\operatorname{im} \varphi$. Then $G(x_1, \ldots, x_d, y) = 0$. Now view G as an element of $k[x_1, \ldots, x_d][T]$, i.e. as a univariate polynomial. Since F is the minimal polynomial, $F \mid G$ in $k(x_1, \ldots, x_d)[T]$, but since F is primitive, we also have $F \mid G$ in $k[x_1, \ldots, x_d][T]$. Thus G vanishes on $\mathscr{H}$, since $\mathscr{H}$ is the zero set of F. $\square$

We have a well-defined comorphism $\varphi^* : K(\mathscr{H}) \to K(Z)$. φ^* is surjective, since $K(Z)$ is generated by $x_1, \ldots, x_d, y$. Hence $K(\mathscr{H}) \cong K(Z)$. Thus $\mathscr{H}$ and Z are bivariate equivalent. $\square$

Illustration:



What does this image tell us?

5 Tangent spaces and Derivatives

5.1 Tangent spaces

Let $F \in k[x_1, \ldots, x_n]$, $p \in k^n$. The first order approximation of F at p is given by the affine linear polynomial

$$F_p^{(1)} := \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(p)(x_i - p_i)$$

which is just the first Taylor-polynomial. Clearly $F_p^{(1)}(p) = 0$. Furthermore we have the rules

$$(F + G)_p^{(1)} = F_p^{(1)} + G_p^{(1)}$$
$$(F \cdot G)_p^{(1)} = F(p) \cdot G_p^{(1)} + F^{(1)} \cdot G(p)$$

5.1 Definition. Let $Z \subseteq \mathbb{A}^n$ a closed subvariety and $p \in Z$. The *embedded tangent space* of Z at p is defined as the affine linear subspace

$$T_p' Z := \left\{ \xi \in \mathbb{A}^n : \forall F \in I(Z). F_p^{(1)}(\xi) = 0 \right\}$$

5.2 Lemma. *If $I(Z) = (F_1, \ldots, F_s)$ and $p \in Z$, then*

$$T'_p Z = Z\left((F_1)_p^{(1)}, \ldots, (F_s)_p^{(1)}\right)$$

Proof.

$$\left(\sum_{i=1}^{s} G_i F_i\right)_p^{(1)} = \sum_{i=1}^{s} G_i(p)(F_i)_p^{(1)} + \sum_{i=1}^{s} \underbrace{F_i(p)}_{=0}(G_i)_p^{(1)}$$

□

5.3 Example. 1. $T'_p\mathbb{A}^n = \mathbb{A}^n$.

2. If $Z \subseteq \mathbb{A}^n$ is a hypersurface with $I(Z) = (F)$ and $p \in Z$, then

$$T'_p Z = Z\left(F_p^{(1)}\right) = \left\{\xi \in \mathbb{A}^n : \sum_i \frac{\partial F}{\partial x_i}(\xi)(\xi_i - p_i) = 0\right\}$$

3. Let $Z = Z(x^2 + y^2 - 1)$, $(\xi, \eta) \in Z$. Then $\partial_x F = 2x$ and $\partial_y F = 2y$.

$$F_{(\xi,\eta)}^{(1)}(x, y) = 2\xi(x - \xi) + 2\eta(y - \eta)$$

equation of the tangent line at (ξ, η).

5.4 Definition. 1. The *differential* of $F \in k[x_1, \ldots, x_n]$ at $p \in k^n$ is defined as the linear form

$$d_p F : k^n \to k \qquad v \mapsto \sum_i \frac{\partial F}{\partial x_i}(p) v_i$$

2. The *tangent space* $T_p Z$ of $Z \subseteq \mathbb{A}^n$ at p is defined as the sub-vectorspace

$$T_p Z := \bigcap_{F \in I(Z)} \ker(d_p F)$$

5.5 Example. 1. $T'_p Z = p + T_p Z$.

2. Let $I(Z) = F_1, \ldots, F_s)$. Then $T_p Z = \ker d_p F_1 \cap \ldots \cap \ker d_p F_s$. I.e. $T_p Z$ is the kernel of the Jacobian matrix $\left[\frac{\partial F_i}{\partial x_j}(p)\right]_{i \leq s, j \leq n}$.

5.6 Proposition. *Let $Z \subseteq A^n$ a closed subvariety, $l \in \mathbb{N}$. Then $p \in \{Z : \dim T_p Z \geq k\}$ is closed in Z.*

Proof. Let $I(Z) = F_1, \ldots, F_s)$. Take the Jacobian matrix $J(p) = \left[\frac{\partial F_i}{\partial x_j}(p)\right]_{i \leq s, j \leq n}$. Then $T_p Z = \ker(J(p))$. Hence $\dim T_p Z = n - \operatorname{rank} J(p)$. Now it is sufficient to show $\{p \in$ □ missin

Let $\varphi : X \to Y$ be a morphism of affine varieties. Take some $p \in X$ and put $q := \varphi(p) \in Y$. Then $\varphi^* : A(Y) \to A(X)$. We have $\varphi^*(m_p) \subseteq m_p$ and thus $\varphi^*(m_p^2) \subseteq m_p^2$. φ^* induces a linear map $m_q/m_q^2 \to m_p/m_p^2$. Take the duals. One calls $d_p\varphi : T_p X \to T_q Y$ the *derivative* of φ at p.

5.7 Theorem. *Let $\varphi : X \to Y$ and $\psi : Y \to Z$ be morphisms of affine varieties and let $p \in X$. Then $d_p(\psi \circ \varphi) = d_{\varphi(p)}\psi \circ d_p\varphi$ ("chain rule"). Moreover $d_p \mathrm{id}_X = \mathrm{id}_{T_p X}$. We have a (covariant) functor from the category of pointed affine varieties to the category of k-vector spaces.*

$$\begin{tikzcd} m_p \arrow[r] \arrow[d] & (T_pZ)^* \\ M_p/M_p^2 \arrow[ur, "\sim"] & \end{tikzcd}$$

$$\begin{tikzcd} (m_p/m_p^2)^* \arrow[r] \arrow[d, no head, "\cong"] & (m_q/m_q^2)^* \arrow[d, no head, "\cong"] \\ T_pX \arrow[r, dashed] & T_qX \end{tikzcd}$$

$$\begin{tikzcd} (m_r/m_r^2)^* \arrow[d, no head, "\cong"] & (m_q/m_q^2)^* \arrow[l] \arrow[d, no head, "\cong"] & (m_p/m_p^2)^* \arrow[l] \\ T_rZ & T_qY \arrow[l, "d_q\psi"'] & T_pX \arrow[l, "d_p\varphi"'] \end{tikzcd}$$

Proof. Put $q := \varphi(p)$ and $r := \psi(q)$. We have

$$A(Z) \xrightarrow{\psi^*} A(Y) \xrightarrow{\varphi^*} A(X)$$

with $\varphi^* \circ \psi^* = (\psi \circ \varphi)^*$. Thus $\square$

5.8 Remark. Let $X \subseteq \mathbb{A}^m$, $Y \subseteq \mathbb{A}^n$ closed and $\varphi : X \to Y$ given by $\varphi(x) = (\Phi_1(x), \dots, \Phi_n(x))$ where $\Phi_i \in k[x_1, \dots, x_m]$. Let $p \in X$. Then $d_p\varphi : T_pX \to T_{\varphi(p)}Y$ is given by

$$d_p\varphi \cdot v = \left(\frac{\partial \Phi_i}{\partial x_j}(p)\right)_{i \le n, j \le m} \cdot v$$

for all $v \in T_pX$.

5.2 Regular points

Let X be some variety and $p \in X$. A local ring $\mathcal{O}_{X,p} := \{\varphi \in K(X) : p \in \operatorname{dom} \varphi\}$ has a maximal ideal $m_{X,p} := \{\varphi \in \mathcal{O}_{X,p} : \varphi(p) = 0\}$. Furthermore $\mathcal{O}_{X,p}/m_{X,p} \cong k$.
Let A be some ring and $1 \in S \subseteq A$, S ??closed from localisation $S^{-1}A = \{\frac{a}{s} : a \in A, s \in S\}$. We can embed $A \to S^{-1}A$ via $a \mapsto \frac{a}{1}$.

1. If $S = \{f^n : n \in \mathbb{N}\}$ we get $S^{-1}A =: A_f$.
2. Let $\mathfrak{p} \subseteq A$ be some prime ideal. Put $S := A \setminus p$. Then $S^{-1}A =: A_p$.

Let $A = A(X)$, for some affine X. Put $\mathfrak{p} := m_p$. Then

$$A_\mathfrak{p} = \mathcal{O}_{X,p} = \left\{\frac{f}{g} : f, g \in A(X), g(p) \neq 0\right\}$$

5.9 Definition. Let X be a variety and $p \in X$. We define the *tangent space* of X at p by $T_pX := \left(m_{X,p}/m_{X,p}^2\right)^*$, where $m_{X,p}$ denotes the maximal ideal if $\mathcal{O}_{X,p}$.

5.10 Lemma. *Let $X \subseteq \mathbb{A}^n$ closed, $p \in X$. The morphism $A(X) \to \mathcal{O}_{X,p}$ induces an isomorphism $m_p/m_p^2 \xrightarrow{\sim} m_{X,p}/m_{X,p}^2$.*

Let $\varphi : X \to Y$ be a morphism of varieties, $p \in X$ and $q := \varphi(p)$. This induces a ring morphism

$$\varphi_p^* : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p} \qquad\qquad [(V,q)] \mapsto \left[(\varphi^{-1}(V), g \circ \varphi)\right]$$

such that $\varphi_p^*(m_{Y,q}) \subseteq m_{X,p}$. This induces a linear map

$$d_p\varphi : \left(m_{X,p}/m_{X,p}^2\right)^* = T_pX \to T_qY$$

We get a functor from the category of pointed varieties over k to the category of k-vector-spaces.

5.11 Remark. Let $p \in U \subseteq X$open. Then $\mathcal{O}_{U,p} = \mathcal{O}_{X,p}$ and therefore $T_pU = T_pX$.

5.12 Lemma. *Let $\mathscr{H} \subseteq \mathbb{A}^{n+1}$ be an irreducible closed hypersurface. Then $\dim T_p\mathscr{H} = \dim \mathscr{H}$ for almost all $p \in \mathscr{H}$.*

Proof. Let $\mathscr{H} = Z(F)$ for $F \in k[x_0, \ldots, x_n]$ irrducible. Then $T_p\mathscr{H} = \ker d_pF$. It is sufficient to show $d_pF \neq 0$ for almost all p. Suppose this does not hold. Since we have an open set, this already means $d_pF = 0$ for all $p \in \mathscr{H}$. Since $I(\mathscr{H}) = (F)$ we have $\forall i. F \mid \frac{\partial F}{\partial x_i}$. Due to the degree, this is only possible if $\frac{\partial F}{\partial x_i} = 0$ for all i. For $\operatorname{char} k = 0$ this is impossible. Assume $\operatorname{char} k = q > 0$. Then (by Algebra I/II) there must exist some $G \in k[x_0, \ldots, x_n]$ such that

$$F(x_0, \ldots, x_n) = G(x_0^q, \ldots, x_n^q)$$

But in the coefficients of G we may take q-th roots and use the Frobenius map. Therefore there exists some $H \in k[x_0, \ldots, x_n]$ such that $F = H^q$, contradicting the irreducibility. $\square$

5.13 Theorem. *Let X be a variety. Then*

1. $\dim X \leq \dim T_pX$ *for all* $p \in X$.
2. $\{p \in X : \dim X = \dim T_pX\}$ *is open and non-empty, and thus dense.*

Proof. Put $\delta := \min \{\dim T_pX : p \in X\}$. Then by Proposition ref

$$\{p \in X : \dim T_pX \geq \delta + 1\}$$

is closed in X. But is also is a proper subset of X by definition of δ. So we have $\dim T_pX = \delta$ for almost all $p \in X$.
For the first part is is sufficient to show $\dim X = \delta$. We know that X is birational equivalent to a hypersurface $\mathscr{H} \subseteq \mathbb{A}^{n+1}$ for some n. Therefore there exist non-empty open set $U \subseteq X$, $V \subseteq \mathscr{H}$ and some isomorphism $\varphi : U \to V$. Hence for all $p \in U$ we have

$$d_p\varphi : T_pX = T_pU \xrightarrow{\sim} T_{\varphi(p)}V = T_{\varphi(p)}\mathscr{H}$$

For their dimensions by Lemma 5.12 we have $\dim T_{\varphi(p)}\mathscr{H} = \dim \mathscr{H} \dim X$ for almost all $p \in X$. Thus $\dim T_pX = \dim X$ for almost all $p \in X$. From this we get $\dim X = \delta$. $\square$

o

5.14 Definition. Let X be a variety. A point $p \in X$ is called *regular* if $\dim T_pX = \dim X$. Otherwise p is called *singular*. Denote by $\operatorname{Reg}(X)$ the set of regular points and by $\operatorname{Sing}(X)$ the set of singular points. The variety is called *non-singular/smooth* if $\operatorname{Sing}(X) = \emptyset$.

5.15 Corollary. $\operatorname{Reg}(X)$ *is a non-empty open subset of* X. $\operatorname{Sing}(X) \subset X$ *is closed.*

Let X be some variety over k, $p \in X$ and $\emptyset \neq \mathrm{Reg}(X) \subseteq X$ open. Thus $\mathrm{Sing}(X) := X \setminus \mathrm{Reg}(X)$ is closed. By definition we had $p \in \mathrm{Reg}(X) \Leftrightarrow \dim X = \dim T_pX$.
More explicitly let $X \subseteq \mathbb{A}^n$ closed and irreducible, say $X = I(f_1, \ldots, f_s)$ for some $s \leq n$. Then

$$\forall p \in X.\, \mathrm{rank} \left[\frac{\partial f_i}{\partial x_j}(p)\right]_{i,j} = s$$

$$T_pX = \ker \left[\frac{\partial f_i}{\partial x_j}(p)\right]_{i,j}$$

$$n - s \leq \dim X \leq \dim T_pX = n - s$$

Now we return to the case $k = \mathbb{C}$ and use calculus.

5.16 Corollary. *Let $X \subseteq \mathbb{C}^N$ be a closed subvariety. Then $\mathrm{Reg}(X)$ is an analytic submanifodl of $\mathbb{C}^N$ of dimension $\dim X$. In particular $\mathrm{Reg}(X)$ is a real C^∞-submanifold of $\mathbb{C}^N \cong \mathbb{R}^{2N}$ of real dimension $2\dim X$.*

5.17 Example. Take $f := y^2 - x^3$ and $X = Z(f)$. Then we simply set the derivative $(0,0) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (-3x^2, 2y)$. This gives us $\mathrm{Sing}(X) = \{(0,0)\}$.

5.3 Non-singularity of fibres

5.18 Lemma. *Let X be a variety with $n = \dim X$. Let $u_1, \ldots, u_n$ algebraically independent over k such that $k(u_1, \ldots, u_s) \subseteq K(X)$ is a separable algebraic extension. Then $d_pu_1, \ldots, d_pu_n$ are linearly independent for almost all $p \in X$.*

Proof. Wlog X affine. Let $A(X) = k[x_1, \ldots, x_N]$. For $p \in X$ we have

$$(T_pX)^* = \mathrm{span}\{d_px_1, \ldots, d_px_N\}$$

which is at least n-dimensional since $\dim T_pX \geq \dim X = n$. Fir $1 \leq i \leq N$. Then x_i is algebraically independent on $u_1, \ldots, u_s$, since

$$\mathrm{trdeg}_k\, k(x_1, \ldots, x_n) = \dim X = n$$

There is a separable irreducible polynomial

$$F(u_1, \ldots, u_n, T) = a_dT^d + a_{d-1}T^{d-1} + \ldots + a_0 \qquad a_i \in k[u_1, \ldots, u_n], a_d \neq 0$$

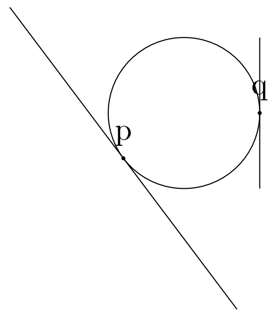
such that $F(u_1, \ldots, u_n, x_i) = 0$. Take the derivative at $p \in X$

$$0 = \sum_{j=1}^{n} \frac{\partial F}{\partial u_j}(u_1(p), \ldots, u_n(p), x_i(p))d_p(a_j \circ u) + \underbrace{\frac{\partial F}{\partial T}(u_1(p), \ldots, u_n(p), x_i(p))}_{\neq 0} \cdot d_px_i$$

We have $\frac{\partial F}{\partial T}(u, x_i) \neq 0$ since F is separable. Therefore for almost all $p \in X$ we have $\frac{\partial F}{\partial T}(u(p), x_i(p)) \neq 0$ for all i. For those p we have $\mathrm{span}\{d_px_1, \ldots, d_p, x_N\} \subseteq \mathrm{span}\{d_p(a_0 \circ u), \ldots, d_p(a_d \circ u)\}$. Moreover for all $p \in X$ since $a_i \in k[u_1, \ldots, u_n]$ we have

$$\mathrm{span}\{d_p(a_0 \circ u), \ldots, d_p(a_d \circ u)\} \subseteq \mathrm{span}\{d_pu_1, \ldots, d_pu_n\}$$

Since $\dim \mathrm{span}\{d_px_1, \ldots, d_px_N\} \geq n$, we know that $d_pu_1, \ldots, d_pu_n$ are linearly independent. $\square$



Example. *An illustration of Lemma 5.18 is given in the following image: Here* $\dim X = 1$. $u = x \in A(X)$*,* $p \in X$*. Take the projection* $d_p x : T_p X \to k$ *via* $(\xi, \eta) \mapsto \xi$*. Then* $d_p x \neq 0$ *but* $d_q x = 0$*. But this is only the case for* q *and* $-q$*.*

5.19 Proposition. *Let* $\varphi : X \to Y$ *be a dominant morphism of varieties. Assume* $\operatorname{char} k = 0$*. Then* $d_p\varphi : T_p X \to T_{\varphi(p)} Y$ *is surjective for almost all* $p \in X$*.*

Proof. Wlog X, Y are affine. Put $m := \dim Y$. Take a transcendence basis, i.e. $v_1, \ldots, v_m \in K(Y)$ which are algebraically independent. Define $u_i \in K(X)$ by $u_i := \varphi^* v_i$. Then the u_i are algebraically independent as well. So we can extend them to a transcendence basis $u_1, \ldots, u_n$ of $K(X)$. Now we apply Lemma 5.18 to $u_1, \ldots, u_n$, which tells us $d_p u_1, \ldots, d_p u_n$ are linearly independent for almost all points $p \in X$. By the chain rule $d_p u_i = d_q v_i \circ d_p \varphi$ where $q := \varphi(p)$. Hence $d_q v_1, \ldots, d_q v_m$ are linearly independent (otherwise we would some a dependency between the u_i). The dual map of $d_p\varphi$ going $\mathfrak{m}_q/\mathfrak{m}_q^2 \to \mathfrak{m}_p/\mathfrak{m}_p^2$ sends $d_q v_i \mapsto d_p u_i$. (Recall $d_q v_i = (v_i - v_i\varphi) \mod \mathfrak{m}_q^2$.) Therefore $(d_p\varphi)^*$ is injective so $d_p\varphi$ is surjective. ☐

5.20 Example. Proposition 5.19 is false in $\operatorname{char} k = p$. As counterexample take $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ given by $x \mapsto x^p$. This map is surjective but $d_p\varphi = 0$.

5.21 Theorem. *Assume* $\operatorname{char} K = 0$ *and* X *is smooth. Let* $\varphi : X \to Y$ *be a dominant morphism of varieties. Then there is a dense subset* $V \subseteq Y$ *such that for all* $y \in V$ *all components of* $\varphi^{-1}(y)$ *are smooth.*

Proof. Wlog assume Y is smooth (replace Y by $\operatorname{Reg}(Y)$, and X by $\varphi^{-1}(\operatorname{Reg}(Y))$, which are both dense subsets). Put

$$U := \{p \in X : d_p\varphi \text{ surjective}\}$$

By Proposition 5.19 U contains a non-empty open subset. check ref

Claim. *We have* $\overline{\varphi(X \setminus U)} \subset Y$*.*

Proof. Otherwise $\varphi(X \setminus U)$ would be dense in Y. Then there is a component Z of $X \setminus U$ such that $\varphi(Z)$ is dense in Y. Apply Propositionto the restriction $\varphi_1 : Z \to Y$ of φ. There is some $p \in Z$ ref such that $d_p\varphi_1 : T_p Z \to T_{\varphi(p)} Y$ is surjective. Therefore $d_p\varphi : T_p X \to T_{\varphi(p)} Y$ is surjective as well. This means $p \in U$, contradicting $p \in Z \subseteq X \setminus U$. ☐

Put $V := Y \setminus \varphi(X \setminus U)$, which is non-empty open. For $q \in V$ we have $\varphi^{-1}(q) \in U$. Let F be a component $\varphi^{-1}(q)$. Let $p \in F$. Then $T_p F \subseteq \ker d_p\varphi$, which means φ is constant on F. Put $n := \dim X$ and $m := \dim Y$. Then we get

$$n - m \leq \dim F \leq \dim T_p F \leq \dim \ker d_p\varphi = \dim T_p X - \dim Y = n - m$$

So we have equality $\dim F = \dim T_p F$, hence p is a regular point of F. ☐

In general, when cutting a variety with a hyperplane we want to keep as many properties as possible. However, the number of components may go up. As example, just intersect a circle with a line. The circle was irreducible, the intersection, 2 points, has 2 components.

Application to general hyperplane sections

Let $Z \subseteq \mathbb{A}^n$ be a closed subvariety. Put $m := \dim Z \geq 1$. For $0 \neq c \in k^n$ and $b \in k$ consider the affine hyperplane

$$H_{b,c} := \left\{ x \in k^n : \sum c_i x_i = b \right\}$$

Consider the morphism

$$\varphi : k^n \times Z \to k^n \times k \qquad\qquad (c, x) \mapsto \left(c, \sum_{i=1}^{n} c_i x_i \right)$$

φ is dominant. For $(c, b) \in k^n \times k$ we get

$$\varphi^{-1}(c, b) \cong \left\{ x \in Z : \sum c_i x_i = b \right\} = Z \cap H_{b,c}$$

Assume Z is smooth (all components). By Theorem 5.21 $Z \cap H_{b,c}$ is smooth for almost all $(b, c) \in k^n \times k$.
This is a theorem by Bertini. A similar result holds for projective subvarieties $Z \subseteq \mathbb{P}^n$.

5.4 Projective Embeddings of smooth varieties

Our goal is the following theorem:

5.22 Theorem. *Any smooth projective variety X is dimension n is isomorphic to a closed subvariety of $\mathbb{P}^{2n+1}$.*

5.23 Corollary. *Any smooth projective curve is isomorphic to a curve in $\mathbb{P}^3$.*

5.24 Remark.
- The bound $2n+1$ for the dimension is optimal.
- However, if we drop the condition of smoothness, the statement becomes false. In this case there is no bound.
- Whitney: Let M be an abstract C^∞-manifold, $n := \dim M$. Then we can embed $M \cong S \subseteq \mathbb{R}^N$, where $N = 2n$ suffices.

For the proof, we quote the following theorem, inspired from calculus (see “Inverse Function Theorem”).

5.25 Theorem. *Let $\varphi : X \to Y$ be a finite morphism of varieties, which is bijective. Assume that the derivatives $d_x\varphi : T_xX \to T_{\varphi(x)}Y$ are bijective for all $x \in X$. Then φ is an isomorphism.*

Proof (idea) of Theorem 5.22. Let $X \subset \mathbb{P}^N$ be a smooth projective subvariety, $n := \dim X$. Suppose $N \geq 2n+1$. We show that for a generic point $\xi \in \mathbb{P}^N \setminus X$ the projection $\varphi : X \to P^{N-1}$ with centre ξ defines an isomorphism onto its image.
We know

- φ is a finite morphism for any $\xi \in \mathbb{P}^N \setminus X$.
- $\varphi(X) =: Y$ is closed in $\mathbb{P}^{N-1}$.

For $\varphi : X \to Y$ beign an isomorphism we need it to be *injective*. We also need that the derivatives $d_x\varphi : T_xX \to T_{\varphi(x)}Y$ is injective for all $x \in X$. $\square$

of Theorem 5.22. Let $X \subseteq \mathbb{P}^N$ be a smooth projective subvariety, $n := \dim X$. For $p \in X$ let $\mathbb{T}_pX$ denote the tangent space of X at p, seen as projective subspace passing through p. Suppose $\xi \in \mathbb{P}^N \setminus X$ is chosen such that:

1. Any line though ξ intersects X in at most one point.
2. $\xi \notin \bigcup_{p\in X} \mathbb{T}_pX$

Then the projection $\varphi : X \to \mathbb{P}^{N-1}$ with centre ξ satisfies the hypothesis the Theorem 5.22.

1. φ is injective.
2. all $d_p\varphi$ are injective.

Therefore by Theorem 5.25 $\varphi : X \to \varphi(X)$ is an isomorphism. Iterate the projection until $N = 2n + 1$.
Now we have to show the existence of such a ξ. Let $U_1 \subseteq \mathbb{P}^N$ denote the set of ξ that violate item 1, and likewise define U_2, violating condition item 2. Show that $\dim \overline{U_1} \leq 2n + 1$ and $\dim \overline{U_2} \leq 2n$. Then $\mathbb{P}^N \setminus (U_1 \cup U_2) \neq \emptyset$, since $N > 2n + 1$, and get the existence of some ξ as above.
Consider the set (which is not a variety)

$$\Gamma := \left\{(a, b, c) : \mathbb{P}^N \times X \times X : a, b, c \text{ collinear}, b \neq c\right\}$$

with the projections $\pi_1 : \Gamma \to \mathbb{P}^N$ and $\pi_2 : \gamma \to X \times X$ via $\pi_2 : (a, b, c) \mapsto (b, c)$. We have $U_1 = \pi_1(\Gamma)$, so $\dim \overline{U_1} \leq \dim \overline{\Gamma}$. Moreover for $(b, c) \in X \times X$ with $b \neq c$ we have $\pi_2^{-1}(b, c) \cong \mathbb{P}^1$. Apply Theorem on dimension of fibres to π_2. We get $\dim \overline{\gamma} \leq \dim(X \times X) + 1$. (Apply Theorem to irreducible component of $\overline{\Gamma}$.) Thus $\dim U_1 \leq 2n + 1$. ref

Where does next paragraph belong to? ref

Let $(b, c) \in X \times X$ with $b \neq c$. The projection $\pi_2 : \overline{\Gamma} \to X \times X$ is dominant, but we want to show $\overline{\Gamma}$ is irreducible.

$$1 = \dim \pi_2^{-1}(b, c) \geq \dim F - \dim(X \times X) = \dim F - 2n \implies \dim F \leq 2n + 1$$

$\varphi : X \to Y$ surjective, Y irreducible. For all $y \in Y$ the fibres $\varphi^{-1}(y)$ are irreducible of the same dimension. Hence X is irreducible.
For U_2 consider the set

$$R := \left\{(a, b) \in \mathbb{P}^N \times X : a \in \mathbb{T}_bX\right\}$$

with the projections p_1, p_2. Then $\overline{U_2} = p_1(\overline{R})$ and for all $b \in X$ we have $p_2^{-1}(b) \cong \mathbb{T}_bX$. Therefore we get $\dim \overline{R} \leq n + n$. Thus $\dim \overline{U_2} \leq \dim \overline{R} \leq 2n$. $\square$

5.26 Example. Take $X = \mathbb{A}^1$ and $Y = Z(y^2 - x^3)$. Define $\varphi : X \to Y$ via $t \mapsto (t^2, t^3)$. Then φ is bijective, finite morphism. But φ is not an isomorphism. The derivatives $d_t\varphi : T_tX \to T_{\varphi(t)}Y$ are injective for $t \neq 0$, but $d_0\varphi = 0$. Moreover Y is not smooth, since $\mathrm{Sing}(Y) = \{(0, 0)\}$.

6 Schemes

Let k algebraically closed. By the Nullstellensatz we have a bijection between $Z \subseteq \mathbb{A}_k^n$ and radical ideals $I \subseteq k[X_1, \dots, X_n]$.
Regard the one-dimensional case in $\mathbb{A}^1$ where $Z = \{0\}$. This corresponds to $I = (X) \subseteq k[X]$. Note that here we have $k[X]/(X) \cong k$. But what if we take $I = (X^2)$. Then if we have some polynomial $\overline{f} \in k[X]/(X^2)$. This corresponds to Taylor expansion up to degree 1. We can go up to arbitrary $I = (X^{m+1})$, so Taylor expansion up to degree m.
Also we want to get rid of the condition that k is algebraically closed. In fact, we won't even need a k-algebra, but a commutative ring with 1 suffices.

6.1 Affine Schemes

Let R be a commutative ring.

6.1 Definition. We put $\operatorname{Spec}(R)$ as the set of prime ideals of R, and call it the *Spectrum* of R, or the *affine scheme* of R. If $\mathfrak{p} \in \operatorname{Spec}(R)$, we have the canonical homomorphism

$$R \to R/\mathfrak{p} \to K(\mathfrak{p}) \qquad\qquad f \mapsto f \mod \mathfrak{p}$$

where $K(\mathfrak{p})$ is the field of fractions of $R/\mathfrak{p}$.
Any $f \in R$ defines a function

$$\operatorname{Spec}(R) \to \coprod_{\mathfrak{p} \in \operatorname{Spec}(R)} K(\mathfrak{p})$$

which we denote $f(\mathfrak{p}) := f \mod \mathfrak{p}$.

6.2 Example.

1. If k is a field, then $\operatorname{Spec}(k) = \{0\}$.
2. $\operatorname{Spec} \mathbb{C}[x] = \{(x-a) : a \in \mathbb{C}\} \cong \mathbb{C}$.
3. $\operatorname{Spec} \mathbb{Z} = \{0\} \cup \{(p) : p \in \mathbb{P}\}$.

6.3 Lemma. *Let $I \subseteq R$ some ideal. Then*

$$\sqrt{I} = \bigcap_{\mathfrak{p} \in \operatorname{Spec} R, I \subseteq \mathfrak{p}} \mathfrak{p}$$

6.4 Definition. For $S \subseteq R$ define

$$Z(S) := \{\mathfrak{p} \in \operatorname{Spec} R : \forall f \in S. f \in \mathfrak{p}\} = \{\mathfrak{p} \in \operatorname{Spec} R : S \subseteq \mathfrak{p}\}$$

6.5 Remark. For a family (I_i) of ideals in R we have

$$\bigcap_i Z(I_i) = Z\left(\sum_i I_i\right)$$
$$U(I_1) \cup Z(I_2) = Z(I_1 \cdot I_2)$$

We define a topology on $\operatorname{Spec} R$ by declaring the $Z(S)$ for subsets $S \subseteq R$ to be the closed sets. Define for $M \subseteq \operatorname{Spec} R$ the vanishing ideal

$$I(M) = \{f \in R : \forall \mathfrak{m} \in M. f(\mathfrak{m}) = 0\} = \bigcap_{\mathfrak{m} \in M} \mathfrak{m}$$

For example if $M = \{\mathfrak{p}\}$, we have $I(M) = \mathfrak{p}$.

6.6 Lemma. *Let $M \subseteq \operatorname{Spec} R$ and $J \subseteq R$ some ideal. Then*

$$I(Z(J)) = \sqrt{J} \tag{4}$$
$$Z(I(M)) = \overline{M} \tag{5}$$

Proof. For the first part we have

$$I(Z(J)) = \bigcap_{\mathfrak{p} \in Z(J)} \mathfrak{p} = \sqrt{J}$$

and the second statement follows by definition. □

6.7 Corollary. *Let $\mathfrak{p} \in \operatorname{Spec} R$. Then $\overline{\{\mathfrak{p}\}} = Z(\mathfrak{p})$, because*

$$M = \{\mathfrak{p}\}, I(M) = \mathfrak{p} \implies \overline{M} = Z(I(M)) = Z(\mathfrak{p})$$

Hence a point $\mathfrak{p}$ is closed iff $\mathfrak{p}$ is a maximal ideal.

6.8 Example. Regard some affine variety X with ring $R = A(X)$. Then we have a bijection

$$\begin{aligned} \{Y : Y \subseteq X \text{ closed}\} &\xrightarrow{\sim} \operatorname{Spec} A(X) \qquad \text{topological space} \\ Y &\mapsto I(Y) \\ Z(\mathfrak{p}) &\leftarrow\!\shortmid \mathfrak{p} \end{aligned}$$

Let $\varphi : R \to S$ be a morphism of commutative rings. This yields a morphism $\varphi^* : \operatorname{Spec} S \to \operatorname{Spec} R$ via $\varphi^*(q) = \varphi^{-1}(q)$.

Claim. *φ^* is continuous.*

Proof. Let $I \subseteq R$ be some ideal. Then

$$(\varphi^*)^{-1}(Z(I)) = \{q \in \operatorname{Spec} S : \varphi^*(q) \in Z(I)\} = \{q : I \subseteq \varphi^*(q)\} = \{q : \varphi(I) \subseteq q\} = Z(\varphi(I))$$ □

6.9 Definition. Let $f \in R$ and regard $X = \operatorname{Spec} R$. Then

$$X_f := \{\mathfrak{p} \in \operatorname{Spec} R : f(p) \neq 0\} = X \setminus Z(f)$$

is called *distinguished open subset*. Then X_f for $f \in R$ form a basis of the topology of $\operatorname{Spec} R$.

Recall the localisation

$$R_f = \left\{ \frac{a}{f^n} : a \in R, n \in \mathbb{N} \right\}$$

If $\mathfrak{p} \in \operatorname{Spec} R$ we have the localisation $R_\mathfrak{p} = \left\{ \frac{a}{b} : a, b \in R, b \notin \mathfrak{p} \right\}$, which is a local ring.

6.10 Definition. Regard $X = \operatorname{Spec} R$. For any open subset $U \subseteq X$ we define

$$\mathcal{O}(U) := \left\{ (\varphi_\mathfrak{p})_{\mathfrak{p} \in U} : \varphi_\mathfrak{p} \in R_\mathfrak{p}, \forall \mathfrak{p} \in U. \exists V \subseteq U \text{ open}. p \in V \wedge \exists f, g \in R. \forall q \in V. g \notin q \wedge \varphi_q = \frac{f}{g} \in R_q \right\}$$

Things to verify:

- $\mathcal{O}(U)$ is a ring.

- $(\mathcal{O}(U))_U$ is a presheaf.
- It is a sheaf.

6.11 Lemma. *For any $\mathfrak{p} \in X$ the stalk $\mathcal{O}_{X,\mathfrak{p}}$ is isomorphic to $R_\mathfrak{p}$.*

Proof. Regard the map $\mathcal{O}_{X,\mathfrak{p}} \to R_\mathfrak{p}$ given by $(U, \varphi) \mapsto \varphi_\mathfrak{p}$. This is well-defined, since for a presheaf, we say $\varphi \sim \psi$ iff they agree on an open neighbourhood. For surjectivity take $\frac{f}{g} \in R_\mathfrak{p}$, where $g \notin \mathfrak{p}$. Put $U := X_g$, define $\varphi = (\varphi_1)_{q \in U} \in \mathcal{O}(U)$ by $\varphi_q := \frac{f}{g} \in R_q$. For injectivity assume $\varphi_\mathfrak{p} = 0 \in R_\mathfrak{p}$. Then we have $\varphi_q = \frac{f}{g}$ for all $q \in V$,where V is some open neighbourhood of p. Since $\varphi_\mathfrak{p} = 0$, there exists some $s \in R \setminus \mathfrak{p}$ such that $sf = 0 \in R$. Consider $q \in V \cap X_s$. There we have $s(q)f(q) = 0$, so $f(q) = 0$. Hence $\varphi = 0$ in $V \cap X_s$. □

6.12 Proposition. *Let $X = \operatorname{Spec} R$. For any $f \in R$we have $\mathcal{O}(X_f) \cong R_f$. In particular $\mathcal{O}(X) \cong R$ (in the global setting).*

6.2 Morphism and Locally Ringed Spaces

6.13 Definition. A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that $\mathcal{O}_{X,P}$ is a local ring for all $P \in X$.

Notation. *Let $\mathfrak{m}_{X,P}$ denote the maximal ideal of $\mathcal{O}_{X,P}$ and $K(P)$ the residue field $\mathcal{O}_{X,P}/\mathfrak{m}_{X,P}$.*

A morphism of locally ringed spaces from $(X; \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is given by

- $f : X \to Y$ is continuous
- for all open $U \subseteq Y$ the function $f^*_U : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ is a ring homomorphism such that the following hold:

  1. for all open $V \subseteq U$ the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}_Y(U) \arrow[r, "f^*_U"] \arrow[d, "\rho_{U,V}"] & \mathcal{O}_X(f^{-1}(U)) \arrow[d, "\rho_{f^{-1}(U),f^{-1}(V)}"] \\
\mathcal{O}_Y(V) \arrow[r, "f^*_V"] & \mathcal{O}_X(f^{-1}(V))
\end{tikzcd}$$

  2. For all $P \in X$ the induced map on stalks given by

$$\begin{aligned} f^*_P : \mathcal{O}_{Y,f(P)} &\to \mathcal{O}_{X,P} \\ [(U, \varphi)]_{f(P)} &\mapsto \left[\left(f^{-1}(U), f^*_U(\varphi)\right)\right]_P \end{aligned}$$

  satisfies $(f^*_P)^{-1} (\mathfrak{m}_{X,P} = \mathfrak{m}_{Y,f(P)}$

6.14 Proposition. *Let R, S be rings. Put $X := \operatorname{Spec} R$ and $Y := \operatorname{Spec} S$. Then we have a one-to-one correspondence between morphisms $X \to Y$ and ring homomorphisms $R \to S$.*

Proof. If $f : X \to Y$, we get a ring homomorphism $f^*_Y : S = \mathcal{O}_Y(Y) \to \mathcal{O}_X(X) = R$.

If $\psi : S \to R$ is a ring homomorphism, then define $f : X \to Y$ via $\mathfrak{p} \mapsto \psi^{-1}(\mathfrak{p})$. Since for every $I \subseteq S$ we have $f^{-1}(Z(I)) = Z(\psi(I))$, f is continuous. For each $\mathfrak{p} \in X$ we localise ψ to get

$$\psi_{\mathfrak{p}} : \mathcal{O}_{Y,f(\mathfrak{p})} = S_{\psi^{-1}(\mathfrak{p})} \to R_{\mathfrak{p}} = \mathcal{O}_{X,\mathfrak{p}}$$
$$(a, b \in S, b \notin \psi^{-1}(\mathfrak{p})) \qquad \frac{a}{b} \mapsto \frac{\psi(a)}{\psi(b)} \qquad (\psi(b) \notin \mathfrak{p}$$

Therefore $\psi_{\mathfrak{p}}^{-1}(\mathfrak{m}_{X,\mathfrak{p}}) = \mathfrak{m}_{Y,f(\mathfrak{p})}$, since

$$\frac{a}{b} \in \psi_{\mathfrak{p}}^{-1}(\mathfrak{m}_{X,\mathfrak{p}}) \Leftrightarrow \psi_{\mathfrak{p}}(a) \in \mathfrak{p} \Leftrightarrow a \in \psi_{\mathfrak{p}}^{-1}(\mathfrak{p}) = f(\mathfrak{p}) \Leftrightarrow \frac{a}{b} \in \mathfrak{m}_{Y,f(\mathfrak{p})}$$

The maps on the stalks give ring homomorphisms $f_U^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ by definition of structure sheaf such that $f_{\mathfrak{p}}^* = \psi_{\mathfrak{p}}$.
We still need to check "one-to-one". For a morphism $f : X \to Y$ and $f_Y^* : S \to R$ we get some $g : X \to Y$ as above. Since $(f_{\mathfrak{p}}^*)^{-1}(\mathfrak{m}_{X,\mathfrak{p}}) = \mathfrak{m}_{Y,f(\mathfrak{p})}$ we have $g(\mathfrak{p}) = (f_Y^*)^{-1}(\mathfrak{p}) = f(\mathfrak{p})$. $\square$

6.15 Definition. Let $X = \operatorname{Spec} R$ and $I \subseteq R$ some ideal. Define $Y := \operatorname{Spec}(R/I)$. Take the canonical projection $\psi : R \to R/I$. Thus by Proposition 6.14, taking $S = R/I$, we have a morphism $f : X \to Y$. Actually we have $f : Y \hookrightarrow X$ an $\operatorname{im}(f) = Z(I)$. Y is called an *affine closed subscheme* of X.

6.16 Definition. Let $Y_i := \operatorname{Spec}(R/I_i)$ for $i = 1, 2$ be closed subschemes of $X = \operatorname{Spec} R$. Then we redefine

$$Y_1 \cup Y_2 := \operatorname{Spec}(R/(I_1 \cap I_2)) \qquad\qquad Y_1 \cap Y_2 := \operatorname{Spec}(R/(I_1 + I_2))$$

6.17 Example. Let $X = \operatorname{Spec} \mathbb{C}[x_1, x_2]$ and take

$$Y_1 = \operatorname{Spec}(\mathbb{C}[x_1, x_2]/\langle x_2 \rangle) \qquad\qquad Y_2 = \operatorname{Spec}(\mathbb{C}[x_1, x_2]/\langle x_2 - x_1^2 + a^2 \rangle)$$

for some $a \in \mathbb{C}$. Then

$$Y_1 \cap Y_2 \cong \operatorname{Spec} \mathbb{C}[x_1]/\langle (x_1 - a)(x_1 + a) \rangle$$

For $a \neq 0$ we have $Y_1 \cap Y_2 \cong \mathbb{C} \times \mathbb{C}$, since the map $\mathbb{C}[x_1] \twoheadrightarrow \mathbb{C} \times \mathbb{C}$ given by $f \mapsto (f(a), f(-a))$ has kernel $\langle (x_1 - a)(x_1 + a) \rangle$. So $Y_1 \cap Y_2$ is the disjoint union of two points $(a, 0)$ and $(-a, 0)$ in $\mathbb{C}^2$. However, for $a = 0$, $Y_1 \cap Y_2 \cong \mathbb{C}[x_1]/\langle x_1^2 \rangle$ has only 1 point $(0, 0)$ in $\mathbb{C}^2$, but with multiplicity 2 We say that $Y_1 \cap Y_2$ is a scheme of length 2.
Note: There is a unique line in $\mathbb{C}^2$, which passes through the scheme, even for $a = 0$. $Y_1 \cap Y_2 = \operatorname{Spec} \mathbb{C}[x_1, x_2]/\langle x_2, x_1^2 \rangle$ is a closed subscheme of the line $L = \operatorname{Spec} \mathbb{C}[x_1, x_2]/\langle c_1 x_1 + c_2 x_2 \rangle$ iff $\langle c_1 x_1 + c_2 x_2 \rangle \subseteq \langle x_2, x_1^2 \rangle$ iff $c_1 = 0$. So Y_1 is the unique scheme, which contains this line. Hence the tangent is unique.

6.3 Schemes and Prevarieties

6.18 Definition. A *scheme* is a locally ringed space $(X, \mathcal{O}_X)$, that can be covered by open subsets $U_i \subseteq X$ such that $(U_i, \mathcal{O}_{X|U_i})$ is isomorphic to an affine scheme $\operatorname{Spec} R_i$ for all i. A morphism of schemes is a morphism of locally ringed spaces.

Remark. *We allow schemes without "closed diagonal". Schemes with "closed diagonal" are called* separated.

If you have a prevariety then it already is a scheme. Now the question is, which additional properties do we need, such that a scheme is a prevariety.

6.19 Definition. Let Y be a scheme. A *scheme over* Y is a scheme X together with a morphism $X \to Y$. A morphisms of schemes X_1, X_2 over Y is a morphism $X_1 \to X_2$ such that the following commutes If R is a ring, a*scheme over* R is a scheme over $\operatorname{Spec} R$.

$$\begin{tikzcd} X_1 \arrow[rr] \arrow[dr] & & X_2 \arrow[dl] \\ & Y & \end{tikzcd}$$

6.20 Example.
- $X = \operatorname{Spec} R$ is a scheme over k iff there is a morphism $k \to R$ iff R is a k-algebra.
- A morphism $\operatorname{Spec} R \to \operatorname{Spec} S$ is a morphism of scheme over k iff the corresponding $S \to R$ is a morphism of k-algebras.
- Regular functions are no longer determined by their values on points, see Example 6.17. $R := \mathbb{C}[x]/\langle x^2 \rangle$, $X = \operatorname{Spec} R$ has 1 point $\langle x \rangle$. The 2 functions $x \in R = \mathcal{O}_X(X)$ and $0 \in R$ have both the value $0 = x \in k[x]/\langle x \rangle \cong k$, but they are not the same as regular functions.

6.21 Definition. A scheme X over Y with a morphism $f : X \to Y$ is of *finite type* over Y if there is a covering of Y by open affine $V_i = \operatorname{Spec} B_i \subseteq V$ such that $f^{-1}(V_i)$ can be covered by finitely many open affines $U_{i,j} = \operatorname{Spec} A_{i,j}$ where $A_{i,j}$ is a finitely generated B_i-algebra.

In particular, a scheme X over k is of finite type over k if it can be covered by finitely many open affine $U_i i = \operatorname{Spec} A_i$, where A_i is a finitely generated k-algebra.

6.22 Example. $\operatorname{Spec} R$ is of finite type over k iff R is a finitely generated k-algebra.

6.23 Definition. Ascheme X is *reduced* if the rings $\mathcal{O}_X(U)$ have no nilpotent element for all open $U \subseteq X$.

6.24 Example. Every $X = \operatorname{Spec} R$ is reduced and irreducible iff R is a domain.

Proof. **$\Leftarrow$:** If R is a domain, then X is reduced. Assume

$$X = Z(I) \cup Z(J) = Z(I \cdot J)$$

and $Z(I), Z(J) \subset X$. Then there exist some $f \in I \setminus J$ and $g \in J \setminus I$. Since R is a domain, we have $\langle 0 \rangle \in X = Z(I \cdot J)$. Hence $\langle 0 \rangle \in I \cdot J \ni fg$,so $fg = 0$. Since R is a domain, we have $f = 0$ or $g = 0$. $\lightning$

$\Rightarrow$: We need to show $\forall f, g \in R. fg = 0 \to f = 0 \lor g = 0$. Assume $fg = 0$ with $g \neq 0 \neq f$. If f and g have a common "root" $f = h^n, g = h^m$, then R is not nilpotent-free. Hence $\operatorname{Spec} R = Z(f) \cup Z(g)$.

$\square$

6.25 Proposition. *Let k be an algebraically closed field. There is a one-to-one correspondence between prevarieties over k (and their morphisms) and reduced, irreducible schemes of finite type over k (and their morphisms).*

6.26 Remark. 1. As before, we can glue together schemes.

2. A morphism from a glued scheme $X = \bigcup_i X_i$ to a scheme Y can be given by morphisms $X_i \to Y$ that agree on the overlaps.

6.4 Projective Schemes

6.27 Definition. Let R be a graded ring, i.e. a ring with a decomposition $R = \bigoplus_{d \in \mathbb{N}} R^{(d)}$ into abelian subgroups such that $R^{(d)} \cdot R^{(e)} \subseteq R^{(d+e)}$. Element of $R^{(d)}$ are called homogeneous of degree d. An ideal $I \subseteq R$ is called homogeneous if it can be generated by homogeneous elements. The *irrelevant* ideal is $R_+ = \bigoplus_{d>0} R^{(d)}$.

$$\operatorname{Proj} R = \{\mathfrak{p} \subseteq R : \mathfrak{p} \text{ hom. prime ideal}, R_+ \not\subseteq \mathfrak{p}\}$$

If $I \subseteq R$ is a homogeneous ideal, we have the zero locus $Z(I) := \{\mathfrak{p} \in \operatorname{Proj} R : I \subseteq \mathfrak{p}\}$.

6.28 Lemma. *Let R be a graded ring.*

1. *If (I_i) is a family of homogeneous ideals of R, then*

$$\bigcap_i Z(I_i) = Z\left(\sum I_i\right) \subseteq \operatorname{Proj} R$$

2. *If $I_1, I_2 \subseteq R$ are homogeneous ideals, then $Z(I_1) \cup Z(I_2) = Z(I_1 I_2)$.*

Hence we have a topology on $\operatorname{Proj} R$ *where the closed sets are those of the form $Z(I)$.*

6.29 Definition. Let R be a graded ring and $\mathfrak{p} \in \operatorname{Proj} R$. Then we put

$$R_{(\mathfrak{p})} := \left\{\frac{f}{g} : \exists d. f, g \in R^{(d)}, f \notin \mathfrak{p}\right\}$$

For $U \subseteq \operatorname{Proj} R$ open, we put

$$\mathcal{O}_{\operatorname{Proj} R}(U) := \left\{(\varphi_\mathfrak{p})_{\mathfrak{p} \in U} : \forall \mathfrak{p} \in U. \varphi_\mathfrak{p} \in R_{(\mathfrak{p})}, \exists V \subseteq U \text{ open}. \mathfrak{p} \in V, \exists d. \exists f, g \in R^{(d)}. \forall \mathfrak{q} \in V. g \notin \mathfrak{q} \to \varphi_\mathfrak{q} = \frac{f}{g}\right\}$$

6.30 Proposition. *Let R be a graded ring, $X = \operatorname{Proj} R$.*

1. *$\mathfrak{O}_X$ is a sheaf and $\forall \mathfrak{p} \in X. \mathcal{O}_{X,\mathfrak{p}} \cong R_{(\mathfrak{p})}$.*

2. *For every homogeneous element $f \in R_+$, the* distinguished open subset *is*

$$X_f := X \setminus Z(f) = \{\mathfrak{p} \in X. f \notin \mathfrak{p}\}$$

These open sets cover X. Furthermore, putting

$$R_{(f)} := \left\{\frac{g}{f^r} : g \in R^{(r \cdot \deg f)}\right\}$$

we have $\left(X_f, (\mathcal{O}_X)_{|X_f}\right) \cong \operatorname{Spec} R_{(f)}$ for all $f \in R_+$.

6.31 Example. Let k algebraically closed. Then $\operatorname{Proj} k[x_0, \dots, x_n]$ "is" $\mathbb{P}^n_k$. Also a variety $X \subseteq \mathbb{P}^n_k$ "is" $\operatorname{Proj} k[x_0, \dots, x_n]/I(X)$.

6.32 Definition. Let k algebraically closed. A *projective subscheme* of $\mathbb{P}_k^n$ is a scheme of the form $\operatorname{Proj} k[x_0, \dots, x_n]/I$ for some ideal I.

6.33 Remark. Projective subschemes do not have to be irreducible (e.g. $\operatorname{Proj} k[x_0, x_1, x_2]/\langle x_1 x_2 \rangle$) not reduced (e.g. $\operatorname{Poj} k[x_0, x_1]/\langle x_1^2 \rangle$).

6.34 Definition. Let $I \subseteq S := k[x_0, \dots, x_n]$ be some homogeneous ideal. The *saturation* $\overline{I}$ of I is

$$\overline{I} := \{s \in S : \forall i \leq n. \exists m \in \mathbb{N}. x_i^m s \in I\}$$

6.35 Example. If f is irreducible and $I = \langle f x_0, \dots, f x_n \rangle$, then $\overline{I} = \langle f \rangle$.

6.36 Lemma. *Let $I, J \subseteq S := k[x_0, \dots, x_n]$ homogeneous ideals.*

1. *$\overline{I}$ is homogeneous. Setting some variable to 1 makes I and $\overline{I}$ equal, i.e. $I[x_i \mapsto 1] = \overline{I}[x_i \mapsto 1]$.*
2. $\operatorname{Proj} R/I = \operatorname{Proj} R/\overline{I}$.
3. $\operatorname{Proj} R/\overline{I} = \operatorname{Proj} /\overline{J}$ *iff* $\overline{I} = \overline{J}$.
4. $I^{(d)} = \overline{I}^{(d)}$ *for d large enough.*

6.37 Definition. If X is a projective subscheme of $\mathbb{P}^n$, then $I(X)$ is the saturation of any ideal $J \subseteq S := k[x_0, \dots, x_n]$ such that $X = \operatorname{Proj} S/J$. $I(X)$ is called the *ideal of* X.

6.38 Corollary. *We have a one-to-one correspondence between projective subschemes of $\mathbb{P}_k^n$ and saturated homogeneous ideals in $k[x_0, \dots, x_n]$.*

$$\begin{aligned} X &\mapsto I(X) \\ \operatorname{Proj} k[x_0, \dots, x_n]/I &\mapsfrom I \end{aligned}$$

7 First Applications of Scheme Theory

7.1 Hilbert Polynomial

Let $I \subseteq k[x_0, \dots, x_n]$ a homogeneous ideal.

$$X := \operatorname{Proj}(\underbrace{k[x_0, \dots, x_n]/I}_{=:R}) = \{\mathfrak{p} : \mathfrak{p} \text{ homogeneous prime ideal}, p \neq (x_0, \dots, x_n)\}$$

This we enhance wit the Zariski topology. For $U \subseteq X$ open we have $\left(X_f, (\mathcal{O}_X)_{|X_f}\right) \cong \operatorname{Spec}\left(R_{(f)}\right)$. X is called the projective subscheme of $\mathbb{P}_k^n$. $I(X)$ is the saturation of I. Put $S(X) := k[x]/I(X)$.

7.1 Definition. Let X be a projective subscheme of $\mathbb{P}_k^n$. The *Hilbert function* of X is

$$h_X : \mathbb{N} \to \mathbb{N} \qquad\qquad h_X(d) = \dim_k S(X)^{(d)}$$

where the latter denotes the degree d part of $S(X)$.

7.2 Example. 1. Let $X = \mathbb{P}_k^n$, $I(X) = 0$ and $S(X) = k[x_0, \dots, x_n]$. Then

$$h_X(d) = \dim k[x_0, \dots, x_n]^{(d)} = \binom{n+d}{n} \in \frac{1}{n!} d^n + \mathcal{O}\left(n^{d-1}\right)$$

For the latter growth, note that n is fixed for the function.

2. Let $X = \{(0:1),(1:0)\} \subseteq \mathbb{P}^1_k$. Then $I(X) = (x_0x_1)$. We have $S(X)^{(0)} \cong k$ and for any $d > 0$ only x_0^d, x_1^d remain as basis. Thus

$$h_X(d) = \begin{cases} 1 & : d = 0 \\ 2 & : d > 0 \end{cases}$$

3. Let $X = \{(1:0),(0:1),(1:1)\} \subseteq P^1_k$. Now $I(X) = (x_1x_1(x_0 - x_1))$. Let $d > 1$. Then $S(X)$ has basis $x_0^d, x_0x_1^{d-1}, x_1^d$, because of our relation $x_0^2x_1 = x_0x_1^2$. Hence

$$h_X(d) = \begin{cases} 1 & : d = 0 \\ 2 & : d = 1 \\ 3 & : d > 1 \end{cases}$$

4. Let $X \subseteq \mathbb{P}^1_k$ be the "double point" given by $I(X) = (x_0^2)$. If $d > 0$, $S(X)^{(d)}$ has basis $x_1^d, x_0x_1^{d-1}$. Thus

$$h_X(d) = \begin{cases} 1 & : d = 0 \\ 2 & : d > 0 \end{cases}$$

So we have the same Hilbert function as in the case of 2 points.

7.3 Lemma. *Let X be a zero-dimensional projective subscheme of $\mathbb{P}^n_k$. Then*

1. *X is affine, i.e. $X \cong \operatorname{Spec}(R)$ for some k-algebra R. (And R is unique up to isomorphism.)*
2. *This R is a finite-dimensional k-vectorspace. Its dimension is called the* length *of X. The interpretation is the number of points of X, counted with multiplicity.*
3. *For all sufficiently large d we have $h_X(d) = \dim_k R$.*

Proof. 1. There is a hyperplane H hat does not intersect X. Therefore $X \subseteq \mathbb{P}^n \setminus H$, the right part is affine, so X is affine.

2. We decompose into irreducible components $X = X_1 \cup \ldots \cup X_m$. Say $X_i = \operatorname{Spec} R_i$. We can show that $X = \operatorname{Spec}(R_1 \times \ldots \times R_m)$. (Is related to Chinese Remainder Theorem.) We can assume wlog that X is irreducible. This basically means we have $\mathbb{A}^n$ and the origin. So we can assume $X = \operatorname{Spec}(k[x]/I)$ with $\sqrt{I} = (x_1 \ldots x_n)$. There is some $\delta \in \mathbb{N}$ such that $x_i^\delta \in I$ for all i. Put $D := \delta \cdot n$. Now any monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ with $\sum \alpha_i \geq D$ lies in I (pigeon-hole). Therefore

$$kx[]/I = \operatorname{span}\left\{x_1^{\alpha_1} \ldots x_n^{\alpha_n} \mod I : \sum \alpha_i < D\right\}$$

which is a finite-dimensional k-vector-space.

3. We can assume $k[x_0, \ldots, x_n] \supseteq I(X) =: J$ is homogenisation of ideal I. $I \subseteq k[x_1, \ldots, x_n]$ is an ideal such that $k[x_1, \ldots, x_n]/I$is generated by x^α for $|\alpha| < D$. Let $J \subseteq k[x]$ be the homogenisation of I.

$$(k[x_0, \ldots, x_n]/J)^{(d)} \to k[x_1, \ldots, x_n]/I =: R \qquad\qquad f \mapsto f[x_0 = 1]$$

This map is a k-linear isomorphism of $d \geq D$. So

$$h_X(d) = (k[x_0, \ldots, x_n]/J)^{(d)} = \dim_k R$$

□

7.4 Proposition. *Let X be an m-dimensional projective subscheme of $\mathbb{P}^n_k$. Then there is a unique polynomial $\chi_X \in \mathbb{Q}[d]$ such that for sufficiently large d we have $h_X(d) = \chi_X(d)$. Moreover*

1. $\deg \chi_X = m = \dim X$.
2. *The leading coefficient of χ_X has the form $\frac{1}{m!}\cdot$positive integer and the latter we call the* degree *of X.*

Proof. Induction on m, where the base case is Lemma 7.3. Now assume $m > 0$. By linear change of coordinates, we can assume that no component of X lies in the hyperplane $H = \{x : x_0 = 0\}$. Then we have an exact sequence

$$0 \to k[x_1, \ldots, x_n]/I(X) \xrightarrow{\cdot x_0} k[x_0, \ldots, x_n]/I(X) \to k[x_0, \ldots, x_n]/\left(I(X) + (x_0)\right) \to 0$$

Claim. *The first map is injective.*

Proof. Suppose $f \notin I(X)$, but $f x_0 \in I(X)$. Then $X = (X \cap Z(f)) \cup (X \cap H)$. But by assumption on H we get $X = X \cap Z(f)$, which would mean X vanishes on f. ↯ □

But then restricting to degree d yields another exact sequence

$$0 \to \left(k[x_1, \ldots, x_n]/I(X)\right)^{(d-1)} \xrightarrow{\cdot x_0} \left(k[x_0, \ldots, x_n]/I(X)\right)^{(d)} \to \left(k[x_0, \ldots, x_n]/\left(I(X) + (x_0)\right)\right)^{(d)} \to 0$$

For any short exact sequence $0 \to U_0 \to U_1 \to U_2 \to 0$ we have $\dim U_1 = \dim U_0 + \dim U_2$. In this case, this yields

$$h_X(d) = h_X(d-1) + h_{X\cap H}(d)$$

By induction hypothesis we know $(h_X(d) - h_X(d-1) = h_{X\cap H} \in \frac{1}{(m-1)!}d^{m-1} + \mathcal{O}\left(d^{m-2}\right)$. Now we can apply discrete summation to recover h_X. First observe that the terms $\binom{d}{i}$ form a basis for our polynomials in d. For the discrete summation, if $F(d) - F(d-1) = \binom{d}{i}$, we obtain $F(d) = \binom{d+1}{i+1} + C$. Since we can write

$$h_{X\cap H} = \sum_{i=0}^{m-1} c_i \binom{d}{i}$$

with $c_i \in \mathbb{Q}$ and $c_{m-1} \in \mathbb{N}$, we recover

$$h_X(d) = \sum_{i=0}^{m-1} c_i \binom{d+1}{i+1} + C$$

which is a polynomial of degree m. □

7.5 Example.

- Let $X = \mathbb{P}^n_k$, so $I(X) = 0$. Then

$$h_X(d) = \binom{d+n}{d} = \frac{1}{n!}d^n + \mathcal{O}\left(d^{n-1}\right) \implies \deg \mathbb{P}^n_k = 1$$

- Let f be homogeneous of degree δ and put $R := k[x_0, \ldots, x_n]$. Let $X = \operatorname{Proj}\left(R/(f)\right)$. Then

$$\dim\left(R/fR\right)^{(d)} = \dim R^{(d)} - \dim(fR)^{(d)} = \binom{n+d}{n} - \dim R^{(d-\delta)} = \binom{n+d}{n} - \binom{n+d-\delta}{n}$$

For n fixed and $d \to \infty$, we get $h_X(d) \sim \frac{1}{(n-1)!}\delta d^{n-1}$, and $n - 1 = \dim X$. Hence $\deg X = \delta = \deg f$.

- If $\dim X = 0$, then $X \cong \operatorname{Spec} R$. We have $\chi_X(d) = \dim_k R$. For $R \cong R_1 \times \ldots \times R_t$ the degrees just add up, e.g.

$$\dim\left(k[x]/(x^2) \times k[x]/(x-1)^3\right) = 2 + 3 = 5$$

7.6 Proposition. *Let X_1 and X_2 be m-dimensional subschemes of $\mathbb{P}^n_k$ such that $\dim(X_1 \cap X_2) < m$. Then $\deg(X_1 \cup X_2) = \deg X_1 + \deg X_2$.*

Proof. Put $S := k[x_0, \ldots, x_n]$. Then

$$X_1 \cap X_2 = \operatorname{Proj}\left(S/(I(X_1) + I(X_2))\right) \qquad X_1 \cup X_2 = \operatorname{Proj}\left(S/(I(X_1) \cap I(X_2))\right)$$

Then we have the following exact sequence

$$\begin{array}{llll} 0 \to S/(I(X_1) \cap I(X_2)) \to S/I(X_1) \times S/I(X_2) & \to S/\left(I(X_1) + I(X_2)\right) \to 0 f & \mapsto (f, f) \\ \qquad\qquad\qquad (f, g) & \mapsto f - g & \end{array}$$

Taking the degree-d-part and using the dimension property from earlier, we get ref

$$h_{X_1}(d) + h_{X_2}(d) = h_{X_1 \cup X_2}(d) + h_{X_1 \cap X_2}(d)$$

In particular this holds for the Hilbert polynomial

$$\chi_{X_1}(d) + \chi_{X_2}(d) = \chi_{X_1 \cup X_2}(d) + \chi_{X_1 \cap X_2}(d)$$

By the assumption, it has a lower degree for $X_1 \cap X_2$, so we can rewrite

$$\chi_{X_1 \cup X_2}(d) = \frac{1}{m!} \deg(X_1) d^m + \frac{1}{m!} \deg(X_2) d^m + \mathcal{O}\left(d^{m-1}\right)$$

so $\deg(X_1 \cup X_2) = \deg(X_1) + \deg(X_2)$. □

7.7 Remark. Let X be a projective subscheme of $\mathbb{P}^n_k$. Then the expression

$$g(X) := (-1)^{\dim X}\left(\chi_X(0) - 1\right)$$

is called the *arithmetic genus* of X.

1. It is an invariant under isomorphisms, i.e. $X \cong Y \implies g(X) = g(Y)$.
2. If X is a smooth projective curve over $\mathbb{C}$, then $g(X)$ is the topological genus of X.

7.8 Example. If $X \subseteq \mathbb{P}^2$ is a planar curve of degree d, then $g(X) = \binom{d-1}{2}$. For $d = 3$ we have $g(X) = 1$, so there is no cubic in $\mathbb{P}^2$, which is isomorphic to $\mathbb{P}^1$.

7.2 Bézout's Theorem

7.9 Theorem. *Let X be a projective subscheme of $\mathbb{P}^n_k$ and let $f \in k[x_0, \ldots, x_n]$ be homogeneous of degree δ such that no component of X is contained in $Z(f)$. Then $\deg(X \cap Z(f)) = \deg X \cdot \deg f$.*

Proof. Put $R := k[x_0, \ldots, x_n]$. We have the exact sequence

$$0 \to R/I(X) \xrightarrow{\cdot f} R/I(X) \to R/(I(X) + (f)) \to 0$$

The multiplication with f is injective, due to the assumption (and a similar argument as last time). ref

Restricting the degree yields the exact sequence

$$0 \to (R/I(X))^{(d-\delta)} \xrightarrow{\cdot f} (R/I(X))^{(d)} \to (R/(I(X)+(f)))^{(d)} \to 0$$

So for the Hilbert function we have

$$h_X(d) = h_X(d-\delta) + h_{X\cap Z(f)}(d) \implies \chi_{X\cap Z(f)}(d) = \chi_X(d) - \chi_X(d-\delta)$$

Writing

$$\chi_X(d) = \frac{1}{m!}\deg X d^m + c_{m-1}d^{m-1} + \mathcal{O}\left(d^{m-2}\right)$$

we get

$$\begin{aligned}\chi_{X\cap Z(f)}(d) &= \frac{\deg X}{m!}\left(d^m - (d-\delta)^m\right) + c_{m-1}\left(d^{m-1} - (d-\delta)^{m-1}\right) + \mathcal{O}\left(d^{m-2}\right)\\ &= \frac{\deg X}{m!}\cdot m\delta d^{m-1} + \mathcal{O}\left(d^{m-2}\right) = \frac{\deg X\cdot\delta}{(m-1)!}d^{m-1} + \mathcal{O}\left(d^{m-2}\right)\end{aligned}$$

□

7.10 Example. Let C_1, C_2 be two curves in $\mathbb{P}^2$ without common component. Let C_i be the zero set of homogeneous polynomials of degree d_i. Then $\deg(C_1 \cap C_2) = d_1 \cdot d_2$.

7.11 Example. Let $P \in C_1 \cap C_2$. We have the following cases.

1. Assume C_1 and C_2 are smooth at P and have different tangent lines. Assume $P = (0,0)$ in affine chart, $C_i \cap \mathbb{A}^2 = Z(f_i)$. Then $\dim_k k[x,y]/(f_1, f_2)$ is the intersection multiplicity at P.
2. Assume P is a smooth point of C_1 and C_2, but the have the same tangents. For simplicity, regard $f_1 = y + \text{h.o.t}$ and $f_2 = y + \text{h.o.t}$. Then $\dim(k[x,y]/(f_1,f_2)) \geq 2$, since 1 and x are linearly independent in this ring.
3. If C_1 and C_2 are singular at P, then both polynomials start with quadratic terms. Hence $1, x, y$ are linearly independent, so $\dim k[x,y]/(f_1,f_2) \geq 3$.

7.12 Corollary. *Every isomorphism $\varphi : \mathbb{P}^n_k \to \mathbb{P}^n_k$ is linear, i.e. of the form $\varphi(x) = Ax$ where $A \in \mathrm{GL}_{n+1}(k)$ and $x = (x_0, \ldots, x_n)^T$.*

Proof. Let $H \subseteq \mathbb{P}^n_k$ be a hyperplane and L be some line, not contained in H. Note $\varphi(H \cap L) = \varphi(H) \cap \varphi(L)$. Then by Bézout

$$1 = \deg(\varphi(H)\cap\varphi(L)) = \deg\varphi(H)\cdot\deg\varphi(L) \implies \deg\varphi(H) = 1$$

So $\varphi(H)$ is another hyperplane. Therefore $x_i \circ \varphi$ is a linear form, say $\sum_j a_{ij}a_j$. Going over all i this lifts up to a matrix. □

7.13 Example. regard the twisted cubic in $\mathbb{P}^3$

$$C = \left\{(s^3 : s^2t : st^2 : t^3) : (s:t) \in \mathbb{P}^1\right\} = Z(f_1, f_2, f_3)$$

for some specific quadratic polynomials f_i. The question is whether there are homogeneous polynomials f, g such that $I(X) = (f,g)$.
Assume we had these f, g. This means $C = Z(f) \cap Z(g)$. If some component of $Z(f)$ would lie in $Z(g)$, then the intersection would not be a curve (too large). Hence we can apply Bézout, so $3 = \deg C = \deg f \cdot \deg g$. So wlog $\deg f = 1$, which means $Z(f)$ is hyperplane. Assume C were contained in a hyperplane. A hyperplane in $\mathbb{P}^3$ is given by $\sum a_i x_i = 0$, for some vector $0 \neq a \in k^4$. For the twisted cubic this means $\forall s, t. \sum a_i s^i t^{3-i} = 0$, which can only be for $a = 0$. So we have our final contradiction.