Algebra 4

Inofficial lecture notes
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Contents

1 Representations of Finite Groups

Notation. *We write* $\mathbb{C}^{d\times d}$ *for the ring of* $d \times d$*-matrices over* $\mathbb{C}$*. Further* $\mathrm{GL}_d = \mathrm{GL}_d(\mathbb{C})$ *is the group of invertible* $d \times d$*-matrices over* $\mathbb{C}$*.*

1.1 Modules and Representations

1.1 Definition. A *matrix representation* of a group G is a homomorphism $M : G \to \mathrm{GL}_d$ for some d. This number d is called the representation degree or dimension of M.

Example.

- *There always is the trivial representation* $G \to \mathrm{GL}_1$ *via* $g \mapsto 1$*.*
- *From the symmetric group we have the sign* $\mathrm{sgn} : S_n \to \{-1, 1\} \leq \mathbb{C}$*.*
- *We have the defining representation* $D : S_n \to \mathrm{GL}_n$ *where*

$$D(\pi)_{ij} = \begin{cases} 1 & : \pi(j) = i \\ 0 & : \textit{else} \end{cases}$$

- *Let* $G = C_n = \langle g \rangle$*. We know* $M(g)^n = M(g^n) = M(1) = 1$*, so* $M(g)$ *is a root of unity and each such choice is a representation.*

1.2 Definition. Let G be a group, V some (finite dimensional) $\mathbb{C}$-vector-space. A *representation of G on V* is a group morphism $D : G \to \mathrm{GL}(V)$. D is called *faithful* if it is injective.

1.3 Definition. A (finite dimensional) G-module is a (finite dimensional) $\mathbb{C}$-vector-space V together with an operation of G on V (i.e. $g.(h.v) = (gh).v$, $1.v = v$) such that $v \mapsto g.v$ is linear for all $g \in G$.

1.4 Remark. If D is a representation of G on V, we define the operation $g.v := D(g)(v)$, which yields a G-module. Conversely any operation defines a representation via $D(g)(v) := g.v$.

Let G operate on some finite set X. Put

$$V = \mathrm{span}_{\mathbb{C}}(X) = \left\{ \sum_{x\in X} \lambda_x \cdot x : \lambda_x \in \mathbb{C} \right\}$$

as formal linear combinations. We extend the operation of G linearly onto V. Then V is a G-module.

Example. *Take the natural operation of* S_n *on* $[n] = \{1, \dots, n\}$*. The* S_n*-module is given by*

$$\pi \left(\sum_{i=1}^{n} \lambda_i \cdot i \right) = \sum_{i=1}^{n} \lambda_i \cdot \pi(i)$$

If we identify i *with* e_i*, then the corresponding matrix is the permutation matrix.*

1.5 Definition. Let G be some finite group. The group algebra is the set

$$\mathbb{C}[G] := \left\{ \sum_{g\in G} \lambda_g \cdot g : \lambda_g \in \mathbb{C} \right\}$$

together with the multiplication

$$\left(\sum_{g\in G}\lambda_g\cdot g\right)\cdot\left(\sum_{h\in G}\mu_h\cdot h\right)=\sum_{g,h\in G}\lambda_g\mu_h\cdot(gh)$$

1.2 Submodules and Reducibility

1.6 Definition. Let V be some G-module. A *submodule* of V is a G-stable subspace, i.e. $\forall g\in G, u\in U. g.u\in U$.

1.7 Remark. U inherits G-module structure from V. We always have the trivial submodules 0 and V.

Example. *Consider $\mathbb{C}^n$ as S_n module. Then $U := \operatorname{span}(e_1+\ldots+e_n)$ is a 1-dimensional submodule. S_n operates trivially on U, since it just permutes the summands.*

1.8 Exercise. Show that U and $U^\perp$ are the only non-trivial S_n-submodules of $\mathbb{C}^n$.

1.9 Definition. A G-module V is called *simple*, if $V\neq 0$ and the only submodules are 0 and V itself. The corresponding representation is called *irreducible*.

1.10 Lemma. *Every simple C_n-module is 1-dimensional.*

Proof. Let $C_n=\langle g\rangle$ and v be an eigenvector of $D(g)\in\mathrm{GL}(V)$. Then $g.v=\lambda v$, and therefore $g^j.v=\lambda^j v$. Hence $\mathbb{C}v$ is a non-empty C_n-submodule. If V is simple, we must have $V=\operatorname{span}(v)$, which is 1-dimensional. □

1.11 Definition. Let V be a G-module and $U\leq V$. A *module-complement* is a submodule W with $V=U\oplus W$.

Let $V=U\oplus W$ with basis $U=\langle u_1,\ldots,u_m\rangle$ and $W=\langle w_1,\ldots,w_n\rangle$. For $g\in G$ let $M(g)$ be the matrix of $D(g)\in\mathrm{GL}(V)$. Then U,W are submodules iff $M(g)$ has block-form for all $g\in G$. So we look for a simultaneous block decomposition.

1.12 Notation. We fix the following notation, unless states otherwise.

- From now on let G be some group, V be some (finite dimensional) $\mathbb{C}$-vector-space.
- A G-module is a map $G\times V:(g,v)\mapsto g.v$.
- A representation is a morphism $D:G\to\mathrm{GL}(V)$.
- A submodule $U\subseteq V$ is a G-invariant subspace.
- Let $\langle\cdot,\cdot\rangle$ be some hermitian inner product on V:

$$\langle\lambda_1u_1+\lambda_2u_2,v\rangle=\lambda_1\langle u_1,v\rangle+\lambda_2\langle u_2,v\rangle$$
$$\overline{\langle v,u\rangle}=\langle u,v\rangle$$

1.13 Lemma. *Every G-module has a G-invariant inner product, i.e. $\forall g\in G, u,v\in V.\langle gu,gv\rangle=\langle u,v\rangle$.*

Proof. Let $\langle \cdot, \cdot \rangle$ be any inner product. Then we define

$$\langle v, w \rangle_G := \frac{1}{|G|} \sum_{g \in G} \langle gv, gw \rangle$$

By construction this is a G-invariant inner product. □

1.14 Lemma. *Let $\langle \cdot, \cdot \rangle$ be G-invariant and $U \subseteq V$ be some G-submodule. Then*

$$U^\perp := \{v \in V : \forall u \in U. \langle u, v \rangle = 0\}$$

is a submodule.

Proof. Let $g \in G, v \in U^\perp$. It remains to show $gv \in U^\perp$. But for any $u \in U$ we have $\langle gv, u \rangle = \langle v, g^{-1}u \rangle = 0$, since $g^{-1}u \in U$. □

1.15 Theorem (Maschke). *There are simple submodules $U_1, \dots, U_t$ of V with $V = U_1 \oplus \dots \oplus U_t$.*

Proof. By induction on $d := \dim V$. For $d = 0$ it is clear, so let $d > 0$.
If V is simple, we are done. Otherwise there is a submodule $0 \neq U_1 \subset V$. By a previous lemma which
$V = U_1 \oplus V'$ for some V'. Then we apply induction on U_1 and V'. □

1.16 Corollary. *Let $M : G \to \mathrm{GL}_d$ be the matrix representation. Then there exists some $T \in \mathrm{GL}_d$ and a decomposition $d = d_1 + \dots + d_t$ such that for all $g \in G$ we have*

$$TM(g)T^{-1} = \begin{pmatrix} M_1(g) & & 0 & \\ & M_2(g) & & \\ & & \ddots & \\ & & & M_t(g) \end{pmatrix}$$

where $M_i : G \to \mathrm{GL}_{d_i}$ are irreducible matrix representations.

Example. *Take $G = C_n = \langle g \rangle$. Then C_n-module V is simple iff $\dim V = 1$. The 1-dimensional C_n-modules are given by the group morphisms*

$$M_k : C_n \to C^* : g^j \mapsto \zeta^{kj}$$

where ζ is an n-th root of unity. For al matrix representation $M : C_n \to \mathrm{GL}_d$ there exist integers $k_1, \dots, k_t$ and there is some $T \in \mathrm{GL}_d$ such that

$$TM(g)T^{-1} = \mathrm{diag}(M_{k_1}(g), \dots, M_{k_t}(g)) = \mathrm{diag}(\zeta^{k_1}, \dots, \zeta^{k_t})$$

1.17 Remark. (i) Maschke's Theorem still holds if $\mathbb{C}$ is replaced by a field K with $\mathrm{char}\, K \nmid |G|$ (because we computer $|G|^{-1}$ when averaging over the group).

(ii) For infinite groups Maschke's theorem does not hold. Take $G = (\mathbb{R}, +)$ with

$$M : \mathbb{R} \to \mathrm{GL}_2 \quad M(r) = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$$

It is in fact a representation since $M(r_1)M(r_2) = M(r_1 + r_2)$. Now take the submodule $U = \mathbb{R}(1, 0)^T$. We claim that this is the only proper submodule: Assume $v_2 \neq 0$,so $v \notin U$. Then $M(1)v = (v_1 + v_2, v_2)$, which is linearly independent of v. So with any of these v we span all of V. Thus U does not have a module complement.

1.3 Morphisms and Schur's Lemma

1.18 Definition. Let V, W be G-modules. A G-module morphism is a linear map

$$\varphi : V \to W \quad \forall g \in G, v \in V. \varphi(g.v) = g.\varphi(v). \tag{1}$$

In matrix language: Choose some basis of V and W with $m = \dim V$ and $n = \dim W$. Let $M : G \to \mathrm{GL}_m$, $N : G \to \mathrm{GL}_n$ be the corresponding matrix representations. Let T be the representation matrix of φ. Then eq. (1) means: $\forall g \in G. TM(g) = N(g)T$.

That's a natural transformation

1.19 Definition. A *G-module isomorphism* $\varphi : V \to W$ is a bijective module morphism. If there exists such an isomorphism, we say V and W are isomorphic, written $V \cong W$.

Now we have a special case of the above scenario. The corresponding modules are isomorphic iff there exists some $T \in \mathrm{GL}_d$ with $\forall g \in G. N(g) = TM(g)T^{-1}$, which means they are conjugate.

Notation.

- *Denote* $\mathrm{Hom}_G(V, W) := \{\varphi : V \to W : G\text{-module morphism}\} \leq \mathrm{Hom}(V, W)$.
- $\mathrm{End}_G(V) := \mathrm{Hom}_G(V, V) \leq \mathrm{End}(V)$*, is a subalgebra.*

Lemma. *Let $\varphi : V \to W$ be some G-module morphism. Then $\ker \varphi \leq V$ and $\operatorname{im} \varphi \leq W$ are submodules.*

Theorem (Schur's Lemma). *Let V, W be simple G-modules.*

(i) If $V \ncong W$ then $\mathrm{Hom}_G(V, W) = 0$.

(ii) If $V \cong W$ then $\dim \mathrm{Hom}_G(V, W) = 1$.

Proof. Let $V \ncong W$ and assume $0 \neq \varphi \in \mathrm{Hom}_G(V, W)$. Then $\ker < V$, but since V is simple, we get $\ker \varphi = 0$. So φ is injective. Additionally $0 \neq \operatorname{im} \varphi \leq W$, so $\operatorname{im} \varphi = W$ since W is simple. Thus φ is surjective, hence bijective, so φ is an isomorphism. $\lightning$
Next we show $\mathrm{End}_G(V) = \mathbb{C} \, \mathrm{id}_V$ (so we have the case $W = V$).
Let $\varphi \in \mathrm{End}_G(V)$. Let $v \neq 0$ be an eigenvector of φ, with $\varphi(v) = \lambda v$. Then

$$\varphi - \lambda \, \mathrm{id}_V \in \mathrm{End}_G(V) \quad v \in \ker(\varphi - \lambda \, \mathrm{id}_V) \neq 0$$

Since V is simple, we get $\ker(\varphi - \lambda \, \mathrm{id}_V) = V$, so $\varphi = \lambda \, \mathrm{id}_V$.
Now let $\alpha : V \to W$ be an isomorphism and $\varphi \in \mathrm{Hom}_G(V, W)$. Then $\alpha^{-1} \circ \varphi \in \mathrm{End}_G(V)$. So there is some $\lambda \in \mathbb{C}$ with $\alpha^{-1} \circ \varphi = \lambda \, \mathrm{id}_V$, which means $\varphi = \lambda \alpha$. $\square$

Remark. *In the proof we didn't use that G is finite.*

Corollary. *Let $V = U_1 \oplus \ldots \oplus U_t$ be a direct sum of simple modules. Take any simple G-module W. Then*

$$|\{i : U_i \cong W\}| = \dim \mathrm{Hom}_G(V, W).$$

This number is called multiplicity of W in V, written $\mathrm{mult}_W(V)$.

Proof. We regard the map

$$\bigoplus_{i=1}^{t} \mathrm{Hom}(W, U_i) \to \mathrm{Hom}(W, V) \qquad (\varphi_1, \ldots, \varphi_t) \mapsto (w \mapsto \varphi_1(w) + \ldots + \varphi_t(w))$$

By taking projections $\varphi_i := \pi_i \circ \varphi$, we get an inverse map, so this is an isomorphism. Restriction onto G-invariant maps yields an isomorphism

$$\bigoplus_{i=1}^{t} \mathrm{Hom}_G(W, U_i) \to \mathrm{Hom}_g(W, V)$$

By Schur's Lemma we have

$$\dim \mathrm{Hom}_G(W, U_i) = \begin{cases} 1 & W \cong U_i \\ 0 & \text{else} \end{cases}$$

Thus $\dim \mathrm{Hom}_G(W, V) = |\{i : W \cong U_i\}|$. □

Corollary. *Let V be some G-module and U_i, U'_j be simple submodules such that*

$$V = U_1 \oplus \ldots \oplus U_t = U'_1 \oplus \ldots \oplus U'_s$$

Then $s = t$ and there is some $\pi \in S_t$ with $U_i \cong U'_{\pi(i)}$ for all i.

Let $M = M(g)$ be some matrix representation. Now we are looking at polynomial functions $f : \mathbb{C}^{d \times d} \to \mathbb{C}$ with $f(TMT^{-1}) = f(M)$ for all $T \in \mathrm{GL}_d$. We can use every elementary symmetric polynomial in the eigenvalues (e.g. trace, determinant).

1.4 Characters

Let $\alpha : V \to V$ be linear, and $a = [a_{ij}]$ a matrix representation with respect to some basis. The function $\mathrm{tr}(a) = \sum_i a_{ii}$ is called the *trace*. It is independent of the basis. In fact $\mathrm{tr}(\alpha)$ is a coefficient of the characteristic polynomial

$$\det(T - \alpha) = T^d - \mathrm{tr}(\alpha) T^{d-1} + \ldots + (-1)^d \det(\alpha)$$

In particular $\mathrm{tr}(g \alpha g^{-1}) = \mathrm{tr}(\alpha)$ for all $g \in \mathrm{GL}(V)$.

Definition. *Let D be a representation of G. The function $\chi_D : G \to \mathbb{C}$ via $g \mapsto \mathrm{tr}(D(g))$ is called a* character *of D. If V is the module corresponding to D, we write χ_V.*

Proposition. *Isomorphic modules have the same characters.*

Proof. Let U, V be G-modules with representations D, F and isomorphism $\alpha : U \to V$. Then by definition $\alpha \cdot D(g) = F(g) \cdot \alpha$, or rather $F(g) = \alpha D(g) \alpha^{-1}$. Thus

$$\chi_F(g) = \mathrm{tr}(F(g)) = \mathrm{tr}(D(g)) = \chi_D(g)$$ □

Remark. *(i) $\chi_V(1) = \dim V$.*

(ii) χ_V is a class function, i.e. constant on conjugacy classes.

(iii) $\chi_{U \oplus V} = \chi_U + \chi_V$

Example. *(i) Suppose X is some finite set and G acts on X. Let $V := \mathrm{span}_{\mathbb{C}}(X)$ with representation D. Then $\chi_V(g) = \mathrm{tr}(D(g))$ is the number of fixed points of g.*

(ii) Let $G = S_n$ acting on $X = [n]$. We regard $\mathbb{C}^n$ as S_n-module. We put $\mathbf{1} = \mathbb{C}\left(\sum e_i\right)$ and $U := \mathbf{1}^\perp = \{x \in \mathbb{C}^n : \sum x_i = 0\}$. Then U is a simple module (exercise). $\chi_U(\pi) = \#(\text{fixed points of } \pi) - 1$, due to the previous remark.

(iii) Take the regular representation: $X = G$ and G acts on the left. In this case, we have a fixed point iff $g = 1$. Hence

$$\chi(g) = \begin{cases} 0 & g \neq 1 \\ |G| & g = 1 \end{cases}$$

Lemma. $\chi_V(g^{-1}) = \overline{\chi_G(g)}$

Proof. Take $H = \langle g \rangle$, say $H \cong C_n$. With a suitable basis, V has a representation of the form $M(g) = \operatorname{diag}(\zeta^{k_1}, \dots, \zeta^{k_d})$, for $\zeta^n = 1$. But since $\zeta^{-1} = \overline{zeta}$ we have

$$\chi_V(g) = \operatorname{tr}(M(g^{-1})) = \sum_{i=1}^{d} \zeta^{-k_i} = \sum_{i=1}^{d} \overline{\zeta}^{k_i} = \overline{\chi_V(g)} \qquad \square$$

1.5 Orthogonality relations

G acts on $\operatorname{Hom}(U, V)$ via $(g.\alpha)(u) := g.\alpha(g^{-1}.u)$. This way, $\operatorname{Hom}(U, V)$ becomes a G-module. Recall $\operatorname{Hom}_G(U, V) \leq \operatorname{Hom}(U, V)$. In fact $\alpha \in \operatorname{Hom}_G(U, V) \Leftrightarrow \forall g \in G.g.\alpha = \alpha$.

Corollary (From Schur's Lemma). *Let $\varphi : V \to U$ be some linear map. We define*

$$\widetilde{\varphi} := |G|^{-1} \cdot \sum_{g \in G} D_V(g) \circ \varphi \circ D_U(g^{-1}).$$

Then we have

(i) If $U \ncong V$ then $\widetilde{\varphi} = 0$.

(ii) If $U \cong V$ then $\widetilde{\varphi} = \frac{1}{n} \operatorname{tr}(\varphi) \cdot \operatorname{id}_V$, where $n = \dim V$.

Proof. Due to the averaging $\widetilde{\varphi} \in \operatorname{Hom}_G(U, V)$, so we can apply Schur's Lemma. This immediately yields the first part. For the second, we know $\widetilde{\varphi} = \lambda \operatorname{id}_V$, so just have to compute λ.

$$\lambda n = \operatorname{tr}(\widetilde{\varphi}) = \frac{1}{|G|} \sum_{g \in G} \operatorname{tr}(D_V(g) \varphi D_V(g)^{-1}) = \operatorname{tr}(\varphi) \qquad \square$$

Corollary. *Let U, V be simple G-modules with representations R, S.*

(i) If $U \ncong V$, then $|G|^{-1} \sum_{g \in G} S_{ij}(g) R_{kl}(g^{-1}) = 0$.

(ii) If $U \cong V$, then $|G|^{-1} \sum_{g \in G} S_{ij}(g) R_{kl}(g^{-1}) = \frac{1}{n} \delta_{il} \delta_{jk}$.

Proof. Apply the corollary to $\varphi : U \to V$ with matrix E_{jk}. In the case $U \ncong V$ we get

$$0 = \frac{1}{|G|} \sum_{g \in G} \sum_{j', k'} S_{ij'}(g) \left(E_{jk}\right)_{j'k'} R_{k'l}(g^{-1}) = \frac{1}{|G|} \sum_{g \in G} S_{ij}(g) R_{kl}(g^{-1})$$

For the case $U \cong V$ we proceed similarly

$$\frac{1}{|G|} \sum_{g \in G} s_{ij}(g) R_{kl}(g^{-1}) = \frac{\overline{tr}(E_{jk})}{n} \delta_{il} = \frac{1}{n} \delta_{jk} \delta_{il} \qquad \square$$

Let $\langle\cdot,\cdot\rangle$ be a Hermitian inner product on H. We say $x_1,\dots,x_k$ form an orthonormal system iff $\langle x_i, x_j\rangle = \delta_{ij}$. If $k = \dim H$, then this is an orthonormal basis. In this case we have the Fourier decomposition

$$x = \sum_{i=1}^{k} \langle x_i, x_i\rangle x_i$$

On our vector space $\mathbb{C}^G$ we define the Hermitian product

$$\langle \varphi, \psi\rangle := \frac{1}{|G|}\sum_{g\in G} \varphi(g)\overline{\psi(g)}$$

Theorem (Orthogonality relations). *Let U, V be simple G-modules. Then*

$$\langle \chi_U, \chi_V\rangle = \begin{cases} 1 & : U \cong V \\ 0 & : U \not\cong V \end{cases}$$

Proof. Take matrix representations R, S of U, V.

$$\begin{aligned}\langle \chi_U, \chi_V\rangle &= \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\overline{\chi_U(g)} \overset{\text{Lemma}}{=} \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\chi_U(g^{-1}) \\ &= \frac{1}{|G|}\sum_{g\in G}\left(\sum_i s_{ii}(g)\right)\left(\sum_j R_{jj}(g)\right) = \sum_{i,j}\frac{1}{|G|}\sum_{g\in G} S_{ii}(g)R_{jj}(g^{-1})\end{aligned}$$

For $U \not\cong V$, the corollary tells us that the inner sum is zero. So in this case $\langle \chi_V, \chi_U\rangle = 0$.
Let $U \cong V$, so wlog $U = V$. By the corollary we get

$$\frac{1}{|G|}\sum_{g\in G} S_{ii}(g)R_{jj}(g^{-1}) = \frac{1}{n}\delta_{ij}$$

Thus $\langle \chi_V, \chi_U\rangle = \frac{1}{n}\sum_{i,j}\delta_{ij} = 1$. ☐

Corollary. *Let W be simple. Then*

$$\text{mult}_W(V) = \dim \text{Hom}_G(W, V) = \langle \chi_V, \chi_W\rangle$$

Proof. We decompose $V = U_1 \oplus \ldots \oplus U_t$, with U_i simple. Thus $\chi_V = \sum \chi_{U_i}$. By linearity

$$\langle \chi_V, \chi_W\rangle = \sum_i \langle \chi_{U_i}, \chi_W\rangle = \#\{i : U_i \cong W\} = \text{mult}_W(V)$$ ☐

Note that this also shows that the multiplicity in independent of the decomposition.

Theorem. *There are only finitely many isomorphism types of simple G-modules. If they are represented by $W_1,\dots,W_k$, then $\sum(\dim W_i)^2 = |G|$. Moreover k is upper bounded by the number of conjugacy classes of G.*

Proof. Characters lie in the subspace of class functions

$$R(G) := \{f \in \mathbb{C}^G : f \text{ constant on conjugacy classes}\}$$

Moreover $\dim R(G) = \#$conjugacy classes. But $\chi_{W_1}, \dots, \chi_{W_k}$ are orthogonal, in particular linearly independent. Hence $k \leq \dim R(G)$.
Let V denote the G-module of the regular representation, i.e. $V = \mathbb{C}[G]$. Then

$$\mathrm{mult}_{W_j}(V) = \langle \chi_V, \chi_{W_j} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi_{W_j}(g)} = \frac{1}{|G|} \cdot \chi_V(1) \overline{\chi_{W_j}(1)} = \overline{\chi_{W_j}(g)} = \dim W_j =: d_j$$

Thus we get the decomposition

$$V \cong \bigoplus_{j=1}^{k} \bigoplus_{*=1}^{d_j} W_j$$

which leads to $|G| = \dim V = \sum_{j=1}^{k} d_j^2$. □

Remark. *In fact, k is the number of conjugacy classes.*

Theorem. *We have $U \cong V \Leftrightarrow \chi_U = \chi_V$.*

Proof. We have already done one direction. So assume $\chi_U = \chi_V$. Let $U \cong \bigoplus_{i=1}^{k} m_i W_i$ and $V \cong \bigoplus_{i=1}^{k} n_i W_i$ with $n_i, m_i \in \mathbb{N}$. Thus we get $\chi_U = \sum_i m_i \chi_{W_i}$ and $\chi_V = \sum_i n_i \chi_{W_i}$. But since the characters are linearly independent (orthogonal system), this decomposition is unique. So $n_i = m_i$ and $U \cong V$. □

Proposition. *Let χ be some character of a G-module. Then*

(i) χ is 1-dimensional iff $\chi : G \to \mathbb{C}^*$ is a group morphism.

(ii) χ is irreducible iff $\langle \chi, \chi \rangle = 1$.

Proof. (i) $\Rightarrow$: Let $\chi = \chi_V$ with representation $D : G \to \mathrm{GL}_1 = \mathrm{GL}(\mathbb{C}) = \mathbb{C}^*$. But then $\chi = D$.
$\Leftarrow$: Define a 1-dimensional module on $\mathbb{C}$ by $g.x = \chi(g)x$. This has character χ.

(ii) Let $\chi = \chi_V$ with decomposition $V \cong \bigoplus m_i W_i$. Then $\chi_V = \sum m_i \chi_{W_i}$. For the inner product we get

$$\langle \chi_V, \chi_V \rangle = \sum m_i^2 \langle \chi_{W_i}, \chi_{W_i} \rangle = \sum m_i^2$$

The latter is a sum of non-negative squares, which can only be one iff there is a single contribution, which means there is a single W_1. □

Example. *Take $G = S_3$. We have the following simple representations.*

- $\mathbb{C} \cong W_1$*: trivial representation,* $\chi_1 = 1$
- $\mathbb{C} \cong W_2$*: sign,* $\chi_2 = \mathrm{sgn}$
- $\mathbb{C}^2 \cong W_3$*: Consider $\mathbb{C}^3 = \mathbb{C}(1,1,1) \oplus U$, then U is our simple S_3-module: We know $\chi_U(\pi) = \mathrm{fix}_G(\pi) - 1$. Then*

$$\langle \chi_U, \chi_u \rangle = \frac{1}{6} \sum_{\pi \in S_3} \chi_U(\pi)^2 = \frac{2^2 + 3 \cdot 0^2 + 2 \cdot (-1)^2}{6} = 1$$

As a short notation we have the character table*:*

	id	(ij)	(ijk)
$\chi_1 = 1$	1	1	1
$\chi_2 = \mathrm{sgn}$	1	-1	1
χ_3	2	0	-1

Example. *Take $G = S_4$. The number of conjugacy classes is the number of partitions of 4, which is 5. As before we have W_1, W_2 and $W_3 = (1,1,1)^\perp$ (check that it is simple). For the remaining ones we get $13 = \dim(W_4)^2 + \dim(W_5)^2$, so $\dim W_4 = 3$ and $\dim W_5 = 2$.*
Define the module U by $(\pi, u) \mapsto \mathrm{sgn}(\pi)\pi.u$, so $\chi_4 = \chi_2 \cdot \chi_3$. This module is simple as well and $\chi_3 \neq \chi_4$, so it is a new module. To find W_5, we take a map $\varphi : S_4 \twoheadrightarrow S_3$ with kernel Klein Four. Let $D_2 : S_3 \to \mathrm{GL}(W_3)$ be the irreducible representation. Then $\varphi \circ D_2$ is an irreducible representation of S_4. When checking the orthogonality we get

	id	(ij)	(ijk)	$(ijkl)$	$(ij)(kl)$
$\chi_1 = 1$	1	1	1	1	1
$\chi_2 = \mathrm{sgn}$	1	-1	1	-1	1
χ_3	3	1	0	-1	-1
$\chi_4 = \mathrm{sgn} \cdot \chi_3$	3	-1	0	1	-1
χ_5	2	0	-1	0	2

$$\frac{1}{24} = \sum_{\pi \in S_4} \chi_i(\pi)\overline{\chi_j(\pi)} = \frac{1}{24} \sum_{K \in conj.\ class} |K| \cdot \chi_i(K)\chi_j(K)$$

Thus the 5×5-matrix $\left[\sqrt{\frac{|K|}{24}}\chi_i(K)\right]_{i,K}$ is orthogonal in the rows and thus also in the columns. This means the columns of the character table are orthogonal.

1.6 Decomposition of the Group Algebra

Let G be some finite group. Recall that we have the group algebra

$$\mathbb{C}[G] = \left\{ \sum_{g \in G} \lambda_g g : \lambda_g \in \mathbb{C} \right\} \qquad \dim \mathbb{C}[G] = |G|$$

Example. *Take $G = C_n = \langle g \rangle$. Starting with the polynomial ring $\mathbb{C}[X]$, we have the surjective map $\varphi : \mathbb{C}[X] \to \mathbb{C}[C_n]$ induced by $X \mapsto g$. We have $\varphi(X^n) = g^n = 1$, so $X^n - 1 \in \ker \varphi$, and in fact this generates the kernel. So $\mathbb{C}[X]/(X^n - 1) \cong \mathbb{C}[G]$.*
Let ζ be some n-th root of unity. Then $X^n - 1 = \prod(X - \zeta^j)$. By the Chinese Remainder Theorem we have $\mathbb{C}[X]/(X^n - 1) \cong \mathbb{C}^n$ via the map $[F] \mapsto (F(\zeta^0), \dots, F(\zeta^{n-1}))$. This map is the Discrete Fourier Transform.

Next we take a look at the centre of group algebra, which is given by

$$Z(\mathbb{C}[G]) := \{a \in \mathbb{C}[G] : \forall b \in \mathbb{C}[G].ab = ba\}$$

The centre is a sub-algebra.
For some $a = \sum \lambda_g g$ conjugation can be written as

$$hah^{-1} = \sum_{g \in G} \lambda_g hgh^{-1} = \sum_{g' \in G} \lambda_{h^{-1}g'h} g'$$

So all that happens is a reordering of the coefficients. Then we have $a \in Z(\mathbb{C}[G])$ iff $\forall g.\lambda_{h^{-1}gh} = \lambda_g$ iff $\lambda : G \to \mathbb{C}$ is constant on conjugacy classes. As a consequence, $\dim Z(\mathbb{C}[G])$ is the number of conjugacy classes.

Remark. *Let $D : G \to \mathrm{GL}(V)$ be some group representation. We extend this to a $\mathbb{C}$-algebra morphism $D' : \mathbb{C}[G] \to \mathrm{End}(V)$ via $\sum \lambda_g g \mapsto \sum \lambda_g D(g)$. This is called a representation of the algebra $\mathbb{C}[G]$.*

Theorem (Wedderburn). *Let G be finite and $W_1, \ldots, W_k$ the isomorphism types of simple G-modules (recall that there are only finitely many), with corresponding representations $D_i : G \to \mathrm{GL}(W_i)$, extended to algebra morphisms $D_i : \mathbb{C}[G] \to \mathrm{End}(W_i)$. Then $\varphi : \mathbb{C}[G] \to \prod \mathrm{End}(W_i)$ via $a \mapsto (D_1(a), \ldots, D_k(a))$ is a $\mathbb{C}$-algebra* isomorphism*. After choosing bases for the W_i, we get $\mathbb{C}[G] \cong \mathbb{C}^{n_1 \times n_1} \times \ldots \times \mathbb{C}^{n_k \times n_k}$.*

Proof.

- First note that by construction φ is a $\mathbb{C}$-algebra morphism.
- Let $a \in \ker \varphi$. This means $D_i(a) = 0$ or $aW_i = \{aw : w \in W_i\} = 0$ for all i. So a annihilates each W_i. But any G-module is a direct sum of these W_i, so a annihilates every G-module. In particular a annihilates $\mathbb{C}[G]$, which means $a.1 = 0$, so $a = 0$. Thus φ is injective.
- Both sides have the same dimension

  $$\dim \mathbb{C}[G] = |G| = \sum (\dim W_i)^2 = \dim \left(\prod \mathrm{End}(W_i)\right)$$

  Therefore φ is an isomorphism. □

Example.

- *Wedderburn's Theorem generalises our above observation on the Discrete Fourier Transform: $\mathbb{C}[C_n] = \mathbb{C} \times \ldots \times \mathbb{C}$.*
- *Take $G = S_3$. Then $\mathbb{C}[G] \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{2\times 2}$, from previous parts.*
- *Similarly $\mathbb{C}[S_4] \cong \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{3\times 3} \times \mathbb{C}^{3\times 3} \times \mathbb{C}^{2\times 2}$.*

Corollary. *The number of isomorphism classes of simple G-modules is the number of conjugacy classes (before we had "$\leq$").*

Proof. First observe that $Z\left(\mathbb{C}^{n\times n}\right) = \mathbb{C}I_n$, which is 1-dimensional. Furthermore

$$Z\left(\mathbb{C}^{n_1\times n_1} \times \ldots \times \mathbb{C}^{n_k \times n_k}\right) = Z\left(\mathbb{C}^{n_1\times n_1}\right) \times \ldots \times Z\left(\mathbb{C}^{n_k\times n_k}\right) \cong \mathbb{C} \times \ldots \times \mathbb{C} \cong \mathbb{C}^k$$

So we have $Z(\mathbb{C}[G]) \cong \mathbb{C}^k$. Finally

$$\#(\text{iso-classes}) = k = \dim Z(\mathbb{C}[G]) = \#(\text{conjugacy classes})$$ □

Corollary. *Let $|G| = n$, G abelian. The simple G-modules are the 1-dimensional modules. There are exactly n pairwise non.isomorphic 1-dimensional G-modules. The group algebra is $\mathbb{C}[G] \cong \mathbb{C}^n$.*

Proof. We have n conjugacy classes, so n isomorphism classes. Further $n = |G| = \sum_{i=1}^n (\dim W_i)^2$. Hence each W_i needs to have dimension 1. □

Corollary.

$$\#(\textit{iso classes of } S_n) = \#(\textit{conjugacy classes of } S_n) = P(n) := \#(\textit{partitions of } n)$$

Example. *Take the dihedral group D_n for n odd. We denote the elements by*

$$D_n = \{1, d, \dots, d^{n-1}, s, sd, \dots, sd^{n-1}\}$$

It is generated by the relations $d^n = s^2 = 1$ and $sds = d^{-1}$.
We have $C_n = \langle d \rangle \trianglelefteq D_n$, as normal subgroup of index 2. Thus $D_n/C_n = \langle \overline{s} \rangle \cong C_2 \cong \{1, -1\}$. So we get the 1-dimensional representation $\chi_1 : D_n \to D_n/C_n \to \{1, -1\}$. There are $2 + \frac{n-1}{2}$ further conjugacy classes $\{1\}, \{d^j, d^{-j}\}$ and $\{sd^j : j < n\}$. For $1 \leq l \leq \frac{n-1}{2}$ we have the C_n-representation $D_l : C_n \to \mathrm{GL}_2$ by $D_l(d) = \mathrm{diag}(\zeta^l, \zeta^{-l})$. We extend this onto D_n by

$$D_l(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad D_l(sd^j) = \begin{pmatrix} 0 & \zeta^{-lj} \\ \zeta^{lj} & 0 \end{pmatrix}$$

which gives rise to character χ_l with $\chi_l(d^j) = \zeta^{lj} + \zeta^{-lj}$ and $\chi_l(sd^j) = 0$.

$$\langle \chi_l, \chi_l \rangle = \frac{1}{2n} \left(|\chi_l(1)|^2 + \sum_{j=1}^{n-1} |\chi_l(d^j)|^2 + 0 \right) = \frac{2^2 + (2n-4)}{2n} = 1$$

Hence the χ_l are irreducible. Moreover $\chi_l \neq \chi_r$ for $l \neq r$.

1.7 Tensor Products

Let k be some field and U, V finite dimensional k-vector-spaces. For $U \cong k^m$ and $V \cong k^n$ we will get $U \otimes V \cong k^{m \times n}$, although it will not be defined this way.

1.20 Definition. A tensor product of U and V is a k-vector-space W together with a bilinear map $\varphi : U \times V \to W$, written $(u, v) \mapsto u \otimes v$ which satisfies the following universal property: For all k-spans W' and $\varphi' : U \times V \to W'$, which is k-linear, there is exactly one k-linear map $\gamma : W \to W'$ such that the following diagram commutes

$$\begin{tikzcd} U \times V \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & W \arrow[d, dashed, "\gamma"] \\ & W' \end{tikzcd}$$

1.21 Theorem. *(i) The tensor product exists and it is unique up to "canonical isomorphism".*

(ii) Let $\varphi : U \times V \to W$ be k-linear, e_i be some basis of U and f_j basis of V. Then (W, φ) is a tensor product of U and V iff $(\varphi(e_i, f_j) : i, j)$ is a basis of W.

Proof. (i) Existence follows from the second part. Taking $W = W'$ we get that there only is the identity map. In the general case, swap W and W', to get $W \xrightarrow{\gamma} W' \xrightarrow{\gamma'} W$. Therefore $\gamma \circ \gamma' = \mathrm{id}$, so it is an isomorphism.

(ii) If we have the basis, then the map is uniquely defined.

□

Notation. *We write $U \times V \to U \otimes_k V$ via $(x, y) \mapsto x \otimes y$ for the tensor product. Every tensor $w \in U \otimes V$ can be written as $w = \sum u_i \otimes v_j$ but this is not unique.*

Example. *The map into matrices given by* $(\xi, \eta) \mapsto (\xi_i \cdot \eta_j)_{ij}$ *is a tensor product.*

1.22 Remark. (i) $\dim(U \otimes V) = (\dim U) \cdot (\dim V)$

(ii) $U \xrightarrow{\sim} U \otimes k$ via $u \mapsto u \otimes 1$, which is a k-isomorphism. There is exactly one isomorphism $U \otimes V \xrightarrow{\sim} V \otimes U$ mapping all $u \otimes v$ to $v \otimes u$.

$$\begin{tikzcd}
 & U \times V \arrow[dl] \arrow[dr] & \\
V \otimes U \arrow[rr, dashed, "\gamma"] & & U \otimes V \arrow[ll, dashed, bend left, "\gamma'"]
\end{tikzcd}$$

(iii) The tensor product is associative in the sense that $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$. There is exactly one such isomorphism and it maps $(u \otimes V) \otimes w \mapsto u \otimes (v \otimes w)$.

1.23 Proposition. *Let* $\alpha : U \to U'$ *and* $\beta : V \to V'$ *be* k*-linear maps. Then there is exactly one* k*-linear map* $\alpha \otimes \beta : U \otimes V \to U' \otimes V'$ *such that* $(\alpha \otimes \beta)(x \otimes y) = \alpha(x) \otimes \beta(y)$*. One calls* $\alpha \otimes \beta$ *the tensor product of* α *and* β*, also called* Kronecker product.

Proof. We have a bilinear map

$$U \times V \xrightarrow{\varphi} U' \otimes V' \quad (u, v) \mapsto \alpha(u) \otimes \beta(v)$$

By universal property there exists exactly one map such that the following commutes:

$$\begin{tikzcd}
U \times V \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & U \otimes V \arrow[d, dashed, "\gamma"] \\
 & U' \otimes V'
\end{tikzcd}$$

So $\gamma(u \otimes v) = \alpha(u) \otimes \beta(v)$. □

More concrete let $\alpha : k^m \to k^{m'}$ and $\beta : k^n \to k^{n'}$ be given by matrices $A \in k^{m' \times m}$ and $B \in k^{n' \times n}$. Then we use $k^m \otimes k^n = k^{m \times n}$ and $k^{m'} \otimes k^{n'} = k^{m' \times n'}$. Then

$$\alpha \otimes \beta : k^{m \times n} \to k^{m' \times n'}$$

has the representation matrix

$$A \otimes B = [a_{ij} b_{ol}]_{(i,o),(j,l)} \qquad \begin{aligned} (i, o) &\in [m'] \times [n'] \\ (j, l) &\in [m] \times [n] \end{aligned}$$

Choose lexicographic order for (i, o), similarly for (j, l). Then $A \otimes B \in k^{m'n' \times mn}$ looks like

$$\begin{pmatrix} a_{11}B & \dots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{m'1}B & \dots & a_{m'm}B \end{pmatrix}$$

1.24 Lemma. *Let $m = m'$ and $n = n'$ in the above scenario. Then* $\operatorname{tr}(A \otimes B) = \operatorname{tr}(A) \cdot \operatorname{tr}(B)$.

Proof. Using the above writing, we obtain

$$\operatorname{tr}(A \otimes B) = \sum_i (a_{ii} \operatorname{tr}(B)) = \left(\sum_i a_{ii}\right) \cdot \operatorname{tr}(B) = \operatorname{tr}(A) \cdot \operatorname{tr}(B) \qquad \square$$

1.25 Definition. On $U \otimes V$ we define by $g(u \otimes v) = (gu) \otimes (gv)$ a G-module structure, which is called the tensor product of the G-modules U and V.
Note that for $g \in G$ we have a well-defined linear map

$$U \otimes V \to U \otimes V \quad u \otimes v \mapsto (gu) \otimes (gv)$$

If D_U and D_V are the corresponding representations, then the representation D of $U \otimes V$ satisfies $D(g) = D_U(g) \otimes D_V(g)$ (Kronecker product).

1.26 Remark. For G-modules we also have

- $U \otimes V \cong V \otimes U$
- $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$
- $U \otimes k \cong U$

1.27 Proposition. *For $U, V \in \operatorname{mod} G$ we have $\chi_{U \otimes V}(g) = \chi_U(g) \cdot \chi_V(g)$, by Lemma 1.24.*

Example. *Let W be some simple S_n-module; $\mathbb{C}_{\text{sgn}}$ the 1-dimensional S_n module given by the sign. Then $\mathbb{C}_{\text{sgn}} \otimes W$ is an S_n-module. As a vector space $\mathbb{C}_{\text{sgn}} \otimes W \cong \mathbb{C} \otimes W \cong W$, however, this isomorphism might not respect the operation. The S_n-operation is $(\pi, w) \mapsto \operatorname{sgn}(\pi) \cdot \pi(w)$. This module is also simple.*
We furthermore have $\mathbb{C}_{\text{sgn}} \otimes W \cong W$ as S_n-modules iff

$$\operatorname{sgn} \cdot \chi_w = \chi_w \Leftrightarrow \forall \pi \in S_n \setminus A_n . \chi_w(\pi)$$

Example. *Some further examples of tensors:*

- $V^* \otimes W \cong \operatorname{Hom}(V, W)$ *via the map*

  $$(\varphi \otimes w) \mapsto (v \mapsto \varphi(v) \cdot w)$$

- $(\mathbb{C}^n)^* \otimes \mathbb{C}^m \cong \mathbb{C}^{n \times m}$ *via*

  $$((\varphi_1, \ldots, \varphi_n), w) \mapsto w \cdot \varphi^T$$

  which is a rank 1 matrix. Every matrix can be written as a sum of rank 1 matrices, and the same holds for tensor elements. But the converse is not true (usually matrices don't have rank 1, also a tensor element usually is just a sum of these terms.)

- *Let's think about the Kronecker-product again. Let $A \in \operatorname{GL}(\mathbb{C}^n)$ and $B \in \operatorname{GL}(\mathbb{C}^m)$. Then on rank 1 matrices it acts the following way:*

  $$(A \otimes B)(wv^T) = B(w) \cdot A(v)^T = Bwv^T A^T$$

which we write as BCA^T. So

$$(A \otimes B)(E_{ij}) = (A \otimes B)(e_i e_j^T) = (Be_i)(e_j^T A^T) = b_i \cdot a_j^T = \sum_{k,l} \left(b_{ik} a_{jl} (e_k e_l^T) \right)$$

if we use the vectors for the columns of the matrices. Now we regard the operation on matrices as operation on vectors (concatenating all columns), then we get the Kronecker product of matrices for the operation.

Maybe we have to transpose some terms. Jesko is not sure.

2 Representations of Algebras

Let k be a field and A some finite-dimensional algebra.

2.1 Definition.
- A *representation* of A is an algebra-morphism $D : A \to \mathrm{End}(M)$ where M is a k-vector-space.
- An A-module is a k-vector-space M together with a map $A \times M \to M$ which satisfies

$$\begin{aligned} &\forall a \in A. \varphi_a : M \to M \quad x \mapsto ax \text{ is } k\text{-linear} \\ &\forall x \in M. \varphi_x : A \to M \quad a \mapsto ax \text{ is } k\text{-linear} \\ &\forall a, b \in A, x \in M. (ab)x = a(bx) \\ &\forall x \in M. 1x = x \end{aligned}$$

2.1 Semi-simple modules

2.2 Remark. A k-module is the same as a k-vector-space.

(i) If $D : A \to \mathrm{End}(M)$ is a representation of A, then $ax := D(a)(x)$ defines an A-module and vice-versa.

(ii) Let G be a finite group, $A = k[G]$. Every representation $D : G \to \mathrm{GL}(M)$ of G extends to a representation $\widetilde{D} : k[G] \to \mathrm{End}(M)$.

Example. *Let $A = \mathrm{End}(V)$ for some k-vector-space and $M := V$. Then the representation is $D = \mathrm{id}$. So the action is the evaluation $(a, v) \mapsto a(v)$.*

2.3 Definition. Let M be an A-module. A *submodule* of M is a linear subspace $N \leq M$ such that $\forall a \in A, x \in M. \varphi(a.x) = a.\varphi(x)$.

$\mathrm{Hom}_A(M, N)$ is a linear subspace of $\mathrm{Hom}_k(M, N)$ and $\mathrm{End}_A(M)$ is a subalgebra of $\mathrm{End}_k(M)$.

2.4 Definition. An A-module M is called *simple* if $M \neq 0$ and M does not have a proper submodule.

Example. *V as $\mathrm{End}(V)$-module is simple.*

2.5 Lemma (Schur's Lemma). *If M and N are simple A-modules and $\varphi : \mathrm{Hom}_A(M, N)$ then either $\varphi = 0$ or φ is an isomorphism.*

2.6 Theorem. *Let M be an A-module. The following are equivalent* *(i) M is a direct sum of simple modules.*

(ii) M is a sum of simple submodules.

(iii) Every submodule of M has a module complement.

Proof. **(i) $\Rightarrow$ (ii):** is clear.

(ii) $\Rightarrow$ (iii): Let $N \leq M$ be a submodule. Let $L \leq M$ be a maximal submodule such that $N \cap L = 0$ (exists since finite-dimensional). Then it suffices to show $N + L = M$. By assumption $M = M_1 + \ldots + M_r$ with M_i simple submodules. If $N + L \neq M$ there is some i with $M_i \nsubseteq N + L$. Thus $M_i \cap (N + L) = 0$, since M_i is simple. This also implies $N \cap (M_i + L) = 0$, which contradicts the maximality of L.

(iii) $\Rightarrow$ (i): induction on the dimension □

cref for item

2.7 Definition. An A-module is called *semi-simple* if it satisfies the conditions of Theorem 2.6.

2.8 Remark. (i) Any $\mathbb{C}[G]$-module is semi-simple (by Maschke, Theorem 1.15).

(ii) Any $k[G]$-module is semi-simple if $\operatorname{char} k \nmid |G|$.

2.9 Definition. If M is an A-module and $N \leq M$, then we define an A-module structure on the k-vector-space M/N by $a.(x + N) = a.x + N$. This is called the *factor-module.*

2.10 Proposition. *Submodules and homomorphic images of semi-simple modules are semi-simple.*

Proof. Let $M = M_1 + \ldots + M_r$, $M_i \leq M$ simple subodules. Let $\varphi : M \to P$ be a surjective morphism of A-modules. Then

$$P = \varphi(M) = \varphi(M_1) + \ldots + \varphi(M_r)$$

By Schur (Lemma 2.5) either $\varphi(M_i) = 0$ or $\varphi(M_i) = M_i$ which is simple. So P is semi-simple.
Let $N \leq M$ be some submodule and $L \leq M$ a module-complement. Then $M = N \oplus L$. Let $\pi : M \to N$ be the projection along L. Then by the first part $N = \pi(M)$ is semi-simple. □

2.11 Theorem (Uniqueness of decomposition). *Assume M is an A-module and*

$$M = S_1 \oplus \ldots \oplus S_m = T_1 \oplus \ldots \oplus T_n$$

with $S_i, T_j \leq M$ simple. Then $n = m$ and there is some π with $S_i \cong T_{\pi(j)}$.

Again we have a well-defined notion of multiplicity $\operatorname{mult}_W(M)$ of a simple module W in M.

continue

2.2 Isotypical Decompositions

2.12 Definition. An A-module M is called *W-isotypical* if it is a (direct) sum of simple submodules $\cong W$.
Let M be semi-simple. We call $M_W := \sum_{N \leq M, N \cong W} N$ the W-isotypical component of M, if $M_W \neq 0$.

Although M_W is defined by a possibly infinite sum, it is well-defined, since only finitely many summands actually contribute.

2.13 Lemma. *Let $M = M_1 \oplus \ldots \oplus M_r$, with M_i simple. Then $M_W = \bigoplus_{m_i \cong W} M_i$.*

Proof. Let $l = \mathrm{mult}_W(M)$. Wlog $M_i \cong W$ for exactly $i \leq l$. Clearly $M_W \supseteq M_1 \oplus \ldots \oplus M_l$. Suppose $W \cong L \leq M$, but $L \nsubseteq M_1 \oplus \ldots \oplus M_l$. Then $L \cap (M_1 \oplus \ldots \oplus M_l) = 0$ since L is simple. Put $N := (M_1 \oplus \ldots \oplus M_l) \oplus L$. Take the complement of N, i.e. $M = N \oplus T$ with $T = S_1 \oplus \ldots \oplus m$ and S_i simple. This yields a decomposition

$$M = M_1 \oplus \ldots \oplus M_l \oplus L \oplus S_1 \oplus \ldots \oplus S_m$$

so $\mathrm{mult}_W(M) \geq l + 1$, which is a contradiction. $\square$

2.14 Corollary. *Let M be semi-simple.*

(i) We have $M = \bigoplus_{i=1}^r M_{W_i}$ is $W_1, \ldots, W_r$ is an isomorphism list of simple modules occurring in M. This is called an isotypical decomposition.

(ii) Let $\varphi : M \to P$ be a module morphism. Then $\varphi(M_W) \leq P_W$ with equality if φ is surjective.

(iii) If $N \leq M$ is a submodule, then $N_W = M_W \cap N$.

Proof. (i) follows directly from Lemma 2.13

(ii) $\varphi(M_W) \leq P_W$ by Schur (Lemma 2.5). If φ is surjective, then by item (i) we have

$$\varphi(M) = \sum_{i=1}^{r} \varphi\left(M_{W_i}\right) \leq \bigoplus_{i=1}^{r} P_{W_i} = P = \varphi(M)$$

(iii) Apply item (ii) to a projection $\pi : M \to N$ along a module complement. $\square$

Now consider the special case $A = \mathbb{C}[G]$.

2.15 Remark. The isotypical component of the trivial module $\chi = 1$ is the submodule of G-invariants $V^G = \{v \in V : \forall g \in G. g.v = v\}$.

Let $W \in \mathrm{mod}\, G$ simple, $\chi := \chi_W$. Consider

$$a := \frac{\dim W}{|G|} \sum_{g \in G} \overline{\chi(g)} g \in \mathbb{C}[G]$$

Since χ is a class function, we even have $a \in Z(\mathbb{C}[G])$. Let $V \in \mathrm{mod}\, G$ with corresponding representation $D : \mathbb{C}[G] \to \mathrm{End}(V)$. Then $D(a) \in \mathrm{End}_G(V)$.
Now suppose V is simple. By Schur there is some $\lambda \in \mathbb{C}$ such that $D(a) = \lambda \cdot \mathrm{id}_V$. Then

$$\lambda \cdot \dim V = \mathrm{tr}(D(a)) = \frac{\dim W}{|G|} \sum_{g \in G} \overline{\chi(g)}\, \underbrace{\mathrm{tr}(D(g))}_{\chi_V(g)} = \dim W \cdot \langle \chi_V, \chi \rangle \implies \lambda = \begin{cases} 0 & : V \ncong W \\ 1 & : V \cong W \end{cases}$$

Now suppose $V = V_1 \oplus \ldots \oplus V_r$ is the isotypical decomposition and V_i is W-isotypical. Then the representation D of V satisfies

$$D(a)_{|V_i} = \mathrm{id}$$
$$\forall j \neq i. D(a)_{|V_j} = 0$$

Therefore $D(a)$is the projection of V onto V_i along $\bigoplus_{j \neq i} V_j$. Furthermore we have an explicit description of this projection in terms of characters.

Example. *For the trivial character the projection $V \to V^G$ is given by*

$$a = \frac{1}{|G|} \sum_{g \in G} g$$

Now we return to the general situation of A-modules; let $W \in \operatorname{mod} A$ simple. Let M be some W-isotypical modules. We investigate the submodules of M and $\operatorname{End}_A(M)$. We decompose $M = M_1 \oplus \ldots \oplus M_m$ with isomorphisms $\sigma_i : W \to M_i$, so $\operatorname{mult}_W(M) = m$. We regard k^m as the trivial A-module and form the A-module $W \otimes k^m$. So $a(w \otimes x) = (aw) \otimes x$ for $a \in A, w \in W, x \in k^m$. Our k-linear map is $D(a) \otimes \operatorname{id}_{k^m} : W \otimes k^m \to W \otimes k^m$.

2.16 Lemma. *The map $\tau : W \otimes k^m \to M$ induced by $w \otimes x \mapsto \sum x_i \sigma_i(w)$ is an A-module isomorphism.*

Proof.
- τ is well-defined. The map $(w, x) \mapsto \sum x_i \sigma_i(w)$ is bilinear, then use universal property.
- τ is an A-module morphism $a.(w \otimes x) = (a.w) \otimes x$ because

$$\tau(a(w \otimes x)) = \sum_i x_i \sigma_i(a.w) = a.\sigma_i x_i \sigma_i(w) = a.\tau(w \otimes x)$$

- $\operatorname{im} \tau = \sum M_i$, so τ is surjective.
- τ is an isomorphism since

$$\dim(W \otimes k^m) = m \cdot \dim W = \dim M \qquad \square$$

2.17 Lemma. *Assume k is algebraically closed. The map $\rho : k^{m \times m} \to \operatorname{End}_A(W \otimes k^m)$ induced by $b \mapsto \operatorname{id}_W \otimes b$ is an algebra-isomorphism.*

Proof.
- $\rho(b)$ is an A-module-isomorphism since

$$\rho(b)(a.(w \otimes x)) = \rho(b)(a.w \otimes x) = (a.w) \otimes (b.x) = \ldots = a.\rho(b)(w \otimes x)$$

- ρ is k-linear,$\rho(I_m) = \operatorname{id}$.

$$\rho(bc)(w \otimes x) = w \otimes (bc.x) = \rho(b).(w \otimes c.x) = \rho(b)\rho(c).(w \otimes x)$$

  So it is an algebra-morphism.
- For $\varphi \in \operatorname{End}_A(\bigoplus_1^m W)$ there exist $\varphi_{ij} \in \operatorname{End}_k(W)$ such that

$$\varphi(w_1 \oplus \ldots \oplus w_m) = \bigoplus_{i=1}^m (\varphi_{i1}(w_1) \oplus \ldots \oplus \varphi_{im}(w_m))$$

  We view (φ_{ij}) as the "representation matrix" of φ. The map $\varphi \mapsto [\varphi_{ij}]$ (as matrix) is a k-linear isomorphism. Moreover

$$\varphi \in \operatorname{End}_A\left(\bigoplus_1^m W\right) \Leftrightarrow \forall i, j. \varphi_{ij} \in \operatorname{End}_A(W)$$

By Schur there exist $b_{ij} \in k$ such that $\varphi_{ij} = b_{ij} \operatorname{id}_W$. Then we get a linear isomorphism

$$k^{m\times m} \to \operatorname{End}_A\left(\bigoplus_1^m W\right) \quad b = [b_{ij}] \mapsto \Big(w_1 \oplus \ldots \oplus w_m \mapsto \bigoplus (b_{i1}w_1 \oplus \ldots \oplus b_{im}w_m)\Big)$$

This is clearly an algebra-isomorphism. We get an algebra-isomorphism

$$k^{m\times m} \xrightarrow{\sim} \operatorname{End}_A\left(\bigoplus_1^m W\right) \xrightarrow{\sim} \operatorname{End}_A(W \otimes k^m) \qquad \square$$

The image of $b \in k^{m\times m}$ is indeed $(w \otimes x \mapsto w \otimes (b.x))$.

$$W \otimes k^m \xrightarrow{\sim} \bigoplus_1^m W \to \bigoplus_1^m W \to W \otimes k^m$$

$$w \otimes x \mapsto x_1 w \oplus \ldots \oplus x_m w \to \bigoplus_i \left(\sum_j b_{ij}x_j\right) w \mapsto w \otimes (b.x)$$

Let V be a W-isotypical module $V \cong W \oplus \ldots \oplus W \cong W \otimes k^m$. Take $w \otimes x \in V$ and some $a \in A$. Then $a(w \otimes x) = (aw) \otimes x$.

2.18 Lemma. *Let $X \subseteq k^n$ be some linear subspace. Then $U := W \otimes X$ is a submodule of $W \otimes k^m$ and all submodules are obtained this way. Moreover X is uniquely determined by the space $U := W \otimes X$.*

Proof. (i) It is clear that $W \otimes X$ is a submodule, because $a \in A$ operates only on the first component.

(ii) Let $U \leq W \otimes k^m$ be a submodule and $p \in \operatorname{End}_A(W \otimes k^m)$ be a projection onto U. By Lemma 2.17 there exists some $b \in k^{m\times m}$ such that $p = \operatorname{id}_W \otimes p$. Then $U = \operatorname{im} p = W \otimes \operatorname{im} b$. So we just take $X = \overline{nimb}$.

(iii) Let $W \otimes_1^X = W \otimes X_2$. Then we have $X_1 = X_2$ for subspaces $X_1, X_2 \subseteq k^m$. [Exercise] $\square$

2.19 Theorem. *Let M be a semi-simple A-module and $M = M_1 \oplus \ldots \oplus M_r$ by isotypical decomposition. Let $M_i \cong W_i \otimes k^{m_i}$ for simple A-modules W_i.*

(i) For subspaces $X_i \leq k^{m_i}$ we have that $\bigoplus_{i=1}^r W_i \cdot \otimes X_i$ is a submodule of M. Every submodule of M is obtained this way (with unique X_i).

(ii) We have an algebra-isomorphism

$$\prod_{i=1}^r k^{m_i \times m_i} \xrightarrow{\sim} \operatorname{End}_A(M)$$

$$(b_1, \ldots, b_r) \mapsto \left(\bigoplus_{i=1}^r w_i \otimes r_i \mapsto \sum_{i=1}^r w_i \otimes b_i r_i\right)$$

Proof. (i) Let $U \leq M$ be some submodule. Then we have the isotypical decomposition

$$U = \bigoplus_{i=1}^r (M_i \cap U)$$

and use Lemma 2.18.

(ii) For $\varphi \in \mathrm{End}_A(M)$ there exist (unique) $\varphi_{ij} \in \mathrm{Hom}_A(M_j, M_i)$ such that

$$\varphi(v_1 \oplus \ldots \oplus v_r) = \bigoplus_{i=1}^{r} (\varphi_{i1}(v_1) \oplus \ldots \oplus \varphi_{ir}(v_r)) = \bigoplus_{i=1}^{r} \varphi_{ii}(v_i)$$

because by Schur $\varphi_{ij} = 0$ if $i \neq j$. So $\varphi_{ii} \in \mathrm{End}_A(M_i)$. By Lemma 2.17 we have $k^{m_i \times m_i} \to \mathrm{End}_A(M_i)$. Hence there exist $b_i \in k^{m_i \times m_i}$ such that

$$\varphi_{ii} = \mathrm{id}_{W_i} \otimes b_i \qquad \varphi_{ii}(w_i \otimes x_i) = w_i \otimes b_i x_i$$

□

2.3 Semi-simple Algebras

2.20 Definition. A k-algebra A is called *semi-simple* if every A-module is semi-simple.

Example. *Let G be a finite group and $k = \mathbb{C}$ (in fact $\operatorname{char} k = 0$ suffices). Then $k[G]$ is semi-simple.*

2.21 Remark. Let $L \leq A$ be a subspace. L is a submodule of A-module A iff

$$\forall a \in A. \forall x \in L. ax \in L$$

Such L is called a *left-ideal* of A. L is called a *minimal left-ideal* if $L \neq \{0\}$ and L does not contain a proper left-ideal.

This means L is a simple A-module. Hence A is semi-simple iff there exist minimal left-ideals $L_1, \ldots, L_r$ such that $A = L_1 \oplus \ldots \oplus L_r$.

2.22 Theorem. *(i) If A-module A is semi-simple then algebra A is semi-simple.*

(ii) Assume $A = L_1 \oplus \ldots \oplus L_r$ for minimal left-ideals. Every simple A-module is isomorphic to some L_i.

Proof. (i) Let M be an A-module and $(f_1, \ldots, f_n)$ a k-basis of M. Consider $\varphi^n_A \to M$ via $(a_1, \ldots, a_n) \mapsto \sum a_i f_i$, which is a surjective module morphism. If A-module A is semisimple, then A-module A^n is semisimple, so M is semisimple.

(ii) Take M and φ as before. Next we regard

$$A^n = A \oplus \ldots \oplus A = (L_1 \oplus \ldots \oplus L_r) \oplus \ldots \oplus (L_1 \oplus \ldots \oplus L_r)$$

as A-module. Assume M is simple. We know $0 \neq M = \varphi(A^n)$, so at least one of the L_i is not mapped to 0. By Schur's Lemma $L_i \cong M$. □

Example. *(i) Let $A = k$, then k is the only minimal left ideal, k is semisimple. There is up to isomorphy only the simple k-module k (see Linear Algebra). Every k-module is isomorphic to some k^n, with $n \in \mathbb{N}$.*

second example

2.23 Definition. Let M be an A-module. Then we define the *annihilator*

$$\mathrm{ann}_A(M) := \{a \in A : \forall x \in M. ax = 0\}$$

2.24 Remark. (i) $\mathrm{ann}_A(M)$ is an ideal of A.

(ii) If $M \xrightarrow{\sim} N$ then $\mathrm{ann}_A(M) = \mathrm{ann}_A(N)$.

2.25 Lemma. *(i) If A and B are semisimple algebras then so is $A \times B$.*

(ii) If $A = \bigoplus_{i=1}^r L_i$ and $B = \bigoplus_{j=1}^s \Lambda_j$ are decompositions into minimal left ideals then

$$A \times B = (L_1 \times 0) \oplus \ldots \oplus (L_r \times 0) \oplus (0 \times \Lambda_1) \oplus \ldots \oplus (0 \times \Lambda_s)$$

is a decomposition into minimal left ideals. Moreover $L_i \times 0 \ncong 0 \times \Lambda_j$ as $A \times B$-modules.

Proof.

- $L_i \times 0$ and $0 \times \Lambda_j$ are minimal left-ideals: clear
- decomposition of $A \times B$ is clear. Thus $A \times B$ is semisimple. For the annihilators we have

$$\begin{aligned}\mathrm{ann}_{A\times B}(L_i \times 0) &= \mathrm{ann}_A(L_i) \times B \\ \mathrm{ann}_{A\times B}(0 \times \Lambda_i) &= A \times \mathrm{ann}_A(\Lambda_i)\end{aligned}$$

Moreover $\mathrm{ann}_A(L_i) \neq A$, since $L_i \neq 0$. Therefore the annihilators are different, which means $L_i \times 0 \ncong 0 \times \Lambda_j$. □

2.26 Theorem. *$\mathscr{A} := k^{n_1\times n_1} \times \ldots \times k^{n_r \times n_r}$ is a semisimple algebra. There are exactly r isomorphism classes of simple $\mathscr{A}$-modules, which are given by $k^{n_1}, \ldots, k^{n_r}$, where $\mathscr{A}$ operates on k^{n_i} by $(a_1, \ldots, a_r).v = a_i v$.*

Proof. For simplicity we just show this for $r = 2$ and use $A = k^{n\times n}$ and $B = k^{m\times m}$. For these we have the decomposition $A = L_1 \oplus \ldots \oplus L_n$ and $B = \Lambda_1 \oplus \ldots \oplus \Lambda_m$, where $L_i \cong k^n$ as A-module and $\Lambda_j \cong k^m$ as B-module. Applying Lemma 2.25 we get that $A \times B$ is semisimple and the isomorphism list is given by $k^n \times 0 \cong k^n$ and $0 \times k^m \cong k^m$:
$A \times B$ opertaes on k^n by $(a, b).v = av$ and similarly on B. □

2.27 Theorem. *Let k be algebraically closed. Any semisimple k-algebra A is isomorphic to an algebra $k^{n_1\times n_1} \times \ldots \times k^{n_r\times n_r}$.*

Example. *Let $k = \mathbb{R}$, consider field $\mathbb{C}$ as $\mathbb{R}$-algebra. The only non-zero ideal is $\mathbb{C}$ itself, so $\mathbb{C}$ is semisimple.*
Can we write $\mathbb{C} = \prod \mathbb{R}^{n_i\times n_i}$? We must have $n_i = 1$, since $\mathbb{C}$ is commutative, but $\mathbb{C} \ncong \mathbb{R} \times \mathbb{R}$. So Wedderburn does not hold over the reals.

2.28 Definition. An A-module M is claled faithful if the corresponding representation $D : A \to \mathrm{End}(M)$ is injective.

2.29 Corollary. *Suppose $A = k^{n_1\times n_1} \times \ldots \times k^{n_r\times n_r}$ and M is an A-module. Suppose in M there occur exactly the simple A-modules $k^{n_1}, \ldots, k^{n_s}$, $s \leq r$. Then*

$$\mathrm{ann}_A(M) = \underbrace{0 \times \ldots \times 0}_{s} \times k^{n_{s+1}\times n_{s+1}} \times \ldots \times k^{n_r\times n_r}$$

In particular M is faithful $\Leftrightarrow \mathrm{ann}_A(M) = 0 iffs = r \Leftrightarrow$ all types of simple A-modules occur in M.

2.30 Theorem. *Assume k is algebraically closed. Let V be a semisimple A-module. Then $\mathrm{End}_A(V)$ is a semisimple algebra.*

Proof. If $V = V_1 \oplus \ldots \oplus V_r$ is the isotypical decomposition, then

$$\operatorname{End}_A(V) \cong k^{n_1 \times n_1} \times \ldots \times k^{n_r \times n_r}$$

where $V_i \cong w_i \otimes k^{m_i}$ with W_i simple. $\square$

2.4 Endomorphism algebras

large gap

3 Representation of S_N and $\mathrm{GL}(V)$

3.1 Polarisation and Restitution

3.4 Definition. Let $\dim_k V = m$, where $\operatorname{char} k = 0$, and $N \geq 1$. Then we define $V^{\otimes N} := V \otimes \ldots \otimes V$. Note that $V^{\otimes N}$ is an S_N-module.

3.5 Remark. Further we put $S^N(V) := \left\{ t \in V^{\otimes N} : \forall \pi \in S_n : \pi.t = t \right\}$ the subspace of symmetric tensors. We have a projection onto isotpical components with respect to the trivial S_n-representation.

$$\varphi : V^{\otimes N} \to S^N(V) \qquad\qquad t \mapsto \frac{1}{N!} \sum_{\pi \in S_N} \pi.t$$

This also gives another map

$$V^N \to S^N(V) \qquad\qquad (v_1, \ldots, v_N) \mapsto \varphi(v_1 \otimes \ldots \otimes v_N)$$

This is a multilinear S_N-invariant map.

3.6 Proposition. *Let $\{e_1, \ldots, e_m\}$ be a basis of V.*

(i) $\left\{ e_1^{\lambda_1} \ldots e_m^{\lambda_m} : \sum \lambda_i = N \right\}$ *is a basis of* $S^N(V)$.

(ii) $\dim S^N(V) = \binom{N+m-1}{m-1}$

We can interpret $S^N(V)$ as the space of homogeneous polynomial of degree N in m variables.

3.7 Definition. An element $t \in V^{\otimes N}$ is called *alternating* if

$$\forall \pi \in S_N : \pi.t = \operatorname{sgn}(\pi) \cdot t$$

We put $\Lambda^N(V) := \left\{ t \in V^{\otimes N} : t \text{ alternating} \right\}$.

3.8 Remark. The map

$$\psi : V^{\otimes N} \to \Lambda^N(V) \qquad\qquad t \mapsto \frac{1}{N!} \sum_{\pi \in S_N} \operatorname{sgn}(\pi) \pi.t$$

is a projection onto the isotypical components of the sgn-module.
This yields the alternating product (wedge product)

$$V^N \to \Lambda^N V) \qquad\qquad (v_1, \ldots, v_N) \mapsto \psi(v_1 \otimes \ldots \otimes v_N) =: v_1 \wedge \ldots \wedge v_N$$

which is multilinear, but antisymmetric.

3.9 Proposition. *(i) $\{e_{i_1} \wedge \ldots \wedge e_{i_N} : 1 \leq i_1 < \ldots < i_N \leq m\}$ is a basis of $\Lambda^N(V)$.*

(ii) For the dimension we have

$$\dim \Lambda^N(V) = \begin{cases} \binom{m}{N} & : N \leq m \\ 0 & : \textit{else} \end{cases}$$

Proof. We only need to check linear independence.

$$0 = \sum \alpha_{i_1 \ldots i_N} e_{i_1} \wedge \ldots \wedge e_{i_N} = \sum \alpha_{i_1 \ldots i_N} \frac{1}{N!} \sum_{\pi \in S_N} e_{i_{\pi(1)}} \otimes \ldots \otimes e_{i_{\pi(N)}}$$

Then the coefficient of $e_{j_1} \otimes \ldots \otimes e_{j_N}$ equals $\pm \frac{\alpha_{j_1 \ldots j_N}}{N!}$, hence $\alpha_{j_1 \ldots j_N} = 0$, since the tensor products of the e_i form a basis of $V^{\otimes N}$. □

Example. *Let $N = 2$, and $\{e_1, \ldots, e_m\}$ basis of V. Then $V^{\otimes 2} = V \otimes V$ has basis $\{e_i \otimes e_j : i, j \leq m\}$. Then we have the linear isomorphism*

$$V \otimes V \to k^{m \times m} \qquad t := \sum_{i,j} \alpha_{i,j} e_i \otimes e_j \mapsto (\alpha_{i,j})_{1 \leq i,j \leq m} =: A$$

For example $(12)t \mapsto A^T$, so we just transpose. Furthermore t symmetric iff A symmetric and t alternating iff A skew-symmetric (i.e. $A^t = -A$).
When observing the dimension we get

$$\dim S^2(V) = \binom{m+1}{2} \qquad \dim \Lambda^2(V) = \binom{m}{2}$$

$$\dim S^2(V) + \dim \Lambda^2(V) = m^2 = \dim(V \otimes V)$$

In particular $V \otimes V = S^2(V) \oplus \Lambda^2(V)$.
In the more general setting $N > 2$ we still have $S^N(V) \cap \Lambda^N(V) = 0$ but $S^N(V) \oplus \Lambda^N(V) \subset V^{\otimes N}$.

3.10 Theorem. *$S^N(V)$ is generated by $\{v^N : v \in V\}$ (symmetric product).*

Proof.

$$\begin{aligned} \left(\sum_{i=1}^m \xi_i e_i \right)^N &= \sum_{i_1 < \ldots < i_N} \xi_{i_1} \ldots \xi_{i_N} e_{i_1} \ldots e_{i_N} \\ &= \sum_{|\lambda|_1 = N} \#\{(i_1, \ldots, i_N) : j \text{ occurs } \lambda_j \text{ times in } (i_1, \ldots, i_N)\} \xi_1^{\lambda_1} \ldots \xi_m^{\lambda_m} e_1^{\lambda_1} \ldots e_m^{\lambda_m} \end{aligned}$$

Next we claim that the coefficient vectors the following vector space

$$k^{\binom{N+m-1}{m-1}} = \left\langle \left(\frac{N!}{\lambda_1! \cdot \ldots \cdot \lambda_m!} \xi_1^{\lambda_1} \ldots \xi_m^{\lambda_m} \right)_\lambda \right\rangle$$

Otherwise there is some $0 \neq \alpha_{\lambda_1 \ldots \lambda_m} \in k^{\binom{N+m-1}{m-1}}$ such that

$$\forall \xi : \sum_\lambda \alpha_{\lambda_1 \ldots \lambda_m} \frac{N!}{\lambda_1! \ldots \lambda_m!} \xi_1^{\lambda_1} \ldots \xi_m^{\lambda_m} = 0$$

Regarding this as a multivariate polynomial, we get $\forall \lambda : \alpha_\lambda = 0$. $\lightning$ □

3.11 Definition. A map $f : V \to k$ is called *polynomial map* if

$$\exists F \in k[Z_1, \dots, Z_m] : \forall \xi \in k^m : f\left(\sum_{i=1} \xi_i e_i\right) = F(\xi)$$

3.12 Remark. (i) The concept of a polynomial map is independent of the basis.

(ii) $k[V] := \{f : V \to k, \text{polynomial}\}$ is a subalgebra of the algebra of maps $V \to k$. With respect to the chosen basis $e_1, \dots, e_m$ we have the algebra isomorphism

$$\mathbb{C}[Z_1, \dots, Z_m] \xrightarrow{\sim} \mathbb{C}[V] \qquad\qquad Z_i \mapsto e_i^*$$

$k[V]$ is generated by V^* as a k-algebra (this is possible, since we have a coordinate-free definition).

(iii) The notion of homogeneous polynomial functions if well-defined (basis-independent), so we have to show

$$\forall t \in k : \forall v \in V : f(tv) = t^M \cdot f(v)$$

To this end we write

$$k[V]_{(M)} := \{f \in k[V] : f \text{ homogeneous of degree } M\}$$

Then we have the decomposition

$$k[V] = \bigoplus k[V]_{(M)} \qquad\qquad k[V]_{(0)} = k \qquad\qquad k[V]_{(1)} = V^*$$

3.13 Lemma. *We have the isomorphism*

$$(V^*)^{\otimes N} \xrightarrow{\sim} \left(V^{\otimes N}\right)^* \qquad\qquad l_1 \otimes \dots \otimes l_N \mapsto (x_1 \otimes \dots \otimes x_N \mapsto l_1(x_1) \otimes \dots \otimes l_N(x_N))$$

Take $f \in \left(V^{\otimes N}\right)^*$. Define $\phi f : V \to k$ by $\phi f(x) := f(x^{\otimes N})$. Then $\phi f \in k[V]_{(N)}$. Thus we obtain a linear map

$$(V^*)^{\otimes N} \xrightarrow{\sim} \left(V^{\otimes N}\right)^* \xrightarrow{\phi} k[V]_{(N)}$$

For this we claim that the restriction of this map yields a linear isomorphism

$$\phi_0 : S^N(V^*) \xrightarrow{\sim} k[V]_{(N)}$$

This map ϕ_0 is called*restitution*, its inverse ϕ_0^{-1} is called *polarisation.*

of claim. It is clear that ϕ_0 is a linear map. Choose basis $\{e_1, \dots, e_m\}$ of V, so we have its dual basis $\{e_1^*, \dots, e_m^*\}$ of V^*. A basis of $S^N(V^*)$ then is

$$\left((e_1^*)^{\lambda_1} \dots (e_m^*)^{\lambda_m}\right)_{|\lambda|=N}$$

□