Discretisation & Control of Irregularly Actuated and Sampled LTI systems

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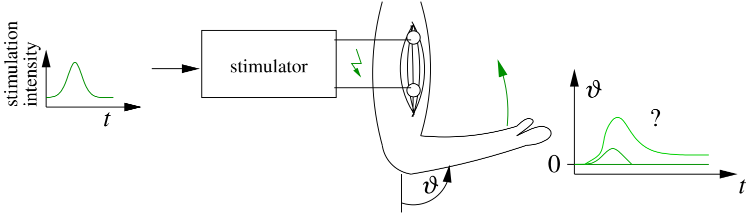
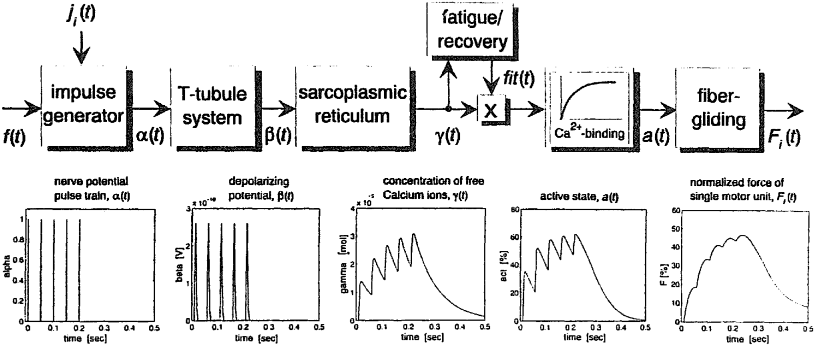
Functional Electrical Stimulation (FES)

- Application of electrical current pulses to a muscle for inducing force / movements in Neuro-Prosthetic Systems.
- The intensity of the pulses can be modulated.
- The time instant for each pulse can be arbitrarily chosen (typical frq. 20 to 60 Hz)





Taken from: R. Riener and J. Quintern, A physiologically based model of muscle activation verified by electrical stimulation, 1997

Muscle fatigue vs. control performance

- The progressing of muscle fatigue significantly depends on the rate of the applied pulses.
- However, more frequently applied pulses typically enable a better performance in discrete-time control systems.
- For temporarily increasing the control performance, a dynamic adaption of the sampling intervals is helpful.

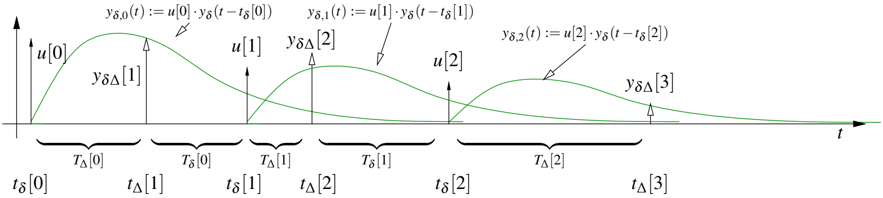
Continuous-time LTI SISO systems are considered.

$$y(t) = \mathbb{S}[v(t)]\,(t), \qquad t \geq 0$$

State-space representation for $t \geq 0$:

$$\mathbb{S} \equiv \left\{ \begin{array}{ll} \dot{\boldsymbol{x}}(t) = \mathcal{A}x(t) + \mathcal{B}v(t), & \boldsymbol{x} \in \mathbb{R}^n,\, v, y \in \mathbb{R} \\ y(t) = \mathcal{C}\boldsymbol{x}(t), & \boldsymbol{x}(t=0) = \boldsymbol{x}_{\mathbf{0}} \end{array} \right\}. \tag{1}$$

- No direct feedthrough



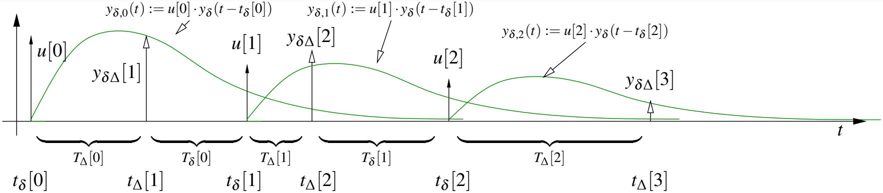
Sequences:

- Actuation: $u = \{u[0], u[1], \cdots\}$
- Sampled output: $y_{\delta\Delta} = \{y_{\delta\Delta}[0], y_{\delta\Delta}[1], \cdots\}$
- Time instants for actuation $t_\delta = \{t_\delta[0], t_\delta[1], \cdots\}$
- Time instants for sampling $t_\Delta = \{t_\Delta[1], t_\Delta[2], \cdots\}$
- Sampling time intervals $T_\delta[k], T_\Delta[k]$ (derived from t_δ, t_Δ)
- Actuation and sampling must be alternating

Continuous-time input signal applied to $\mathbb{S}$:

$$v_\delta(t) := \sum_{j=0}^{\infty} u[j]\, \delta(t - t_\delta[j]), \qquad j \in \mathbb{N}, \tag{2}$$

$\delta(t)$: Dirac delta distribution



A superposition of the individual responses and the initial dynamics $y_\delta^P(t)$ yields the system's output

$$y_\delta(t) = \sum_{j=0}^{\infty} \underbrace{u[j] y_\delta(t - t_\delta[j])}_{y_{\delta,j}(t)} + y_\delta^P(t).$$

Herein, y_δ is the impulse response.
Sampling of $y_\delta(t)$ at $t = t_\Delta[k]$:

$$\begin{aligned} y_{\delta\Delta}[k] &= y_\delta(t_\Delta[k]) = \\ &= \mathbb{S}\left[\sum_{j=0}^{\infty} u[j]\delta(t - t_\delta[j])\right](t_\Delta[k]) \\ &=: \mathcal{Z}_{\delta\Delta}\left[\mathbb{S}, u, t_\delta, t_\Delta\right][k] \end{aligned}$$

The transformation $\mathcal{Z}_{\delta\Delta}$ maps the actuation sequence and the time instants to the output sequence.

Linearity of $\mathcal{Z}_{\delta\Delta}$ w.r.t to the relationship between u and y

If the system $\mathbb{S}$ is a linear system in the sense of

$$\mathbb{S}\left[av_1(t) + bv_2(t)\right](t) = a\,\mathbb{S}\left[v_1(t)\right](t) + b\,\mathbb{S}\left[v_2(t)\right](t),$$

whereby $a, b \in \mathbb{R}$, then the I/O relationship of the sampled system is also linear:

$$\begin{aligned} & \mathcal{Z}_{\delta\Delta}\left[\mathbb{S}, au_1 + bu_2, t_\delta, t_\Delta\right][k] \\ = \quad & a\,\mathcal{Z}_{\delta\Delta}\left[\mathbb{S}, u_1, t_\delta, t_\Delta\right][k] + b\,\mathcal{Z}_{\delta\Delta}\left[\mathbb{S}, u_2, t_\delta, t_\Delta\right][k]. \end{aligned}$$

The proof is straight forward.

A first order system $\mathbb{S}^1$

$$\mathbb{S}^1 \equiv \left\{ \begin{array}{ll} \dot{x}(t) = s_\infty x(t) + v(t), & x, y, v, s_\infty, x_o \in \mathbb{C} \\ y(t) = x(t), & x(t=0) = x_0. \end{array} \right\}$$

Performed steps to obtain a relationship between the in- and output sequence:

- Derived an analytical solution for the system output.
- Sampled the output at $t_\Delta[k]$.
- Derived a recursive formulation for the relation between t_Δ and t_δ.

Resulting state-space system

$$\left\{ y_{\delta\Delta} = \mathcal{Z}_{\delta\Delta}\left[\mathbb{S}^1, u, t_\delta, t_\Delta\right][k] \right\}$$

$$\equiv \left\{ \begin{array}{l} y_{\delta\Delta}[k+1] = y_{\delta\Delta}[k] \cdot e^{s_\infty[T_\delta[k-1]+T_\Delta[k]]} + u[k] e^{s_\infty T_\Delta[k]} \\ y_{\delta\Delta}[0] = x_0 \end{array} \right\}$$

- The coefficients depend on time-intervals.

Second order system $\mathbb{S}^{cc}$:

$$\mathbb{S}^{cc} \equiv \left\{ \begin{array}{ll} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \lambda + \overline{\lambda} & -\lambda\overline{\lambda} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \boldsymbol{B}v(t), \quad \boldsymbol{B} \in \mathbb{R}^2 \\ y(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} \boldsymbol{x}(t) \qquad \boldsymbol{x}(0) = \begin{pmatrix} x_{1,0} \\ x_{2,0} \end{pmatrix} \end{array} \right\}$$

Eigenvalues: $\lambda = \sigma + j\omega$, $\overline{\lambda} = \sigma - j\omega$
Ongoing, two different classes of systems are considered:

A) $\mathbb{S}^{cc}_A$ that derives from $\mathbb{S}^{cc}$ by choosing $\boldsymbol{B} = \boldsymbol{B_A} := [1, 0]^T$ and realises the transfer function $1/((s-\lambda)(s-\overline{\lambda}))$.
B) $\mathbb{S}^{cc}_B$ as derived from $\mathbb{S}^{cc}$ by choosing $\boldsymbol{B} = \boldsymbol{B_B} := [-a + \lambda + \overline{\lambda}, 1]^T$ that realises $(s-a)/((s-\lambda)(s-\overline{\lambda}))$.

- Transformation into diagonal form and re-usage of the results for first order systems.

The I/O relationship of the discretisation of $\mathbb{S}^{cc}$ is described by

$$\begin{aligned}
\boldsymbol{z}^{\boldsymbol{cc}}[k+1] &= e^{\sigma T_s[k]} \begin{bmatrix} \cos(\omega T_s[k]) & -\sin(\omega T_s[k]) \\ \sin(\omega T_s[k]) & \cos(\omega T_s[k]) \end{bmatrix} \cdot \boldsymbol{z}^{\boldsymbol{cc}}[k] \\
&+ e^{\sigma T_\Delta[k]} \begin{bmatrix} \cos(\omega T_\Delta[k]) \\ \sin(\omega T_\Delta[k]) \end{bmatrix} u[k] \\
y_{\delta\Delta}[k] &= [2r\cos\phi, 2r\sin\phi] \cdot \boldsymbol{z}^{\boldsymbol{cc}}[k], \qquad T_s[k] := T_\delta[k-1] + T_\Delta[k] \\
\boldsymbol{z}_{\boldsymbol{cc}}[0] &= \boldsymbol{z}_{\boldsymbol{0}}^{\boldsymbol{cc}}, \quad \boldsymbol{z}_{\boldsymbol{cc}} \in \mathbb{R}^2
\end{aligned}$$

A) $r = r_A = 1/(2\omega),\ \phi = \phi_A = \pi/2$

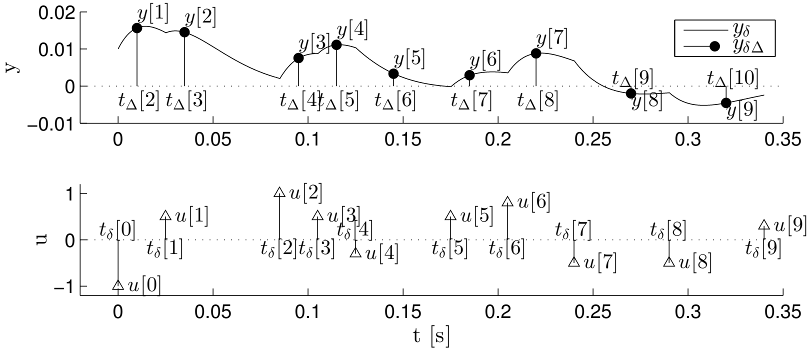
B) $r = r_B = \frac{1}{2}\sqrt{1 + [(\sigma - a)/\omega]^2},\ \phi = \phi_B = \tan^{-1}[(\sigma - a)/\omega]$

Initial states

A) $\boldsymbol{z}^{\boldsymbol{cc}}_{\boldsymbol{A,0}} = \begin{bmatrix} x_{1,0} - \sigma x_{2,0} \\ \omega x_{2,0} \end{bmatrix}$

B) $\boldsymbol{z}^{\boldsymbol{cc}}_{\boldsymbol{B,0}} = \begin{bmatrix} z^R[0] \\ z^I[0] \end{bmatrix} = \begin{bmatrix} \sin\phi_B & \cos\phi_B \\ -\cos\phi_B & \sin\phi_B \end{bmatrix} \begin{bmatrix} x_{1,0} - \sigma x_{2,0} \\ \omega x_{2,0} \end{bmatrix} / (2\omega r_B)$

- Discretisation of a second order system.
- $\lambda = -40 + j30$, $x_{1,0} = 2$, $x_{2,0} = 0.01$, Case A) (without zeros)
- Predefined sequences for t_δ, t_Δ and u were applied.



A discrete-time plant in state-space representation is considered:

$$\begin{aligned} \boldsymbol{z}[k+1] &= \boldsymbol{A}_{\delta\boldsymbol{\Delta}}\boldsymbol{z}[k] + \boldsymbol{B}_{\delta\boldsymbol{\Delta}}u[k] \\ y_{\delta\Delta}[k] &= \boldsymbol{C}_{\delta\boldsymbol{\Delta}}\boldsymbol{z}[k]. \end{aligned}$$

The output of the discrete-time system for the next time step is:

$$y_{\delta\Delta}[k+1] = \boldsymbol{C}_{\delta\boldsymbol{\Delta}}\boldsymbol{A}_{\delta\boldsymbol{\Delta}}\boldsymbol{z}[k] + \boldsymbol{C}_{\delta\boldsymbol{\Delta}}\boldsymbol{B}_{\delta\boldsymbol{\Delta}}u[k].$$

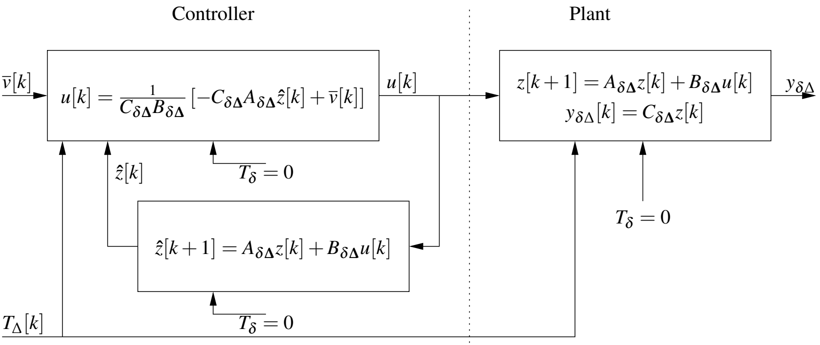
Applying the linearising controller

$$u[k] = \frac{1}{\boldsymbol{C}_{\delta\boldsymbol{\Delta}}\boldsymbol{B}_{\delta\boldsymbol{\Delta}}}\left[-\boldsymbol{C}_{\delta\boldsymbol{\Delta}}\boldsymbol{A}_{\delta\boldsymbol{\Delta}}\boldsymbol{z}[k] + \overline{v}[k]\right],$$

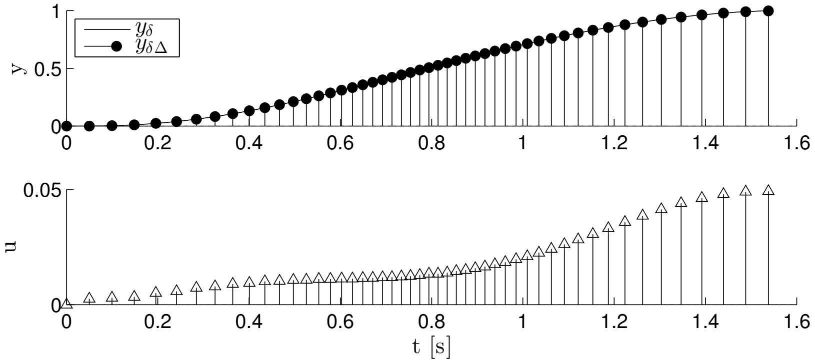
yields $y_{\delta\Delta}[k+1] = \overline{v}[k]$. The virtual input $\overline{v}$ linearly influences the sampled output.

Linearising controller: Example

- Continuous-time model described by the transfer function (rise time 0.3 s, 0.25 % overshoot) $G(s) = \frac{c}{(s-s_\infty)(s^2-2\sigma s+\sigma^2+\omega^2)}$, $\sigma = -3$, $\omega = 10$, $c = 1090$, $s_\infty = -10$
- Jordan transformation yielding: A first order and a second order system (Case B)
- An internal model is used to calculate the states $\hat{z}[k]$.



Results



- Sinusoidal shape of the actuation frequency.
- The sampled output matches the reference trajectory.

Summary & Outlook

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- FES-stimulation pulses are approximated using delta-pulses at arbitrarily time instants.
- The output is sampled at irregular sampling times.
- Discrete-time state-space systems result wherein the coefficients depend on time-intervals.
- Feed-forward control is used to linearise the effects introduced by irregular time intervals.

Outlook

- State estimation using model-based observers
- Linear feedback control on top of the linearising controller