Blind Deconvolution and Polynomial Factorization

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Motivation and Background

Blind deconvolution is within reach for applications when using cyclic extension, lifting and convex programming [Ahmed:2012] or even with gradient-based methods [Li:2016] - when the data is in lower-dimensional random subspaces. The recent demands for sporadic and short messages in next generation wireless networks have put blind strategies back into the focus in communication engineering.

Prototypical problem: infer on \((x, h)\) given noisy observation of:
\[ y_k = (h \ast x)_k = \sum h_l x_{k-l}. \]

assumptions & open questions:
- single-channel baseband: \(x\) and \(h\) are finite complex sequences
- sporadic and short messages: length of \(x\) and \(h\) are similar
- derandomized and even deterministic constructions

z-transform / conjugate-symmetry / factorization:
- \(X(z)\) of \(x\) and its reciprocal \(X^\ast(z)\) are the polynomials (in \(z^{-1} \in \mathbb{F}\)):
\[ X(z) = \sum_{k=0}^{L-1} x_k z^{-k} \quad \text{and} \quad X^\ast(z) := z^{-1} X(1/z) = \sum_{k=0}^{L-1} x^\ast_{L-1-k} z^{-k}, \]
and for convolution \(y = h \ast x\) it holds \(Y(z) = H(z)X(z)\).  
- \(X = X^\ast\) (conjugate-symmetric) \(\iff\) self-reciprocal polynomials
- Wiener-Hopf (WH) factorization \(PQ = (P, Q)\) with \(\xi_P \in \mathbb{F}\) and \(\xi_Q \in \mathbb{F}\) (fast Newton method [Boettcher:2013:WienerHopf:Real] gives directly \(p = q \Rightarrow (p, q)\)).
- For autocorrelation \(a = x \ast x^\ast\) therefore \(A = XX^\ast\) is self-reciprocal polynomial with even zeros on \(1\), \(A(1) \geq 0\)

idea / principle: avoid ambiguities (zeros separated in pre-defined regions)

Using the Wiener-Hopf Factorization

The following ideas are applicable if \(h\) is minimum phase, i.e., \(\forall \xi \in \mathbb{F}^\ast\)
- a sufficient condition: \(\forall \omega: |\sum_{k=1}^K h_k e^{i\omega k}| \leq |h_0|\) (LOS channels)

method (a) - maximum-phase signaling:
- Tx data signal \(x\) is maximum-phase (\(X^\ast\) is minimum-phase)
- Rx WH factorization of \(Y = HX\) provides \((H, X)\) up-scaling

method (b) - autocorrelation signaling:
- Tx encode information directly into \(a = x \ast x^\ast\), i.e., \(A = XX^\ast\)
- Rx receive noisy observations of \(h \ast a = h \ast (x \ast x^\ast)\)
- WH factorization for \(Y = HA = HX^\ast\) giving \((HX, X^\ast)\).
- compute \(A = XX^\ast\) and decode \(A\)

- methods are straightforward, fast WH implementation can be used
- empirically, maximum-phase constructions have undesired PAPR

Using Autocorrelations

If \(h\) is not minimum-phase but \(H\) is co-prime from \(X\) then a factorization can be obtained by an semidefinite program [JH:16]. Stack \(u = [x, h]\) and:

\[ A(\bar{u}) = A_x(\bar{x}^\ast \bar{x}) = \begin{pmatrix} x & \bar{x} \\ x^\ast & \bar{x}^\ast \end{pmatrix} A \begin{pmatrix} \bar{x} \bar{x}^\ast \\ \bar{x}^\ast \bar{x} \end{pmatrix} = b \in \mathbb{F}^{2N-4}. \]

Theorem: ([JH:16] and deterministic case [WJ:17]) With measurements \(b\) above: For co-prime polynomials \(X\) and \(H\) the convex program (SDP) gives uniquely:

\[ U = \arg \min \|A(\bar{V}) - b\|_0 \quad 0 \leq V \in \mathbb{F}^{N \times N}. \]

where \(U = u u^\ast\) and \(u = [x, h] \in \mathbb{F}^n\).

method (c) - signaling with \(A(i) = \text{const} > 0\): (see [WJ:17])

- From Wiener-Lee relation for \(d := y \ast y^\ast\):
\[ d = (h \ast H^\ast)^\ast (x \ast x^\ast) \quad \text{and} \quad Y^\ast = (HH^\ast)XX^\ast A \]
allowing to compute \((h \ast H^\ast)^\ast\) from \(y \ast y^\ast\) if \(A(1) = \text{const}\)
- Huffman sequences [Huf:62] have autocorrelation \(a = x \ast x^\ast\)
\[ a = [-1, 0, \ldots, E, 0, \ldots, -1] \quad \text{and} \quad |A(e^{i\theta})| = |E - 2 \cos(\theta (L - 1))| \]
For \(K \leq 1/2\) it follows \(d = [-f, 0, \ldots, Ef, 0, \ldots, -f]\) with \(f = h \ast H^\ast\)
- zeros of \(A(\bar{\xi})\) are \(k = 1 \ldots L - 1\) conjugated pairs:
\[ \xi_k^\pm = R^\ast \exp(i2\pi k/L - 1) \quad \text{with} \quad R^\ast = (E + \sqrt{E^2 - 4})/2 \]
- Tx choose \(E > 2\), encode \(x\) from \(L - 1\) bits by choosing from pairs \(\xi_k^\pm\)
- Rx estimate \(f = h \ast H^\ast\) and \(E\) from \(d = y \ast y^\ast\), solve SDP
- perform rank-one projection of \(U\) (SVD) to get \(u = [h, x]\)
- decode \(x\) for the estimated \(E\), e.g., using zeros of \(X\)

Figure: MSE for 13000 runs with \(L = 32\) and \(K = 8\), and BER over rSNR. Red-dashed curve is with unknown and blue-solid with known \(E\).

- special structure allows to estimate first \(E\) independent of \(h\)
- reduce PAPR \(E \rightarrow 2\), robust zeros requires \(E > 2\)
- complexity in SDP, sequence detection and root finding

Conclusions: all methods so far seems to suffer from a PAPR problem