Robotics
Exercise 10

Marc Toussaint
Machine Learning & Robotics lab, U Stuttgart
Universitätsstraße 38, 70569 Stuttgart, Germany

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1 Local linearization and Algebraic Riccati equation

The state of the cart-pole is given by $x = (p, \dot{p}, \theta, \dot{\theta})$, with $p \in \mathbb{R}$ the position of the cart, $\theta \in \mathbb{R}$ the pendulums angular deviation from the upright position and $\dot{p}, \dot{\theta}$ their respective temporal derivatives. The only control signal $u \in \mathbb{R}$ is the force applied on the cart. The analytic model of the cart pole is

$$\ddot{\theta} = \frac{g \sin(\theta) + \cos(\theta) \left[-c_1 u - c_2 \dot{\theta}^2 \sin(\theta) \right]}{\frac{4}{3} l - c_2 \cos^2(\theta)} \tag{1}$$

$$\ddot{p} = c_1 u + c_2 \left[\dot{\theta}^2 \sin(\theta) - \ddot{\theta} \cos(\theta) \right] \tag{2}$$

with $g = 9.8ms^2$ the gravitational constant, $l = 1m$ the pendulum length and constants $c_1 = (M_p + M_c)^{-1}$ and $c_2 = l M_p (M_p + M_c)^{-1}$ where $M_p = M_c = 1kg$ are the pendulum and cart masses respectively.

a) Derive the local linearization of these dynamics around $x^* = (0, 0, 0, 0)$. The eventual dynamics should be in the form

$$\dot{x} = Ax + Bu$$

Note that

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\partial \ddot{p}}{\partial p} & \frac{\partial \ddot{p}}{\partial \dot{p}} & \frac{\partial \ddot{p}}{\partial \theta} & \frac{\partial \ddot{p}}{\partial \dot{\theta}} \\ 0 & 0 & 0 & 1 \\ \frac{\partial \ddot{\theta}}{\partial p} & \frac{\partial \ddot{\theta}}{\partial \dot{p}} & \frac{\partial \ddot{\theta}}{\partial \theta} & \frac{\partial \ddot{\theta}}{\partial \dot{\theta}} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \frac{\partial \ddot{p}}{\partial u} \\ 0 \\ \frac{\partial \ddot{\theta}}{\partial u} \end{pmatrix}$$

where all partial derivatives are taken at the point $p = \dot{p} = \theta = \dot{\theta} = 0$.

The solution (to continue with the other parts) is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{\frac{4}{3} l - c_2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ c_1 \\ 0 \\ \frac{-c_1}{\frac{4}{3} l - c_2} \end{pmatrix}$$

b) We assume a stationary infinite-horizon cost function of the form

$$J^\pi = \int_0^\infty c(x(t), u(t))\, dt$$
$$c(x, u) = x^\top Q x + u^\top R u$$
$$Q = \text{diag}(c, 0, 1, 0)\ , \quad R = \mathbf{I}\ .$$

That is, we penalize position offset $c\|p\|^2$ and pole angle offset $\|\theta\|^2$. Choose $c = \varrho = 1$ to start with.

Solve the Algebraic Riccati equation

$$0 = A^\top P + P^\top A - PBR^{-1}B^\top P + Q$$

by initializing $P = Q$ and iterating using the following iteration:

$$P_{k+1} = P_k + \epsilon[A^\top P_k + P_k^\top A - P_k B R^{-1} B^\top P_k + Q]$$

Choose $\epsilon = 1/1000$ and iterate until convergence. Output the gains $K = -R^{-1}B^\top P$. (Why should this iteration converge to the solution of the ARE?)

c) Solve the same Algebraic Riccati equation by calling the `are` routine of the octave control package (or a similar method in Matlab). For Octave, install the Ubuntu packages `octave3.2`, `octave-control`, and `qtoctave`, perhaps use `pkg load control` and `help are` in octave to ensure everything is installed, use `P=are(A,B*inverse(R)*B',Q)` to solve the ARE. Output $K = -R^{-1}B^\top P$ and compare to b).

(I found the solution $K = (1.00000, 2.58375, 52.36463, 15.25927.)$

d) Implement the optimal Linear Quadratic Regulator $u^* = -R^{-1}B^\top P x$ on the cart pole. Increase ϱ (e.g. to 100) and observe how the control strategy changes.