Introduction to Robotics

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This is a direct concatenation and reformatting of all lecture slides and exercises from the Robotics *course (winter term 2013/14, U Stuttgart), including a topic list to prepare for exams. Sorry for the sometimes imperfect formatting.*

Contents

1 Introduction

Why Robotics?

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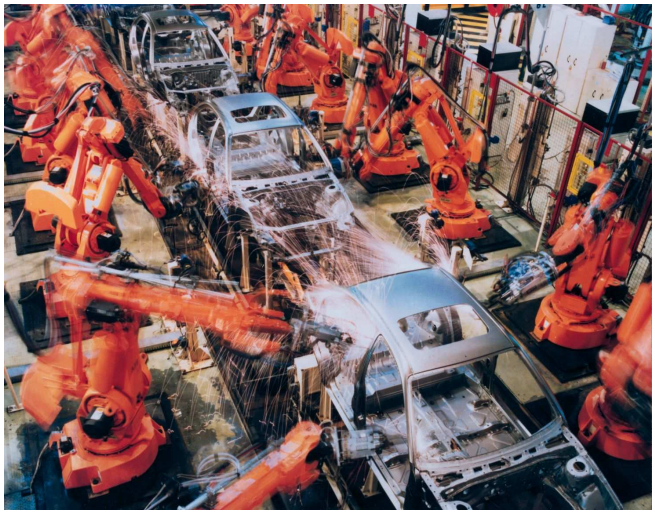

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http://www.saintpatrick.org/index.aspx/Health_Services/da_Vinci_Surgery

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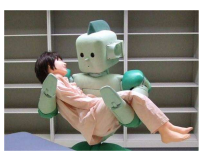
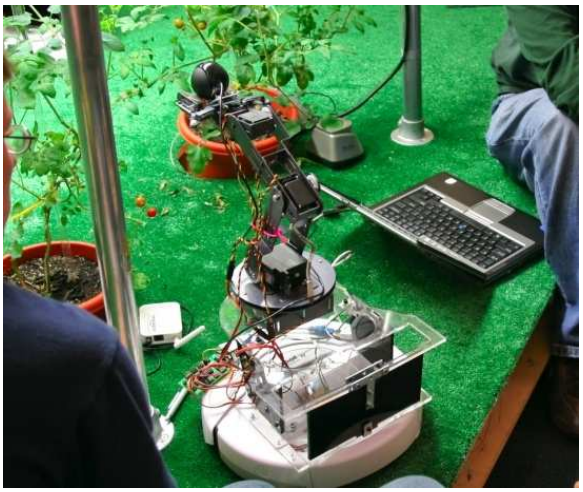



(robot “wife” aico)

1:4



http://people.csail.mit.edu/nikolaus/drg/

1:5

Why Robotics?

- Commercial:

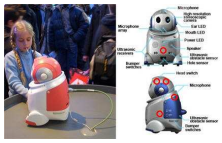
  Industrial, health care, entertainment, agriculture, surgery, etc

- Critical view:

  – International Committee for Robot Arms Control

  http://www.icrac.co.uk/

  – Noel Sharkey’s articles on robot ethics (Child care robots PePeRo.



http://www.nec.co.jp/products/robot/en/

1:6

Robotics as *intelligence research*

AI in the real world

AI: Machine Learning, probabilistic reasoning, optimization

Real World: Interaction, manipulation, perception, navigation, etc

1:7

Why AI needs to go real world

Tunicates digest their brain once they settled!

- **Motion** was *the* driving force to develop intelligence
  - motion needs control & decision making $\leftrightarrow$ fast information processing
  - motion needs anticipation & planning
  - motion needs perception
  - motion needs spatial representations
- **Manipulation** requires to acknowledge the structure (geometry, physics, objects) of the real world. Classical AI does not

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Robotics as *intelligence research*

- Machine Learning and AI are **computational** disciplines, which had great success with statistical modelling, analysis of data sets, symbolic reasoning. But they have not solved *autonomous learning, acting & reasoning in real worlds.*
- Neurosciences and psychology are **descriptive** sciences, either on the biological or cognitive level, e.g. with geat sucesses to describe and cure certain deceases. But they are not sufficient to create intelligent systems.
- Robotics is the only **synthetic** discipline to understand intelligent behavior in natural worlds. Robotic tells us what the actual problems are when trying to organize behavior in natural worlds.

1:9

History

- little movie...

(http://www.csail.mit.edu/videoarchive/history/aifilms http://www.ai.sri.com/shakey/)

1:10

Four chapters

- **Kinematics & Dynamics**

  goal: orchestrate joint movements for desired movement in task spaces

  Kinematic map, Jacobian, optimality principle of inverse kinematics, singularities, configuration/operational/null space, multiple simultaneous tasks, special task variables, trajectory interpolation, motion profiles; 1D point mass, damping & oscillation, PID, general dynamic systems, Newton-Euler, joint space control, reference trajectory following, optimal operational space control

- **Planning & optimization**

  goal: planning around obstacles, optimizing trajectories

  Path finding vs. trajectory optimization, local vs. global, Dijkstra, Probabilistic Roadmaps, Rapidly Exploring Random Trees, differential constraints, metrics; trajectory optimization, general cost function, task variables, transition costs, gradient methods, 2nd order methods, Dynamic Programming

- **Control Theory**

  theory on designing optimal controllers

  Topics in control theory, optimal control, HJB equation, infinite horizon case, Linear-Quadratic optimal control, Riccati equations (differential, algebraic, discrete-time), controllability, stability, eigenvalue analysis, Lyapunov function

- **Mobile robots**

  goal: localize and map yourself

  State estimation, Bayes filter, odometry, particle filter, Kalman filter, Bayes smoothing, SLAM, joint Bayes filter, EKF SLAM, particle SLAM, graph-based SLAM

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- Is this a practical or theoretical course?

  "There is nothing more practical than a good theory."
  (Vapnik, others...)

- Essentially, the whole course is about

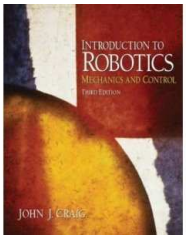
  reducing real-world problems to mathematical problems

  that can be solved efficiently

1:12

Books

There is no reference book for this lecture. But a basic well-known standard text book is:

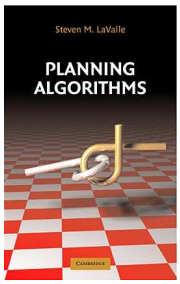


Craig, J.J.: *Introduction to robotics: mechanics and control*. Addison-Wesley New York, 1989. (3rd edition 2006)

1:13

Books

An advanced text book on planning is this:



Steven M. LaValle: *Planning Algorithms*. Cambridge University Press, 2006.
online:
http://planning.cs.uiuc.edu/

1:14

Online resources

- VideoLecture by Oussama Khatib: http://academicearth.org/courses/introduction-to-robotics http://www.virtualprofessors.com/introduction-to-robotics-stanford-cs223a-khatib
  (focus on kinematics, dynamics, control)
- Oliver Brock's lecture http://courses.robotics.tu-berlin.de/mediawiki/index.php/Robotics:_Schedule_WT09
- Stefan Schaal's lecture Introduction to Robotics: http://www-clmc.usc.edu/Teaching/TeachingIntroductionToRoboticsSyllabus
  (focus on control, useful: Basic Linear Control Theory (analytic solution to simple dynamic model → PID), chapter on dynamics)
- Chris Atkeson's "Kinematics, Dynamic Systems, and Control" http://www.cs.cmu.edu/~cga/kdc/
  (uses Schaal's slides and LaValle's book, useful: slides on 3d kinematics http://www.cs.cmu.edu/~cga/kdc/ewhitman1.pptx)
- CMU lecture "introduction to robotics" http://www.cs.cmu.edu/afs/cs.cmu.edu/academic/class/16311/www/current/syllabus.html
  (useful: PID control, simple BUGs algorithms for motion planning, non-holonomic constraints)

- *Handbook of Robotics* (partially online at Google books) http://tiny.cc/u6tzl
- LaValle's *Planning Algorithms* http://planning.cs.uiuc.edu/

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Organization

- Course webpage:

http://ipvs.informatik.uni-stuttgart.de/mlr/marc/teaching/

  - Slides, exercises & software (C++)
  - Links to books and other resources

- Secretary, admin issues:

  Carola Stahl, Carola.Stahl@ipvs.uni-stuttgart.de, room 2.217

- Rules for the tutorials:
  - Doing the exercises is crucial!
  - At the beginning of each tutorial:
    – sign into a list
    – mark which exercises you have (successfully) worked on
  - Students are randomly selected to present their solutions
  - You need 50% of completed exercises to be allowed to the exam

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2 Kinematics

Kinematic map, Jacobian, inverse kinematics as optimization problem, motion profiles, trajectory interpolation, multiple simultaneous tasks, special task variables, configuration/operational/null space, singularities

- Two "types of robotics":

  1) Mobile robotics – is all about localization & mapping

  2) Manipulation – is all about interacting with the world

  [0) Kinematic/Dynamic Motion Control: same as 2) without ever making it to interaction..]

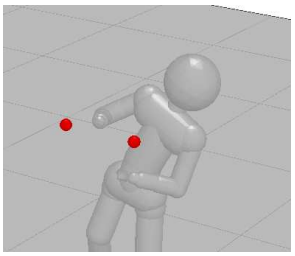
- Typical manipulation robots (and animals) are kinematic trees

  Their pose/state is described by all joint angles

2:1

Basic motion generation problem

- Move all joints in a coordinated way so that the endeffector makes a desired movement



`01-kinematics: ./x.exe -mode 2/3/4`

2:2

Outline

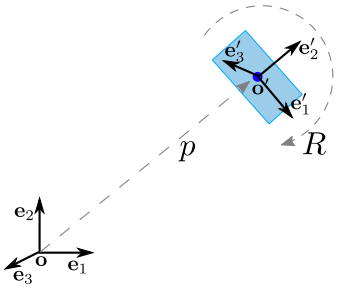
- Basic 3D geometry and notation
- Kinematics: $\phi : q \mapsto y$
- Inverse Kinematics: $y^* \mapsto q^* = \text{argmin}_q \|\phi(q) - y^*\|_C^2 + \|q - q_0\|_W^2$
- Basic motion heuristics: Motion profiles
- Additional things to know
  - Many simultaneous task variables
  - Singularities, null space,

2:3

Basic 3D geometry & notation

2:4

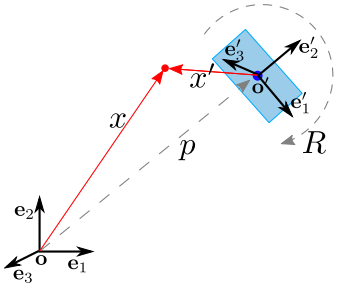
Pose (position & orientation)



- A *pose* is described by a translation $p \in \mathbb{R}^3$ and a rotation $R \in SO(3)$
  - R is an *orthonormal* matrix (orthogonal vectors stay orthogonal, unit vectors stay unit)
  - $R^{-1} = R^\top$
  - columns and rows are orthogonal unit vectors
  - $\det(R) = 1$
  - $R = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}$

2:5

Frame and coordinate transforms



- Let $(\boldsymbol{o}, \boldsymbol{e}_{1:3})$ be the world frame, $(\boldsymbol{o}', \boldsymbol{e}'_{1:3})$ be the body's frame.

  The new basis vectors are the *columns* in R, that is, $\boldsymbol{e}'_1 = R_{11}\boldsymbol{e}_1 + R_{21}\boldsymbol{e}_2 + R_{31}\boldsymbol{e}_3$, etc,
- x = coordinates in world frame $(\boldsymbol{o}, \boldsymbol{e}_{1:3})$

  x' = coordinates in body frame $(\boldsymbol{o}', \boldsymbol{e}'_{1:3})$

  p = coordinates of $\boldsymbol{o}'$ in world frame $(\boldsymbol{o}, \boldsymbol{e}_{1:3})$

$$x = p + Rx'$$

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Rotations

- Rotations can alternatively be represented as
  - Euler angles – NEVER DO THIS!
  - Rotation vector
  - Quaternion – default in code
- See the "geometry notes" for formulas to convert, concatenate & apply to vectors

2:7

Homogeneous transformations

- x^A = coordinates of a point in frame A

  x^B = coordinates of a point in frame B

- Translation and rotation: $x^A = t + Rx^B$

- Homogeneous transform $T \in \mathbb{R}^{4\times4}$:

$$T_{A\to B} = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}$$
$$x^A = T_{A\to B}\, x^B = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^B \\ 1 \end{pmatrix} = \begin{pmatrix} Rx^B + t \\ 1 \end{pmatrix}$$

in homogeneous coordinates, we append a 1 to all coordinate vectors

2:8

Is $T_{A\to B}$ forward or backward?

- $T_{A\to B}$ describes the translation and rotation of *frame* B relative to A

  That is, it describes the forward FRAME transformation (from A to B)

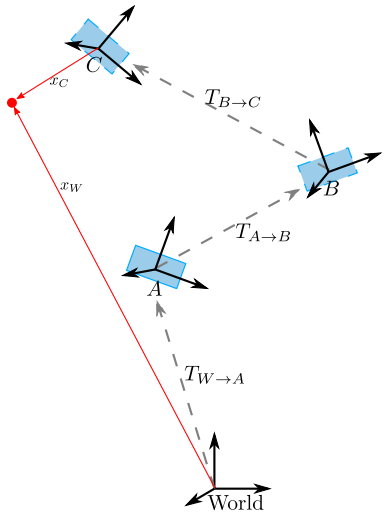
- $T_{A\to B}$ describes the coordinate transformation from x^B to x^A

  That is, it describes the backward COORDINATE transformation

- Confused? Vectors (and frames) transform *covariant*, coordinates *contra-variant*. See "geometry notes" or Wikipedia for more details, if you like.

2:9

Composition of transforms



$$T_{W\to C} = T_{W\to A}\, T_{A\to B}\, T_{B\to C}$$
$$x^W = T_{W\to A}\, T_{A\to B}\, T_{B\to C}\, x^C$$

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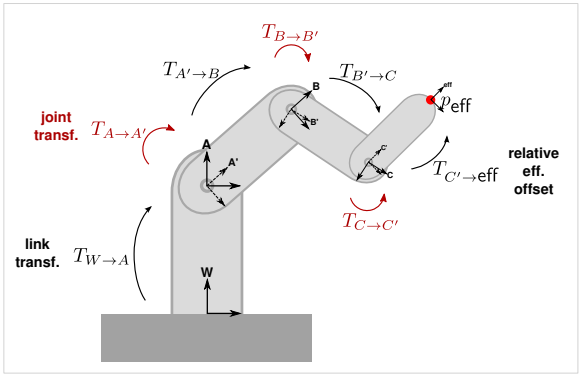
Kinematics

2:11

Kinematics



- A *kinematic structure* is a graph (usually tree or chain) of rigid **links** and **joints**

$$T_{W\to\text{eff}}(q) = T_{W\to A}\, T_{A\to A'}(q)\, T_{A'\to B}\, T_{B\to B'}(q)\, T_{B'\to C}\, T_{C\to C'}(q)\, T_{C'}$$

2:12

Joint types

- Joint transformations: $T_{A\to A'}(q)$ depends on $q \in \mathbb{R}^n$

revolute joint: joint angle $q \in \mathbb{R}$ determines rotation about x-axis:

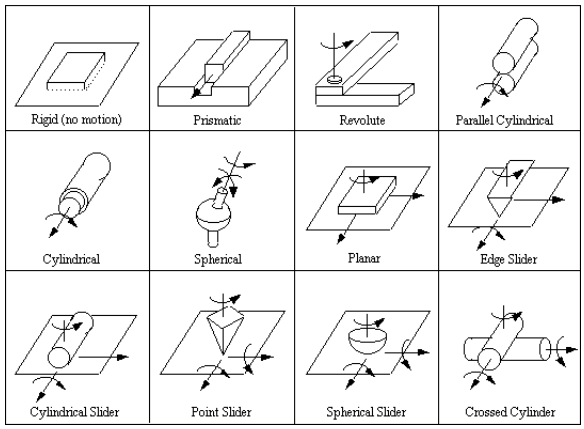
$$T_{A\to A'}(q) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(q) & -\sin(q) & 0 \\ 0 & \sin(q) & \cos(q) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

prismatic joint: offset $q \in \mathbb{R}$ determines translation along x-axis:

$$T_{A\to A'}(q) = \begin{pmatrix} 1 & 0 & 0 & q \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

others: screw (1dof), cylindrical (2dof), spherical (3dof), universal (2dof)

2:13



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Kinematic Map

- For any joint angle vector $q \in \mathbb{R}^n$ we can compute $T_{W\to\text{eff}}(q)$ by *forward chaining* of transformations

  $T_{W\to\text{eff}}(q)$ gives us the *pose* of the endeffector in the world frame

- The two most important examples for a *kinematic map* ϕ are

  1) A point v on the endeffector transformed to world coordinates:

  $$\phi^{\text{pos}}_{\text{eff},v}(q) = T_{W\to\text{eff}}(q)\ v \quad \in \mathbb{R}^3$$

  2) A direction $v \in \mathbb{R}^3$ attached to the endeffector transformed to world:

  $$\phi^{\text{vec}}_{\text{eff},v}(q) = R_{W\to\text{eff}}(q)\ v \quad \in \mathbb{R}^3$$

  Where $R_{A\to B}$ is the rotation in $T_{A\to B}$.

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Kinematic Map

- In general, a kinematic map is *any* (differentiable) mapping

  $$\phi:\ q \mapsto y$$

  that maps to *some arbitrary feature* $y \in \mathbb{R}^d$ of the pose $q \in \mathbb{R}^n$

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Jacobian

- When we change the joint angles, δq, how does the effector position change, δy?

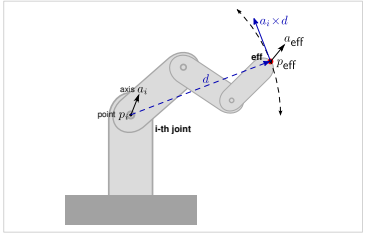
- Given the kinematic map $y = \phi(q)$ and its Jacobian $J(q) = \frac{\partial}{\partial q}\phi(q)$, we have:

  $$\delta y = J(q)\ \delta q$$

$$J(q) = \frac{\partial}{\partial q}\phi(q) = \begin{pmatrix} \frac{\partial\phi_1(q)}{\partial q_1} & \frac{\partial\phi_1(q)}{\partial q_2} & \cdots & \frac{\partial\phi_1(q)}{\partial q_n} \\ \frac{\partial\phi_2(q)}{\partial q_1} & \frac{\partial\phi_2(q)}{\partial q_2} & \cdots & \frac{\partial\phi_2(q)}{\partial q_n} \\ \vdots & & & \vdots \\ \frac{\partial\phi_d(q)}{\partial q_1} & \frac{\partial\phi_d(q)}{\partial q_2} & \cdots & \frac{\partial\phi_d(q)}{\partial q_n} \end{pmatrix} \in \mathbb{R}^{d\times n}$$

2:17

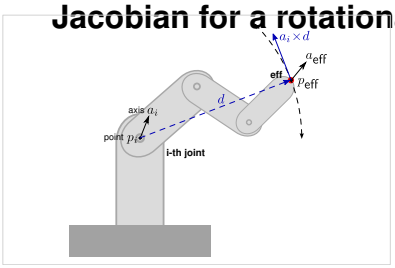
Jacobian for a rotational joint



- The i-th joint is located at $p_i = t_{W\to i}(q)$ and has rotation axis $a_i = R_{W\to i}(q)\begin{pmatrix}1\\0\\0\end{pmatrix}$
- We consider an infinitesimal variation $\delta q_i \in \mathbb{R}$ of the ith joint and see how an endeffector position $p_{\text{eff}} = \phi^{\text{pos}}_{\text{eff},v}(q)$ and attached vector $a_{\text{eff}} = \phi^{\text{vec}}_{\text{eff},v}(q)$ change.

2:18

Jacobian for a rotational joint



Consider a variation δq_i $\to$ the whole sub-tree rotates

$$\delta p_{\text{eff}} = [a_i \times (p_{\text{eff}} - p_i)]\ \delta q_i$$
$$\delta a_{\text{eff}} = [a_i \times a_{\text{eff}}]\ \delta q_i$$

$\Rightarrow$ Position Jacobian:

$$J^{\text{pos}}_{\text{eff},v}(q) = \begin{pmatrix} [a_1 \times (p_{\text{eff}} - p_1)] & [a_2 \times (p_{\text{eff}} - p_2)] & \cdots & [a_n \times (p_{\text{eff}} - p_n)] \end{pmatrix} \in \mathbb{R}^{3\times n}$$

$\Rightarrow$ Vector Jacobian:

$$J^{\text{vec}}_{\text{eff},v}(q) = \begin{pmatrix} [a_1 \times a_{\text{eff}}] & [a_2 \times a_{\text{eff}}] & \cdots & [a_n \times a_{\text{eff}}] \end{pmatrix} \in$$

2:19

Jacobian

- To compute the Jacobian of some endeffector position or vector, we only need to know the position and rotation axis of each joint.

- The two kinematic maps ϕ^{pos} and ϕ^{vec} are the most important two examples – more complex geometric features can be computed from these, as we will see later.

2:20

Inverse Kinematics

2:21

Inverse Kinematics problem

- Generally, the aim is to find a robot configuration q such that $\phi(q) = y^*$
- Iff ϕ is invertible

  $$q^* = \phi^{-1}(y^*)$$

- But in general, ϕ will not be invertible:

  1) The pre-image $\phi^{-1}(y^*) =$ may be empty: No configuration can generate the desired y^*

  2) The pre-image $\phi^{-1}(y^*)$ may be large: many configurations can generate the desired y^*

2:22

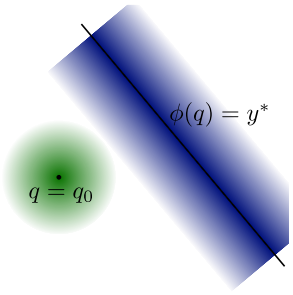
Inverse Kinematics as optimization problem

- We formalize the inverse kinematics problem as an optimization problem

$$q^* = \underset{q}{\operatorname{argmin}} \|\phi(q) - y^*\|_C^2 + \|q - q_0\|_W^2$$

- The 1st term ensures that we find a configuration even if y^* is not exactly reachable

  The 2nd term disambiguates the configurations if there are many $\phi^{-1}(y^*)$



2:23

Inverse Kinematics as optimization problem

$$q^* = \underset{q}{\operatorname{argmin}} \|\phi(q) - y^*\|_C^2 + \|q - q_0\|_W^2$$

- The formulation of IK as an optimization problem is very powerful and has many nice properties
- We will be able to take the limit $C \to \infty$, enforcing exact $\phi(q) = y^*$ if possible
- Non-zero C^{-1} and W corresponds to a regularization that ensures numeric stability
- Classical concepts can be derived as special cases:
  - Null-space motion
  - regularization; singularity robustness
  - multiple tasks
  - hierarchical tasks

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Solving Inverse Kinematics

- The obvious choice of optimization method for this problem is Gauss-Newton, using the Jacobian of ϕ
- We first describe just one step of this, which leads to the classical equations for inverse kinematics using the local Jacobian...

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Solution using the local linearization

- When using the local linearization of ϕ at q_0,

$$\phi(q) \approx y_0 + J\,(q - q_0)\,, \quad y_0 = \phi(q_0)$$

- We can derive the optimum as

$$\begin{aligned} f(q) &= \|\phi(q) - y^*\|_C^2 + \|q - q_0\|_W^2 \\ &= \|y_0 - y^* + J\,(q - q_0)\|_C^2 + \|q - q_0\|_W^2 \\ \frac{\partial}{\partial q} f(q) &= 0^\top = 2(y_0 - y^* + J\,(q - q_0))^\top C J + 2(q - q_0)^T W \\ J^\top C\,(y^* - y_0) &= (J^\top C J + W)\,(q - q_0) \end{aligned}$$

$$q^* = q_0 + J^\sharp (y^* - y_0)$$

with $J^\sharp = (J^\top C J + W)^{-1} J^\top C = W^{-1} J^\top (J W^{-1} J^\top + C^{-1})^{-1}$ (*Woodbury identity*)

  - For $C \to \infty$ and $W = \mathbf{I}$, $J^\sharp = J^\top (J J^\top)^{-1}$ is called *pseudo-inverse*
  - W generalizes the metric in q-space
  - C regularizes this pseudo-inverse (see later section on singularities)

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"Small step" application

- This approximate solution to IK makes sense
  - if the local linearization of ϕ at q_0 is "good"
  - if q_0 and q^* are close
- This equation is therefore typically used to iteratively compute small steps in configuration space

$$q_{t+1} = q_t + J^\sharp (y^*_{t+1} - \phi(q_t))$$

where the target y^*_{t+1} moves smoothly with t

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Example: Iterating IK to follow a trajectory

- Assume initial posture q_0. We want to reach a desired endeff position y^* in T steps:

Input: initial state q_0, desired y^*, methods ϕ^{pos} and J^{pos}
Output: trajectory $q_{0:T}$

1: Set $y_0 = \phi^{\text{pos}}(q_0)$ *// starting endeff position*
2: **for** $t = 1 : T$ **do**
3: $y \leftarrow \phi^{\text{pos}}(q_{t\text{-}1})$ *// current endeff position*
4: $J \leftarrow J^{\text{pos}}(q_{t\text{-}1})$ *// current endeff Jacobian*
5: $\hat{y} \leftarrow y_0 + (t/T)(y^* - y_0)$ *// interpolated endeff target*
6: $q_t = q_{t\text{-}1} + J^\sharp(\hat{y} - y)$ *// new joint positions*
7: Command q_t to all robot motors and compute all $T_{W \to i}(q_t)$
8: **end for**

```
01-kinematics:  ./x.exe -mode 2/3
```

- Why does this not follow the interpolated trajectory $\hat{y}_{0:T}$ exactly?

  – What happens if $T = 1$ and y^* is far?

2:28

Two additional notes

- What if we linearize at some arbitrary q' instead of q_0?

$$\begin{aligned} \phi(q) &\approx y' + J\,(q - q')\,, \quad y' = \phi(q') \\ q^* &= \underset{q}{\operatorname{argmin}} \|\phi(q) - y^*\|_C^2 + \|q - q' + (q' - q_0)\|_W^2 \end{aligned}$$

$$= q' + J^\sharp\,(y^* - y') + (I - J^\sharp J)\,h\,, \quad h = q_0 - q' \qquad (1)$$

Note that h corresponds to the classical concept of *null space motion*

- What if we want to find the *exact* (local) optimum? E.g. what if we want to compute a big step (where q^* will be remote from q) and we cannot not rely only on the local linearization approximation?
  - Iterate equation (1) (optionally with a step size < 1 to ensure convergence) by setting the point y' of linearization to the current q^*
  - This is equivalent to the Gauss-Newton algorithm

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Where are we?

- We've derived a basic motion generation principle in robotics from
  - an understanding of robot geometry & kinematics
  - a basic notion of optimality
- In the remainder:
  A. Heuristic motion profiles for simple trajectory generation
  B. Extension to multiple task variables
  C. Discussion of classical concepts
     - Singularity and singularity-robustness
     - Nullspace, task/operational space, joint space
     - "inverse kinematics" $\leftrightarrow$ "motion rate control"

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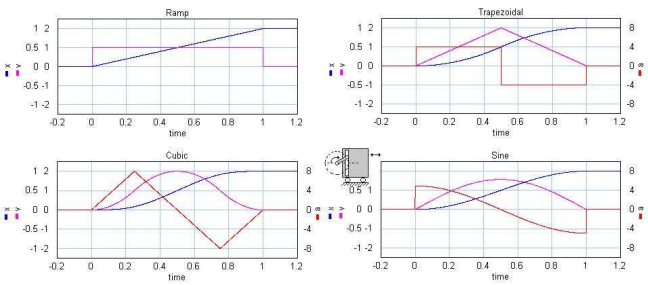
Heuristic motion profiles

2:31

Heuristic motion profiles

- Assume initially $x = 0, \dot{x} = 0$. After 1 second you want $x = 1, \dot{x} = 0$.

  How do you move from $x = 0$ to $x = 1$ in one second?



The sine profile $x_t = x_0 + \frac{1}{2}[1 - \cos(\pi t/T)](x_T - x_0)$ is a compromise for low max-acceleration and max-velocity

Taken from http://www.20sim.com/webhelp/toolboxes/mechatronics_toolbox/motion_profile_wizard/motionprofiles.htm

2:32

Motion profiles

- Generally, let's define a motion profile as a mapping

$$\text{MP} : [0,1] \mapsto [0,1]$$

with $\text{MP}(0) = 0$ and $\text{MP}(1) = 1$ such that the interpolation is given as

$$x_t = x_0 + \text{MP}(t/T)\,(x_T - x_0)$$

- For example

$$\text{MP}_{\text{ramp}}(s) = s$$
$$\text{MP}_{\text{sin}}(s) = \frac{1}{2}[1 - \cos(\pi s)]$$

2:33

Joint space interpolation

1) Optimize a desired final configuration q_T:
   Given a desired final task value y_T, optimize a final joint state q_T to minimize the function

$$f(q_T) = \|q_T - q_0\|^2_{W/T} + \|y_T - \phi(q_T)\|^2_C$$

   – The metric $\frac{1}{T}W$ is consistent with T cost terms with step metric W.
   – In this optimization, q_T will end up remote from q_0. So we need to iterate Gauss-Newton, as described on slide 2.

2) Compute $q_{0:T}$ as interpolation between q_0 and q_T:
   Given the initial configuration q_0 and the final q_T, interpolate on a straight line with a some motion profile. E.g.,

$$q_t = q_0 + \text{MP}(t/T)\,(q_T - q_0)$$

2:34

Task space interpolation

1) Compute $y_{0:T}$ as interpolation between y_0 and y_T:
   Given a initial task value y_0 and a desired final task value y_T, interpolate on a straight line with a some motion profile. E.g,

$$y_t = y_0 + \text{MP}(t/T)\,(y_T - y_0)$$

2) Project $y_{0:T}$ to $q_{0:T}$ using inverse kinematics:
   Given the task trajectory $y_{0:T}$, compute a corresponding joint trajectory $q_{0:T}$ using inverse kinematics

$$q_{t+1} = q_t + J^\sharp(y_{t+1} - \phi(q_t))$$

   (As steps are small, we should be ok with just using this local linearization.)

2:35

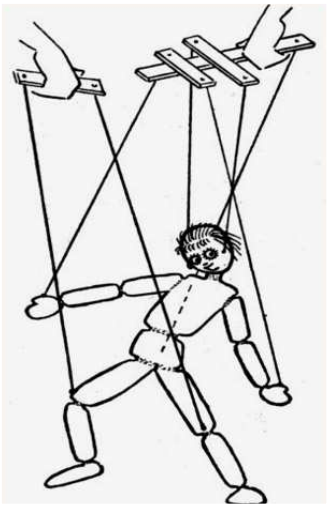
`peg-in-a-hole demo`

2:36

Multiple tasks

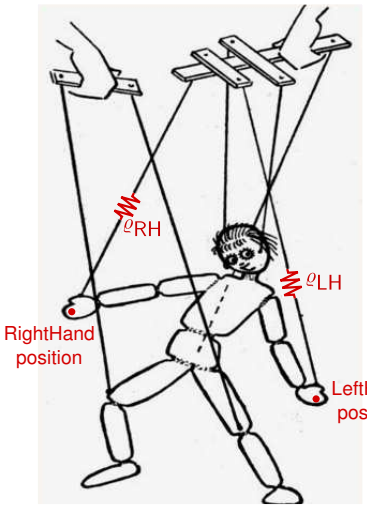
2:37

Multiple tasks



2:38

Multiple tasks



2:39

Multiple tasks

- Assume we have m simultaneous tasks; for each task i we have:
  - a kinematic mapping $y_i = \phi_i(q) \in \mathbb{R}^{d_i}$
  - a current value $y_{i,t} = \phi_i(q_t)$
  - a desired value y_i^*
  - a precision ϱ_i (implying a task cost metric $C_i = \varrho_i \mathbf{I}$)

- Each task contributes a term to the objective function

$$q^* = \underset{q}{\operatorname{argmin}} \|q - q_0\|_W^2 + \varrho_1 \|\phi_1(q) - y_1^*\|^2 + \varrho_2 \|\phi_2(q) - y_2^*\|^2 + \cdots$$

which we can also write as

$$q^* = \underset{q}{\operatorname{argmin}} \|q - q_0\|_W^2 + \|\Phi(q)\|^2$$

where $\Phi(q) := \begin{pmatrix} \sqrt{\varrho_1}\,(\phi_1(q) - y_1^*) \\ \sqrt{\varrho_2}\,(\phi_2(q) - y_2^*) \\ \vdots \end{pmatrix} \in \mathbb{R}^{\sum_i d_i}$

2:40

Multiple tasks

- We can "pack" together all tasks in one "big task" Φ.

  Example: We want to control the 3D position of the left hand and of the right hand. Both are "packed" to one 6-dimensional task vector which becomes zero if both tasks are fulfilled.

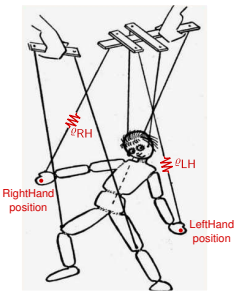
- The big Φ is scaled/normalized in a way that
  - the desired value is always zero
  - the cost metric is $\mathbf{I}$

- Using the local linearization of Φ at q_0, $J = \frac{\partial \Phi(q_0)}{\partial q}$, the optimum is

$$\begin{aligned} q^* &= \underset{q}{\operatorname{argmin}} \|q - q_0\|_W^2 + \|\Phi(q)\|^2 \\ &\approx q_0 - (J^\top J + W)^{-1} J^\top \Phi(q_0) = q_0 - J^\# \Phi(q_0) \end{aligned}$$

2:41

Multiple tasks



- We learnt how to "puppeteer a robot"
- We can handle many task variables (but specifying their precisions ϱ_i becomes cumbersome...)
- In the remainder:
  A. Classical limit of "hierarchical IK" and nullspace motion
  B. What are interesting task variables?

2:42

Hierarchical IK & nullspace motion

- In the classical view, tasks should be executed *exactly*, which means taking the limit $\varrho_i \to \infty$ in some prespecified hierarchical order.
- We can rewrite the solution in a way that allows for such a hierarchical limit:
- One task plus "nullspace motion":

$$\begin{aligned} f(q) &= \|q - a\|_W^2 + \varrho_1 \|J_1 q - y_1\|^2 \\ &\propto \|q - \hat{a}\|_{\widehat{W}}^2 \\ \widehat{W} &= W + \varrho_1 J_1^\top J_1\,, \quad \hat{a} = \widehat{W}^{-1}(W a + \varrho_1 J_1^\top y_1) = J_1^\# y_1 + (\mathbf{I} - J_1^\# J_1) a \\ J_1^\# &= (W/\varrho_1 + J_1^\top J_1)^{-1} J_1^\top \end{aligned}$$

- Two tasks plus nullspace motion:

$$\begin{aligned} f(q) &= \|q - a\|_W^2 + \varrho_1 \|J_1 q - y_1\|^2 + \varrho_2 \|J_2 q - y_2\|^2 \\ &= \|q - \hat{a}\|_{\widehat{W}}^2 + \|J_1 q + \Phi_1\|^2 \\ q^* &= J_1^\# y_1 + (\mathbf{I} - J_1^\# J_1)[J_2^\# y_2 + (\mathbf{I} - J_2^\# J_2) a] \\ J_2^\# &= (W/\varrho_2 + J_2^\top J_2)^{-1} J_2^\top\,, \quad J_1^\# = (\widehat{W}/\varrho_1 + J_1^\top J_1)^{-1} J_1^\top \end{aligned}$$

- etc...

2:43

Hierarchical IK & nullspace motion

- The previous slide did nothing but rewrite the nice solution $q^* = -J^{\#}\Phi(q_0)$ (for the "big" Φ) in a strange hierarchical way that allows to "see" nullspace projection
- The benefit of this hierarchical way to write the solution is that one can take the hierarchical limit $\varrho_i \to \infty$ and retrieve classical hierarchical IK
- The drawbacks are:
  - It is somewhat ugly
  - In practise, I would recommend regularization in any case (for numeric stability). Regularization corresponds to NOT taking the full limit $\varrho_i \to \infty$. Then the hierarchical way to write the solution is unnecessary. (However, it points to a "hierarchical regularization", which might be numerically more robust for very small regularization?)
  - The general solution allows for arbitrary blending of tasks

2:44

What are interesting task variables?

The following slides will define 10 different types of task variables.

This is meant as a reference and to give an idea of possibilities...

2:45

Position

Position of some point attached to link i	
dimension	$d = 3$
parameters	link index i, point offset v
kin. map	$\phi^{\text{pos}}_{iv}(q) = T_{W\to i}\, v$
Jacobian	$J^{\text{pos}}_{iv}(q)_{\cdot k} = [k \prec i]\; a_k \times (\phi^{\text{pos}}_{iv}(q) - p_k)$

Notation:

- a_k, p_k are axis and position of joint k
- $[k \prec i]$ indicates whether joint k is between root and link i
- $J_{\cdot k}$ is the kth column of J

2:46

Vector

Vector attached to link i	
dimension	$d = 3$
parameters	link index i, attached vector v
kin. map	$\phi^{\text{vec}}_{iv}(q) = R_{W\to i}\, v$
Jacobian	$J^{\text{vec}}_{iv}(q) = A_i \times \phi^{\text{vec}}_{iv}(q)$

Notation:

- A_i is a matrix with columns $(A_i)_{\cdot k} = [k \prec i]\; a_k$ containing the joint axes or zeros
- the short notation "$A \times p$" means that each *column* in A takes the cross-product with p.

2:47

Relative position

Position of a point on link i relative to point on link j	
dimension	$d = 3$
parameters	link indices i, j, point offset v in i and w in j
kin. map	$\phi^{\text{pos}}_{iv\|jw}(q) = R_j^{-1}(\phi^{\text{pos}}_{iv} - \phi^{\text{pos}}_{jw})$
Jacobian	$J^{\text{pos}}_{iv\|jw}(q) = R_j^{-1}[J^{\text{pos}}_{iv} - J^{\text{pos}}_{jw} - A_j \times (\phi^{\text{pos}}_{iv} - \phi^{\text{pos}}_{jw})]$

Derivation:
For $y = Rp$ the derivative w.r.t. a rotation around axis a is $y' = Rp' + R'p = Rp' + a \times Rp$. For $y = R^{-1}p$ the derivative is $y' = R^{-1}p' - R^{-1}(R')R^{-1}p = R^{-1}(p' - a \times p)$. (For details see http://ipvs.informatik.uni-stuttgart.de/mlr/marc/notes/3d-geometry.pdf)

2:48

Relative vector

Vector attached to link i relative to link j	
dimension	$d = 3$
parameters	link indices i, j, attached vector v in i
kin. map	$\phi^{\text{vec}}_{iv\|j}(q) = R_j^{-1}\phi^{\text{vec}}_{iv}$
Jacobian	$J^{\text{vec}}_{iv\|j}(q) = R_j^{-1}[J^{\text{vec}}_{iv} - A_j \times \phi^{\text{vec}}_{iv}]$

2:49

Alignment

Alignment of a vector attached to link i with a reference v^*	
dimension	$d = 1$
parameters	link index i, attached vector v, world reference v^*
kin. map	$\phi^{\text{align}}_{iv}(q) = v^{*\top}\, \phi^{\text{vec}}_{iv}$
Jacobian	$J^{\text{align}}_{iv}(q) = v^{*\top}\, J^{\text{vec}}_{iv}$

Note: $\phi^{\text{align}} = 1 \leftrightarrow$ align $\quad \phi^{\text{align}} = -1 \leftrightarrow$ anti-align $\quad \phi^{\text{align}} = 0 \leftrightarrow$ orthog.

2:50

Relative Alignment

Alignment a vector attached to link i with vector attached to j	
dimension	$d = 1$
parameters	link indices i, j, attached vectors v, w
kin. map	$\phi^{\text{align}}_{iv\|jw}(q) = (\phi^{\text{vec}}_{jw})^\top\, \phi^{\text{vec}}_{iv}$
Jacobian	$J^{\text{align}}_{iv\|jw}(q) = (\phi^{\text{vec}}_{jw})^\top\, J^{\text{vec}}_{iv} + \phi^{\text{vec}\top}_{iv}\, J^{\text{vec}}_{jw}$

2:51

Joint limits

Penetration of joint limits	
dimension	$d = 1$
parameters	joint limits $q_{\text{low}}, q_{\text{hi}}$, margin m
kin. map	$\phi_{\text{limits}}(q) = \frac{1}{m}\sum_{i=1}^{n}[m - q_i + q_{\text{low}}]^+ + [m + q_i - q_{\text{hi}}]^+$
Jacobian	$J_{\text{limits}}(q)_{1,i} = -\frac{1}{m}[m - q_i + q_{\text{low}} > 0] + \frac{1}{m}[m + q_i - q_{\text{hi}} > 0]$

$[x]^+ = x > 0 ? x : 0 \qquad [\cdots]$: indicator function

2:52

Collision limits

Penetration of collision limits	
dimension	$d = 1$
parameters	margin m
kin. map	$\phi_{\text{col}}(q) = \frac{1}{m}\sum_{k=1}^{K}[m - \lvert p_k^a - p_k^b\rvert]^+$
Jacobian	$J_{\text{col}}(q) = \frac{1}{m}\sum_{k=1}^{K}[m - \lvert p_k^a - p_k^b\rvert > 0] (-J^{\text{pos}}_{p_k^a} + J^{\text{pos}}_{p_k^b})^\top \frac{p_k^a - p_k^b}{\lvert p_k^a - p_k^b\rvert}$

A collision detection engine returns a set $\{(a, b, p^a, p^b)_{k=1}^K\}$ of potential collisions between link a_k and b_k, with nearest points p_k^a on a and p_k^b on b.

2:53

Center of gravity

Center of gravity of the whole kinematic structure	
dimension	$d = 3$
parameters	(none)
kin. map	$\phi^{\text{cog}}(q) = \sum_i \text{mass}_i\ \phi^{\text{pos}}_{ic_i}$
Jacobian	$J^{\text{cog}}(q) = \sum_i \text{mass}_i\ J^{\text{pos}}_{ic_i}$

c_i denotes the center-of-mass of link i (in its own frame)

2:54

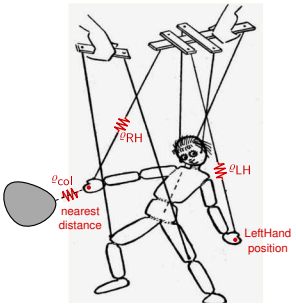
Homing

The joint angles themselves	
dimension	$d = n$
parameters	(none)
kin. map	$\phi_{\text{qitself}}(q) = q$
Jacobian	$J_{\text{qitself}}(q) = \mathbf{I}_n$

Example: Set the target $y^* = 0$ and the precision ϱ very low $\rightarrow$ this task describes posture comfortness in terms of deviation from the joints' zero position. In the classical view, it induces "nullspace motion".

2:55

Task variables – conclusions



- There is much space for creativity in defining task variables! Many are extensions of ϕ^{pos} and ϕ^{vec} and the Jacobians combine the basic Jacobians.
- What the *right* task variables are to design/describe motion is a very hard problem! In what task space do humans control their motion? Possible to learn from data ("task space retrieval") or perhaps via Reinforcement Learning.
- In practice: Robot motion design (including grasping) may require cumbersome hand-tuning of such task variables.

2:56

Discussion of classical concepts

- Singularity and singularity-robustness
- Nullspace, task/operational space, joint space
- "inverse kinematics" $\leftrightarrow$ "motion rate control"

2:57

Singularity

- In general: A matrix J **singular** $\iff$ $\text{rank}(J) < d$
  - rows of J are linearly dependent
  - dimension of image is $< d$
  - $\delta y = J\delta q \ \Rightarrow$ dimensions of δy limited
  - Intuition: arm fully stretched

- Implications:

  $\det(JJ^\top) = 0$

  $\rightarrow$ pseudo-inverse $J^\top(JJ^\top)^{-1}$ is ill-defined!

  $\rightarrow$ inverse kinematics $\delta q = J^\top(JJ^\top)^{-1}\delta y$ computes "infinite" steps!

- **Singularity robust pseudo inverse** $J^\top(JJ^\top + \epsilon\mathbf{I})^{-1}$

  The term $\epsilon\mathbf{I}$ is called **regularization**

- Recall our general solution (for $W = \mathbf{I}$)

  $$J^\sharp = J^\top(JJ^\top + C^{-1})^{-1}$$

  is already singularity robust

2:58

Null/task/operational/joint/configuration spaces

- The space of all $q \in \mathbb{R}^n$ is called **joint/configuration space**

  The space of all $y \in \mathbb{R}^d$ is called **task/operational space**

  Usually $d < n$, which is called **redundancy**

- For a desired endeffector state y^* there exists a whole manifold (assuming ϕ is smooth) of joint configurations q:

  $$\textbf{nullspace}(y^*) = \{q \mid \phi(q) = y^*\}$$

- We found earlier that

  $$\begin{aligned} q^* &= \underset{q}{\text{argmin}} \|q - a\|_W^2 + \varrho\|Jq - y^*\|^2 \\ &= J^\# y^* + (\mathbf{I} - J^\# J)a\ , \quad J^\# = (W/\varrho + J^\top J)^{-1}J^\top \end{aligned}$$

  In the limit $\varrho \to \infty$ it is guaranteed that $Jq = y^*$ (we are exacty on the manifold). The term a introduces additional "nullspace motion".

2:59

Inverse Kinematics and Motion Rate Control

Some clarification of concepts:

- The notion "kinematics" describes the mapping $\phi : \ q \mapsto y$, which usually is a many-to-one function.

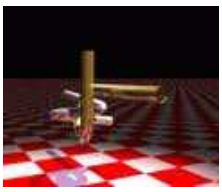
- The notion "inverse kinematics" in the strict sense describes some mapping $g : \ y \mapsto q$ such that $\phi(g(y)) = y$, which usually is non-unique or ill-defined.
- In practice, one often refers to $\delta q = J^\sharp \delta y$ as **inverse kinematics**.

- When iterating $\delta q = J^\sharp \delta y$ in a control cycle with time step τ (typically $\tau \approx 1 - 10$ msec), then $\dot{y} = \delta y / \tau$ and $\dot{q} = \delta q / \tau$ and $\dot{q} = J^\sharp \dot{y}$. Therefore the control cycle effectively controls the endeffector velocity—this is why it is called **motion rate control**.

3 Path Planning

Path finding vs. trajectory optimization, local vs. global, Dijkstra, Probabilistic Roadmaps, Rapidly Exploring Random Trees, non-holonomic systems, car system equation, path-finding for non-holonomic systems, control-based sampling, Dubins curves

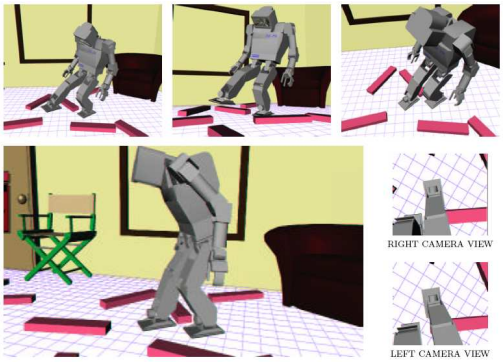
Path finding examples



Alpha-Puzzle, solved with James Kuffner's RRTs

3:1

Path finding examples



J. Kuffner, K. Nishiwaki, S. Kagami, M. Inaba, and H. Inoue. Footstep Planning Among Obstacles for Biped Robots. Proc. IEEE/RSJ Int. Conf. on Intelligent Robots and Systems (IROS), 2001.

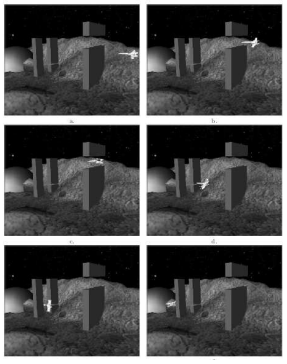
3:2

Path finding examples

T. Bretl. Motion Planning of Multi-Limbed Robots Subject to Equilibrium Constraints: The Free-Climbing Robot Problem. International Journal of Robotics Research, 25(4):317-342, Apr 2006.

3:3

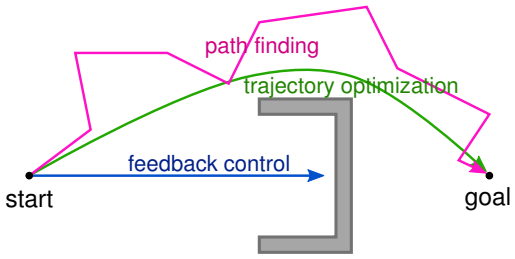
Path finding examples



S. M. LaValle and J. J. Kuffner. Randomized Kinodynamic Planning. International Journal of Robotics Research, 20(5):378–400, May 2001.

3:4

Feedback control, path finding, trajectory optim.



- Feedback Control: E.g., $q_{t+1} = q_t + J^\sharp(y^* - \phi(q_t))$
- Trajectory Optimization: $\text{argmin}_{q_{0:T}} f(q_{0:T})$
- Path Finding: Find some $q_{0:T}$ with only valid configurations

3:5

Control, path finding, trajectory optimization

- Combining methods:

  1) Path Finding

  2) Trajectory Optimization ("smoothing")

  3) Feedback Control

- Many problems can be solved with only feedback control (though not optimally)
- Many more problems can be solved *locally* optimal with only trajectory optimization
- Tricky problems need path finding: *global* search for valid paths

3:6

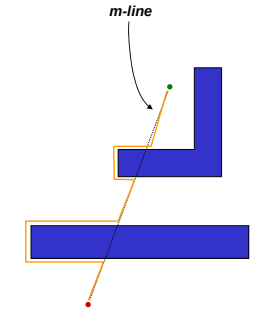
Outline

- Heuristics & Discretization (slides from Howie CHoset's CMU lectures)
  - Bugs algorithm
  - Potentials to guide feedback control
  - Dijkstra
- Sample-based Path Finding
  - Probabilistic Roadmaps
  - Rapidly Exploring Random Trees

3:7

CarnegieMellon

A better bug?



"Bug 2" Algorithm

1) head toward goal on the *m-line*

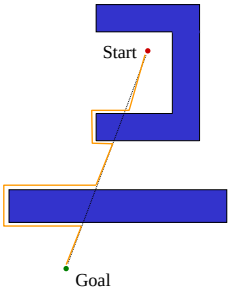
2) if an obstacle is in the way, follow it until you encounter the m-line again.

3) Leave the obstacle and continue toward the goal

3:8

CarnegieMellon

A better bug?



"Bug 2" Algorithm

1) head toward goal on the *m-line*

2) if an obstacle is in the way, follow it until you encounter the m-line again.

3) Leave the obstacle and continue toward the goal

Better or worse than Bug1?

3:9

CarnegieMellon

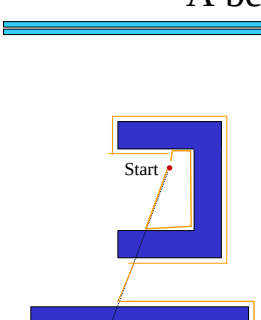
A better bug?

"Bug 2" Algorithm

1) head toward goal on the *m-line*

2) if an obstacle is in the way, follow it until you encounter the m-line again.

3) Leave the obstacle and continue toward the goal



NO! How do we fix this?

3:10

CarnegieMellon

A better bug?

"Bug 2" Algorithm

1) head toward goal on the *m-line*

2) if an obstacle is in the way, follow it until you encounter the m-line again ***closer to the goal***.

3) Leave the obstacle and continue toward the goal

Better or worse than Bug1?

3:11

BUG algorithms – conclusions

- Other variants: TangentBug, VisBug, RoverBug, WedgeBug, ...
- only 2D! (TangentBug has extension to 3D)
- Guaranteed convergence
- Still active research:

  K. Taylor and S.M. LaValle: *I-Bug: An Intensity-Based Bug Algorithm*

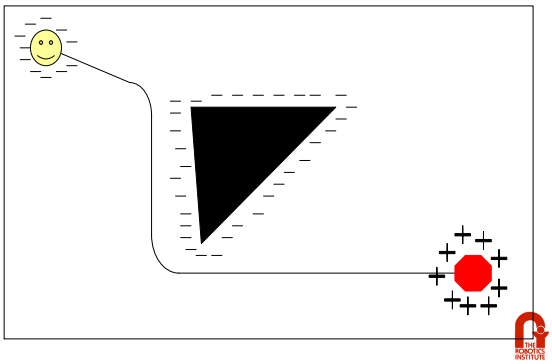
⇒ Useful for minimalistic, robust 2D goal reaching

  – not useful for finding paths in joint space

3:12

CarnegieMellon

Start-Goal Algorithm: Potential Functions



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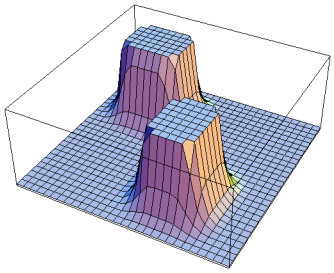
Repulsive Potential



3:14

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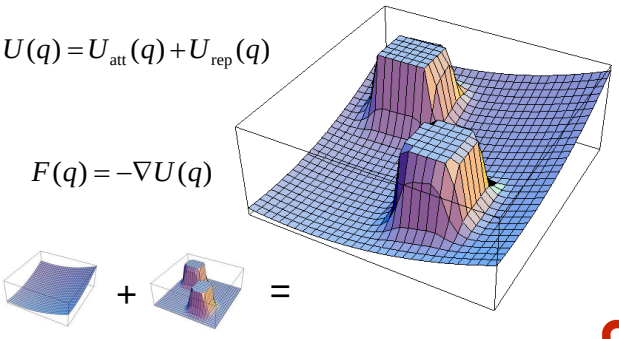
Total Potential Function

$$U(q) = U_{att}(q) + U_{rep}(q)$$

$$F(q) = -\nabla U(q)$$



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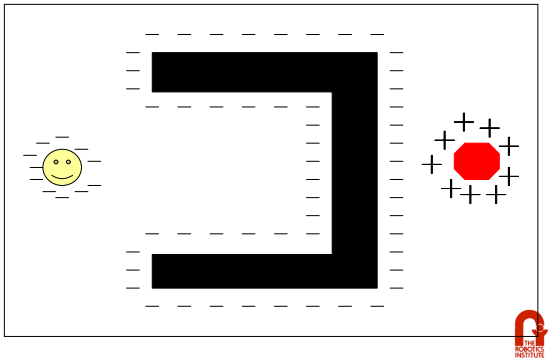
Local Minimum Problem with the Charge Analogy



3:16

Potential fields – conclusions

- Very simple, therefore popular
- In our framework: Combining a goal (endeffector) task variable, with a constraint (collision avoidance) task variable; then using inv. kinematics is *exactly* the same as "Potential Fields"

⇒ Does not solve locality problem of feedback control.

3:17

CarnegieMellon

The Wavefront in Action (Part 2)

- Now repeat with the modified cells
  - This will be repeated until no 0's are adjacent to cells with values >= 2
    - 0's will only remain when regions are unreachable

7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0
3	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4
1	0	0	0	0	0	0	0	0	0	0	0	0	4	3	3
0	0	0	0	0	0	0	0	0	0	0	0	0	4	3	2
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

3:18

CarnegieMellon

The Wavefront in Action (Part 1)

- Starting with the goal, set all adjacent cells with "0" to the current cell + 1
  - 4-Point Connectivity or 8-Point Connectivity?
  - Your Choice. We'll use 8-Point Connectivity in our example

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
3	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	2

3:19

CarnegieMellon

The Wavefront in Action (Part 2)

- Now repeat with the modified cells
  - This will be repeated until no 0's are adjacent to cells with values >= 2
    - 0's will only remain when regions are unreachable

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
3	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0	4	4	4	
1	0	0	0	0	0	0	0	0	0	0	0	0	4	3	3	
0	0	0	0	0	0	0	0	0	0	0	0	0	4	3	2	

3:20

CarnegieMellon

The Wavefront in Action (Part 3)

- Repeat again...

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0
3	0	0	0	0	1	1	1	1	1	1	1	1	5	5	5	5
2	0	0	0	0	0	0	0	0	0	0	0	0	5	4	4	4
1	0	0	0	0	0	0	0	0	0	0	0	0	5	4	3	3
0	0	0	0	0	0	0	0	0	0	0	0	0	5	4	3	2

3:21

CarnegieMellon

The Wavefront in Action (Part 4)

- And again...

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	1	1	1	1	1	1	1	1	6	6	6	6
3	0	0	0	0	1	1	1	1	1	1	1	1	5	5	5	5
2	0	0	0	0	0	0	0	0	0	0	0	0	6	5	4	4
1	0	0	0	0	0	0	0	0	0	0	0	0	6	5	4	3
0	0	0	0	0	0	0	0	0	0	0	0	0	6	5	4	3	2

3:22

CarnegieMellon

The Wavefront in Action (Part 5)

- And again until...

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	7	7	7	7	7
4	0	0	0	0	1	1	1	1	1	1	1	1	6	6	6	6
3	0	0	0	0	1	1	1	1	1	1	1	1	5	5	5	5
2	0	0	0	0	0	0	0	0	0	0	0	0	7	6	5	4	4
1	0	0	0	0	0	0	0	0	0	0	0	0	7	6	5	4	3	3
0	0	0	0	0	0	0	0	0	0	0	0	0	7	6	5	4	3	2

3:23

CarnegieMellon

The Wavefront in Action (Done)

- You're done
  - Remember, 0's should only remain if unreachable regions exist

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
7	18	17	16	15	14	13	12	11	10	9	9	9	9	9	9	9
6	17	17	16	15	14	13	12	11	10	9	8	8	8	8	8	8
5	17	16	16	15	14	13	12	11	10	9	8	7	7	7	7	7
4	17	16	15	15	1	1	1	1	1	1	1	1	6	6	6	6
3	17	16	15	14	1	1	1	1	1	1	1	1	5	5	5	5
2	17	16	15	14	13	12	11	10	9	8	7	6	5	4	4	4
1	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	3
0	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2

3:24

The Wavefront, Now What?

- To find the shortest path, according to your metric, simply always move toward a cell with a lower number
  - The numbers generated by the Wavefront planner are roughly proportional to their distance from the goal

Two possible shortest paths shown

3:25

Dijkstra Algorithm

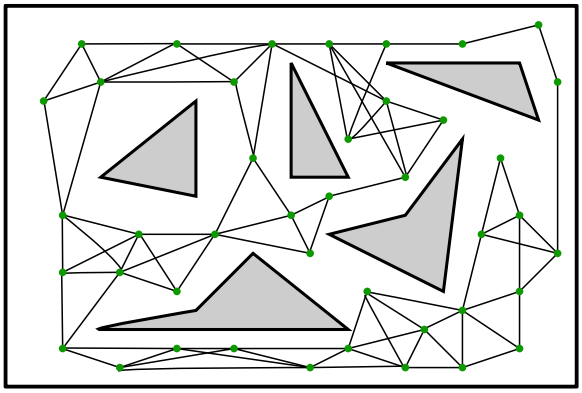
- Is efficient in **discrete domains**
  - Given start and goal node in an arbitrary graph
  - Incrementally label nodes with their distance-from-start
- Produces optimal (shortest) paths
- Applying this to continuous domains requires discretization
  - Grid-like discretization in high-dimensions is daunting! *(curse of dimensionality)*
  - What are other ways to "discretize" space more efficiently?

3:26

Sample-based Path Finding

3:27

Probabilistic Road Maps



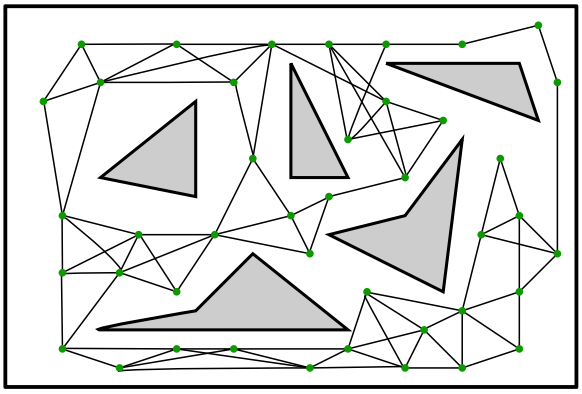
[Kavraki, Svetska, Latombe,Overmars, 95]

$q \in \mathbb{R}^n$ describes configuration

Q_{free} is the set of configurations without collision

3:28

Probabilistic Road Maps



[Kavraki, Svetska, Latombe,Overmars, 95]

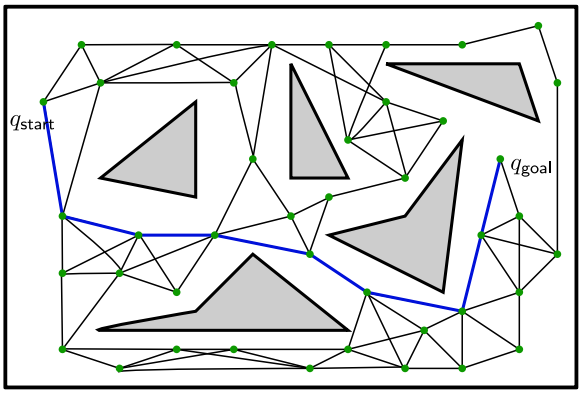
Probabilistic Road Map

– generates a graph $G = (V, E)$ of configurations

– such that configurations along each edges are $\in Q_{\text{free}}$

3:29

Probabilistic Road Maps



Given the graph, use (e.g.) Dijkstra to find path from q_{start} to q_{goal}.

3:30

Probabilistic Road Maps – generation

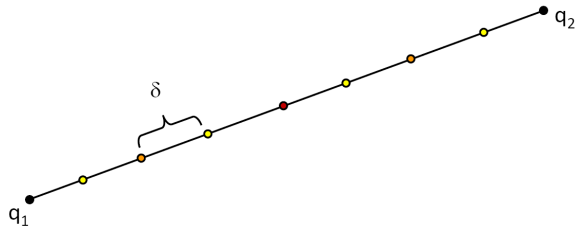
Input: number n of samples, number k number of nearest neighbors
Output: PRM $G = (V, E)$

```
initialize V = ∅, E = ∅
while |V| < n do                    // find n collision free points q_i
    q ← random sample from Q
    if q ∈ Q_free then V ← V ∪ {q}
end while
for all q ∈ V do  // check if near points can be connected
    N_q ← k nearest neighbors of q in V
    for all q' ∈ N_q do
        if path(q, q') ∈ Q_free then E ← E ∪ {(q, q')}
    end for
end for
```

where path(q, q') is a local planner (easiest: straight line)

3:31

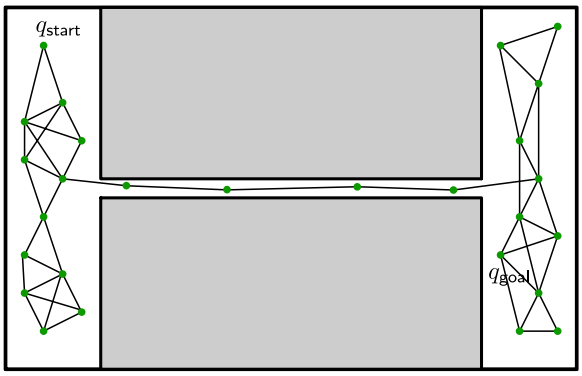
Local Planner



tests collisions up to a specified resolution δ

3:32

Problem: Narrow Passages



The smaller the gap (clearance ϱ) the more unlikely to sample such points.

3:33

PRM theory

(for uniform sampling in d-dim space)

- Let $a, b \in Q_{\text{free}}$ and γ a path in Q_{free} connecting a and b.

Then the probability that PRM found the path after n samples is

$$P(\text{PRM-success} \,|\, n) \geq 1 - \frac{2|\gamma|}{\varrho}\, e^{-\sigma \varrho^d n}$$

$\sigma = \frac{|B_1|}{2^d |Q_{\text{free}}|}$

$\varrho =$ clearance of γ (distance to obstacles)

(roughly: the exponential term are "volume ratios")

- This result is called *probabilistic complete* (one can achieve any probability with high enough n)
- For a given success probability, n needs to be exponential in d

3:34

Other PRM sampling strategies

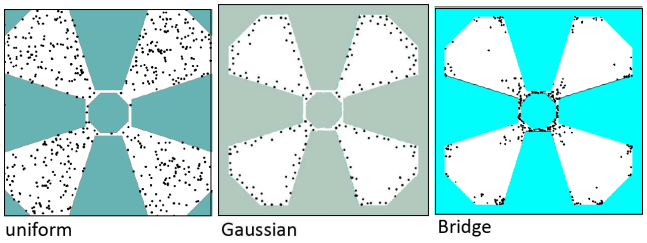


illustration from O. Brock's lecture

Gaussian: $q_1 \sim \mathcal{U}$; $q_2 \sim \mathcal{N}(q_1, \sigma)$; if $q_1 \in Q_{\text{free}}$ and $q_2 \notin Q_{\text{free}}$, add q_1 (or vice versa).

Bridge: $q_1 \sim \mathcal{U}$; $q_2 \sim \mathcal{N}(q_1, \sigma)$; $q_3 = (q_1 + q_2)/2$; if $q_1, q_2 \notin Q_{\text{free}}$ and $q_3 \in Q_{\text{free}}$, add q_3.

- Sampling strategy can be made more intelligence: "utility-based sampling"
- Connection sampling

  (once earlier sampling has produced connected components)

3:35

Probabilistic Roadmaps – conclusions

- Pros:
  - Algorithmically very simple
  - Highly explorative
  - Allows probabilistic performance guarantees
  - Good to answer many queries in an *unchanged* environment
- Cons:
  - Precomputation of exhaustive roadmap takes a long time

    (but not necessary for "Lazy PRMs")

3:36

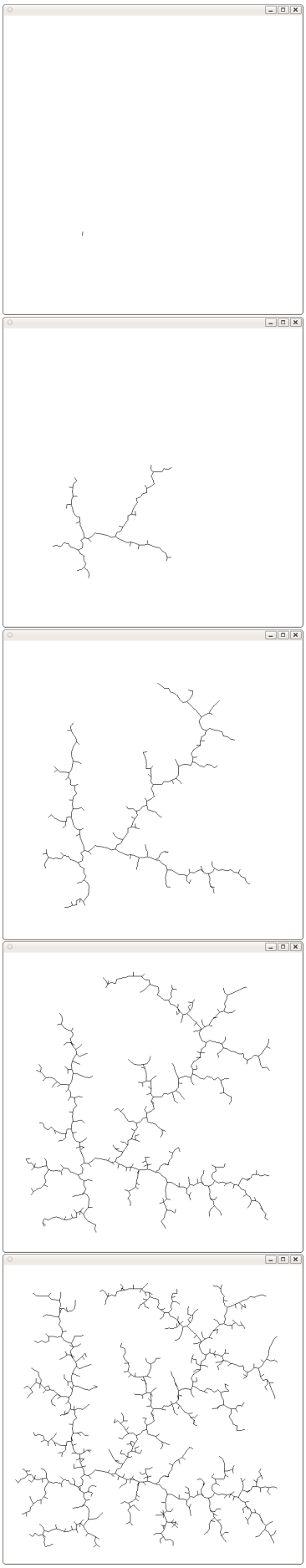
Rapidly Exploring Random Trees

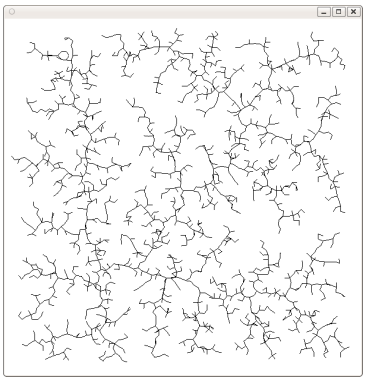
2 motivations:

- Single Query path finding: Focus computational efforts on paths for specific $(q_{\text{start}}, q_{\text{goal}})$
- Use actually controllable DoFs to incrementally explore the search space: *control-based* path finding.

(Ensures that RRTs can be extended to handling differential constraints.)

3:37





$n = 1$ $n = 100$ $n = 300$ $n = 600$ $n = 1000$ $n = 2000$

3:38

Rapidly Exploring Random Trees

Simplest RRT with straight line local planner and step size α

Input: q_{start}, number n of nodes, stepsize α
Output: tree $T = (V, E)$

initialize $V = \{q_{\text{start}}\}$, $E = \emptyset$
for $i = 0 : n$ **do**
  $q_{\text{target}} \leftarrow$ random sample from Q
  $q_{\text{near}} \leftarrow$ nearest neighbor of q_{target} in V
  $q_{\text{new}} \leftarrow q_{\text{near}} + \frac{\alpha}{|q_{\text{target}} - q_{\text{near}}|}(q_{\text{target}} - q_{\text{near}})$
  if $q_{\text{new}} \in Q_{\text{free}}$ **then** $V \leftarrow V \cup \{q_{\text{new}}\}$, $E \leftarrow E \cup \{(q_{\text{near}}, q_{\text{new}})\}$
end for

3:39

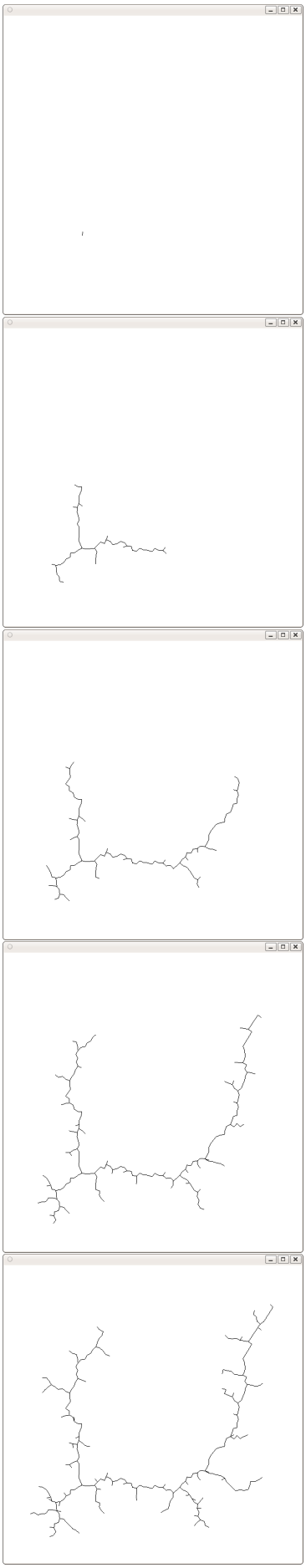
Rapidly Exploring Random Trees

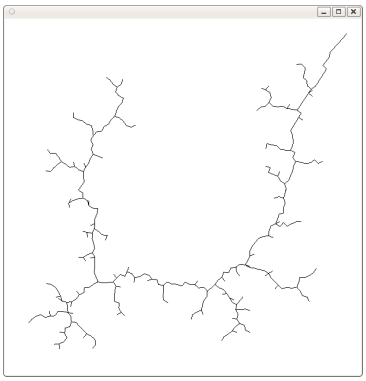
RRT growing directedly towards the goal

Input: q_{start}, q_{goal}, number n of nodes, stepsize α, β
Output: tree $T = (V, E)$

initialize $V = \{q_{\text{start}}\}$, $E = \emptyset$
for $i = 0 : n$ **do**
  if rand$(0, 1) < \beta$ **then** $q_{\text{target}} \leftarrow q_{\text{goal}}$
  else $q_{\text{target}} \leftarrow$ random sample from Q
  $q_{\text{near}} \leftarrow$ nearest neighbor of q_{target} in V
  $q_{\text{new}} \leftarrow q_{\text{near}} + \frac{\alpha}{|q_{\text{target}} - q_{\text{near}}|}(q_{\text{target}} - q_{\text{near}})$
  if $q_{\text{new}} \in Q_{\text{free}}$ **then** $V \leftarrow V \cup \{q_{\text{new}}\}$, $E \leftarrow E \cup \{(q_{\text{near}}, q_{\text{new}})\}$
end for

3:40



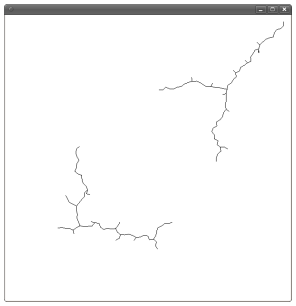


$n = 1$ $n = 100$ $n = 200$ $n = 300$ $n = 400$ $n = 500$

3:41

Bi-directional search

- grow two trees starting from q_{start} and q_{goal}



let one tree grow towards the other

(e.g., "choose q_{new} of T_1 as q_{target} of T_2")

3:42

Summary: RRTs

- Pros (shared with PRMs):
  - Algorithmically very simple
  - Highly explorative
  - Allows probabilistic performance guarantees
- Pros (beyond PRMs):
  - Focus computation on single query $(q_{\text{start}}, q_{\text{goal}})$ problem
  - Trees from multiple queries can be merged to a roadmap
  - Can be extended to differential constraints (nonholonomic systems)
- To keep in mind (shared with PRMs):
  - The metric (for nearest neighbor selection) is sometimes critical
  - The local planner may be non-trivial

3:43

References

Steven M. LaValle: *Planning Algorithms*, http://planning.cs.uiuc.edu/.

Choset et. al.: *Principles of Motion Planning*, MIT Press.

Latombe's "motion planning" lecture, http://robotics.stanford.edu/~latombe/cs326/2007/schedule.htm

3:44

Non-holonomic systems

3:45

Non-holonomic systems

- We define a **nonholonomic system** as one with **differential constraints**:

$$\dim(u_t) < \dim(x_t)$$

$\Rightarrow$ *Not all degrees of freedom are directly controllable*

- Dynamic systems are an example!

- General definition of a differential constraint:

For any given state x, let U_x be the tangent space that is generated by controls:

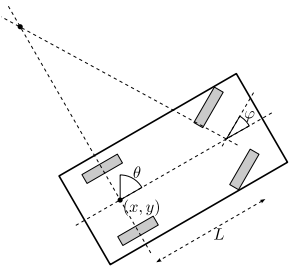
$$U_x = \{\dot{x} \ : \ \dot{x} = f(x,u),\ u \in U\}$$

(non-holonomic $\iff$ $\dim(U_x) < \dim(x)$)

The elements of U_x are elements of T_x subject to additional *differential constraints*.

3:46

Car example



$$\dot{x} = v\ \cos\theta$$
$$\dot{y} = v\ \sin\theta$$
$$\dot{\theta} = (v/L)\ \tan\varphi$$
$$|\varphi| < \Phi$$

State $q = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix}$ **Controls** $u = \begin{pmatrix} v \\ \varphi \end{pmatrix}$

Sytem equation $\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} v\ \cos\theta \\ v\ \sin\theta \\ (v/L)\ \tan\varphi \end{pmatrix}$

3:47

Car example

- The car is a *non-holonomic* system: not all DoFs are controlled, $\dim(u) < \dim(q)$

We have the *differential constraint* $\dot{q}$:

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$

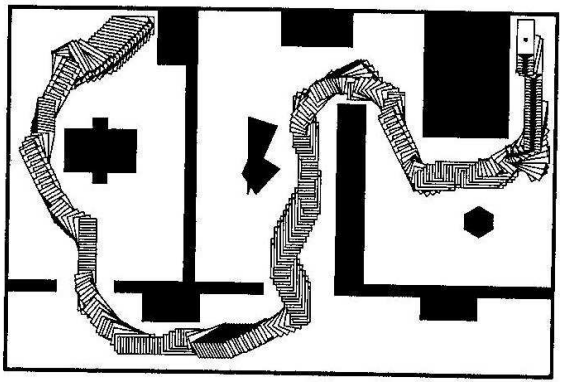
"A car cannot move directly lateral."

- Analogy to dynamic systems: Just like a car cannot instantly move sidewards, a dynamic system cannot instantly change its position q: the current change in position is *constrained* by the current velocity $\dot{q}$.

3:48

Path finding with a non-holonomic system

Could a car follow this trajectory?

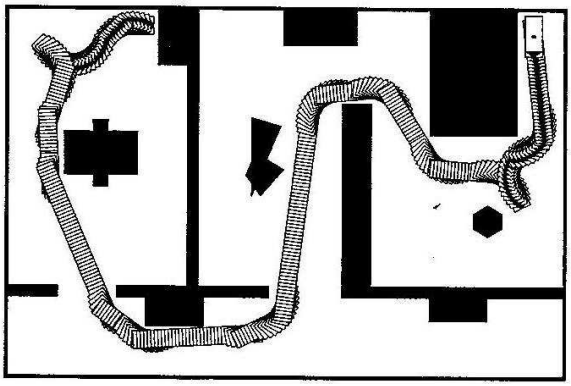


This is generated with a PRM in the state space $q = (x, y, \theta)$ *ignoring the differntial constraint*.

3:49

Path finding with a non-holonomic system

This is a solution we would like to have:



The path respects **differential constraints.**

Each step in the path corresponds to setting certain controls.

3:50

Control-based sampling to grow a tree

- Control-based sampling: fulfils none of the nice exploration properties of RRTs, but fulfils the differential constraints:

  1) Select a $q \in T$ from tree of current configurations

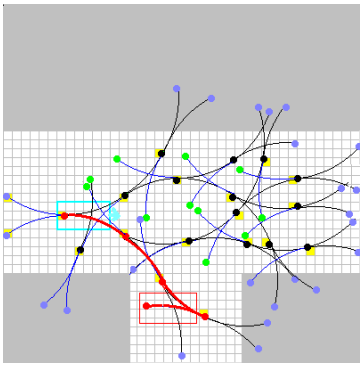
  2) Pick control vector u at random

  3) Integrate equation of motion over short duration (picked at random or not)

  4) If the motion is collision-free, add the endpoint to the tree

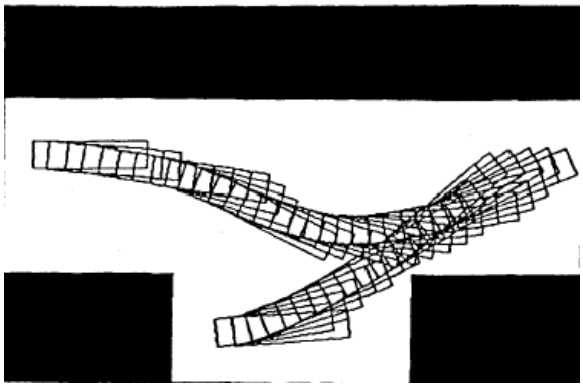
3:51

Control-based sampling for the car



1) Select a $q \in T$
2) Pick v, ϕ, and τ
3) Integrate motion from q
4) Add result if collision-free

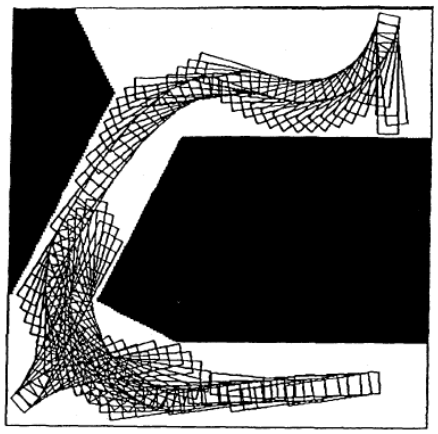
3:52



J. Barraquand and J.C. Latombe. Nonholonomic Multibody Robots: Controllability and Motion Planning in the Presence of Obstacles. Algorithmica, 10:121-155, 1993.

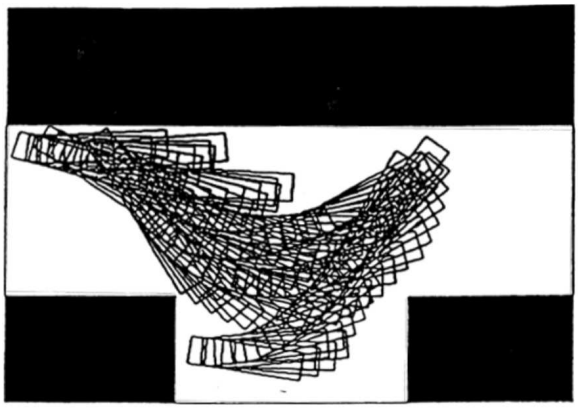
car parking

3:53



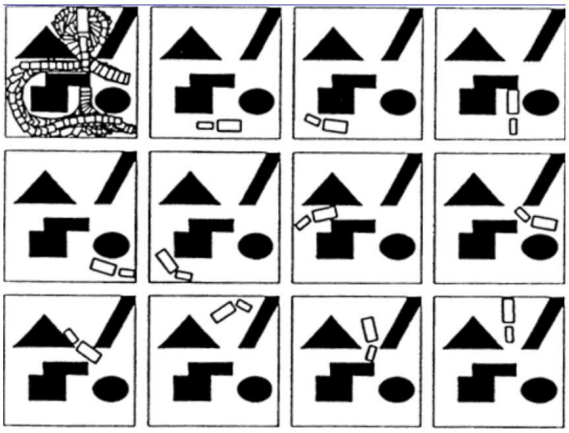
car parking

3:54



parking with only left-steering

3:55



with a trailer

3:56

Better control-based exploration: RRTs revisited

- RRTs with differential constraints:

Input: q_{start}, number k of nodes, time interval τ
Output: tree $T = (V, E)$
1: initialize $V = \{q_{\text{start}}\}$, $E = \emptyset$
2: **for** $i = 0 : k$ **do**
3: $q_{\text{target}} \leftarrow$ random sample from Q
4: $q_{\text{near}} \leftarrow$ nearest neighbor of q_{target} in V
5: use local planner to compute controls u that steer q_{near} towards q_{target}
6: $q_{\text{new}} \leftarrow q_{\text{near}} + \int_{t=0}^{\tau} \dot{q}(q, u)dt$
7: **if** $q_{\text{new}} \in Q_{\text{free}}$ **then** $V \leftarrow V \cup \{q_{\text{new}}\}$, $E \leftarrow E \cup \{(q_{\text{near}}, q_{\text{new}})\}$
8: **end for**

- Crucial questions:
  – How meassure *near* in nonholonomic systems?
  – How find controls u to steer towards target?

3:57

Metrics

Standard/Naive metrics:

- Comparing two 2D rotations/orientations $\theta_1, \theta_2 \in SO(2)$:
  a) Euclidean metric between $e^{i\theta_1}$ and $e^{i\theta_2}$
  b) $d(\theta_1, \theta_2) = \min\{|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|\}$
- Comparing two configurations $(x, y, \theta)_{1,2}$ in $\mathbb{R}^2$:
  Eucledian metric on $(x, y, e^{i\theta})$
- Comparing two 3D rotations/orientations $r_1, r_2 \in SO(3)$:
  Represent both orientations as unit-length quaternions $r_1, r_2 \in \mathbb{R}^4$:
  $$d(r_1, d_2) = \min\{|r_1 - r_2|, |r_1 + r_2|\}$$
  where $|\cdot|$ is the Euclidean metric.
  (Recall that r_1 and $-r_1$ represent exactly the same rotation.)

- **Ideal metric:**
  Optimal cost-to-go between two states x_1 and x_2:
  – Use optimal trajectory cost as metric
  – This is as hard to compute as the original problem, of course!!
  → Approximate, e.g., by neglecting obstacles.

3:58

Dubins curves

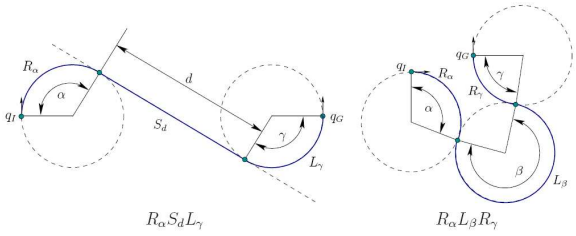
- Dubins car: constant velocity, and steer $\varphi \in [-\Phi, \Phi]$

- Neglecting obstacles, there are only **six** types of trajectories that connect any configuration $\in \mathbb{R}^2 \times \mathbb{S}^1$:
  $$\{LRL, RLR, LSL, LSR, RSL, RSR\}$$

- annotating durations of each phase:
  $\{L_\alpha R_\beta L_\gamma, , R_\alpha L_\beta R_\gamma, L_\alpha S_d L_\gamma, L_\alpha S_d R_\gamma, R_\alpha S_d L_\gamma, R_\alpha S_d R_\gamma\}$
  with $\alpha \in [0, 2\pi), \beta \in (\pi, 2\pi), d \geq 0$

3:59

Dubins curves



→ By testing all six types of trajectories for (q_1, q_2) we can define a Dubins metric for the RRT – and use the Dubins curves as controls!

- **Reeds-Shepp curves** are an extension for cars which can drive back.
  (includes 46 types of trajectories, good metric for use in RRTs for cars)

3:60

4 Path Optimization

very briefly

Outline

- These are only some very brief notes on path optimization
- The aim is to explain how to *formulate* the optimization problem. Concerning the optimization algortihm itself, refer to the *Optimization* lecture.

4:1

From inverse kinematics to path costs

- Recall our optimality principle of inverse kinematics

$$\operatorname*{argmin}_q \|q - q_0\|_W^2 + \|\Phi(q)\|^2$$

- A trajectory $q_{0:T}$ is a sequence of robot configurations $q_t \in \mathbb{R}^n$
- Consider the cost function

$$f(q_{0:T}) = \sum_{t=0}^{T} \|\Psi_t(q_{t\text{-}k}, .., q_t)\|^2 + \sum_{t=0}^{T} \|\Phi_t(q_t)\|^2$$

(where $(q_{-k}, .., q_{\text{-}1})$ is a given prefix)

- $\Psi_t(q_{t\text{-}k}, .., q_t)$ represents **control costs**

  k denotes the **order** of the control costs

  $\Phi_t(q_t)$ represents **task costs**

  (More generally, task costs could depend on $\Phi_t(q_{t\text{-}k}, .., q_t)$)

4:2

Control costs

- The $\Psi_t(q_{t\text{-}k}, .., q_t)$ can penalize various things:

$k=0$	$\Psi_t(q_t) = q_t - q_0$	penalize offset from zero
$k=1$	$\Psi_t(q_{t\text{-}1}, q_t) = q_t - q_{t\text{-}1}$	penalize velocity
$k=2$	$\Psi_t(q_{t\text{-}2}, .., q_t) = q_t - 2q_{t\text{-}1} + q_{t\text{-}2}$	penalize acceleration
$k=3$	$\Psi_t(q_{t\text{-}3}, .., q_t) = q_t - 3q_{t\text{-}1} + 3q_{t\text{-}2} - q_{t\text{-}3}$	penalize jerk

- The big $\Phi_t(q_t)$ imposes tasks as for inverse kinematics

4:3

Choice of optimizer

$$f(q_{0:T}) = \sum_{t=0}^{T} \|\Psi_t(q_{t\text{-}k}, .., q_t)\|^2 + \sum_{t=0}^{T} \|\Phi_t(q_t)\|^2$$

Is in the form of the so-called **Gauss-Newton** optimization problem, and can be solved using such 2nd order methods.

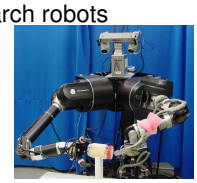
(Note that the pseudo Hessian is a banded, symmetric, positive-definite matrix.)

- Alternativ: formulate hard constraints in the framework of constrained optimization

4:4

5 Dynamics

1D point mass, damping & oscillation, PID, dynamics of mechanical systems, Euler-Lagrange equation, Newton-Euler recursion, general robot dynamics, joint space control, reference trajectory following, operational space control

Kinematic	Dynamic
instantly change joint velocities $\dot{q}$: $\delta q_t \stackrel{!}{=} J^\sharp\,(y^* - \phi(q_t))$	instantly change joint torques u: $u \stackrel{!}{=} ?$
accounts for kinematic coupling of joints but **ignores inertia, forces, torques**	accounts for dynamic coupling of joints and full Newtonian physics
gears, **stiff**, all of industrial robots 	future robots, **compliant**, few research robots 

5:1

When velocities cannot be changed/set arbitrarily

- Examples:
  - An air plane flying: You cannot command it to hold still in the air, or to move straight up.
  - A car: you cannot command it to move side-wards.
  - Your arm: you cannot command it to throw a ball with arbitrary speed (force limits).
  - A *torque controlled* robot: You cannot command it to instantly change velocity (infinite acceleration/torque).
- What all examples have in comment:
  - One can set **controls** u_t (air plane's control stick, car's steering wheel, your muscles activations, torque/voltage/current send to a robot's motors)
  - But these controls only indirectly influence the **dynamics of state**, $x_{t+1} = f(x_t, u_t)$

5:2

Dynamics

- The dynamics of a system describes how the controls u_t influence the change-of-state of the system

$$x_{t+1} = f(x_t, u_t)$$

  - The notation x_t refers to the *dynamic state* of the system: e.g., joint positions *and velocities* $x_t = (q_t, \dot{q}_t)$.
  - f is an arbitrary function, often smooth

5:3

Outline

- We start by discussing a **1D point mass** for 3 reasons:
  - The most basic force-controlled system with inertia
  - We can introduce and understand **PID control**
  - The behavior of a point mass under PID control is a *reference* that we can also follow with arbitrary dynamic robots (if the dynamics are known)
- We discuss computing the dynamics of general robotic systems
  - Euler-Lagrange equations
  - Euler-Newton method
- We derive the dynamic equivalent of inverse kinematics:
  - operational space control

5:4

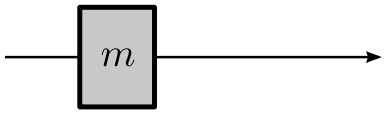
PID and a 1D point mass

5:5

The dynamics of a 1D point mass

- Start with simplest possible example: 1D point mass

  (no gravity, no friction, just a single mass)



- The state $x(t) = (q(t), \dot{q}(t))$ is described by:
  - position $q(t) \in \mathbb{R}$
  - velocity $\dot{q}(t) \in \mathbb{R}$
- The controls $u(t)$ is the force we apply on the mass point
- The system dynamics is:

$$\ddot{q}(t) = u(t)/m$$

5:6

1D point mass – proportional feedback

- Assume current position is q.

  The goal is to move it to the position q^*.

  What can we do?

- **Idea 1:**

"Always pull the mass towards the goal q^*:"

$$u = K_p\ (q^* - q)$$

5:7

1D point mass – proportional feedback

- What's the effect of this control law?

$$m\ \ddot{q} = u = K_p\ (q^* - q)$$

$q = q(t)$ is a function of time, this is a second order differential equation

- Solution: assume $q(t) = a + be^{\omega t}$

(a "non-imaginary" alternative would be $q(t) = a + b\,\epsilon^{-\lambda t}\ \cos(\omega t)$)

$$\begin{aligned}
&m\ b\ \omega^2\ e^{\omega t} = K_p\ q^* - K_p\ a - K_p\ b\ e^{\omega t} \\
&(m\ b\ \omega^2 + K_p\ b)\ e^{\omega t} = K_p\ (q^* - a) \\
&\Rightarrow (m\ b\ \omega^2 + K_p\ b) = 0\ \wedge\ (q^* - a) = 0 \\
&\Rightarrow\ \omega = i\sqrt{K_p/m} \\
&q(t) = q^* + b\ e^{i\sqrt{K_p/m}\ t}
\end{aligned}$$

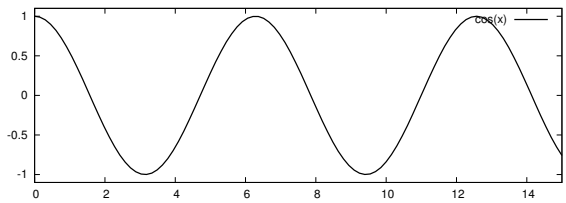
This is an oscillation around q^* with amplitude $b = q(0) - q^*$ and frequency $\sqrt{K_p/m}$!

5:8

1D point mass – proportional feedback

$$\begin{aligned}
&m\ \ddot{q} = u = K_p\ (q^* - q) \\
&q(t) = q^* + b\ e^{i\sqrt{K_p/m}\ t}
\end{aligned}$$

Oscillation around q^* with amplitude $b = q(0) - q^*$ and frequency $\sqrt{K_p/m}$



5:9

1D point mass – derivative feedback

- **Idea 2**

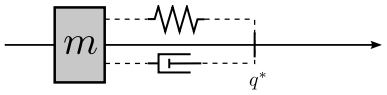
"Pull less, when we're heading the right direction already:"

"Damp the system:"

$$u = K_p(q^* - q) + K_d(\dot{q}^* - \dot{q})$$

$\dot{q}^*$ is a desired goal velocity

For simplicity we set $\dot{q}^* = 0$ in the following.



5:10

1D point mass – derivative feedback

- What's the effect of this control law?

$$m\ddot{q} = u = K_p(q^* - q) + K_d(0 - \dot{q})$$

- Solution: again assume $q(t) = a + be^{\omega t}$

$$\begin{aligned}
&m\ b\ \omega^2\ e^{\omega t} = K_p\ q^* - K_p\ a - K_p\ b\ e^{\omega t} - K_d\ b\ \omega e^{\omega t} \\
&(m\ b\ \omega^2 + K_d\ b\ \omega + K_p\ b)\ e^{\omega t} = K_p\ (q^* - a) \\
&\Rightarrow (m\ \omega^2 + K_d\ \omega + K_p) = 0\ \wedge\ (q^* - a) = 0 \\
&\Rightarrow \omega = \frac{-K_d \pm \sqrt{K_d^2 - 4mK_p}}{2m} \\
&q(t) = q^* + b\ e^{\omega\ t}
\end{aligned}$$

The term $-\frac{K_d}{2m}$ in ω is real $\leftrightarrow$ exponential decay (damping)

5:11

1D point mass – derivative feedback

$$q(t) = q^* + b\ e^{\omega\ t}\ ,\quad \omega = \frac{-K_d \pm \sqrt{K_d^2 - 4mK_p}}{2m}$$

- Effect of the second term $\sqrt{K_d^2 - 4mK_p}/2m$ in ω:

$K_d^2 < 4mK_p$ $\Rightarrow$ ω has imaginary part oscillating with frequency $\sqrt{K_p/m - K_d^2/4m^2}$
$q(t) = q^* + be^{-K_d/2m\ t}\ e^{i\sqrt{K_p/m - K_d^2/4m^2}\ t}$

$K_d^2 > 4mK_p$ $\Rightarrow$ ω real
strongly damped

$K_d^2 = 4mK_p$ $\Rightarrow$ second term zero
only exponential decay

5:12

1D point mass – derivative feedback

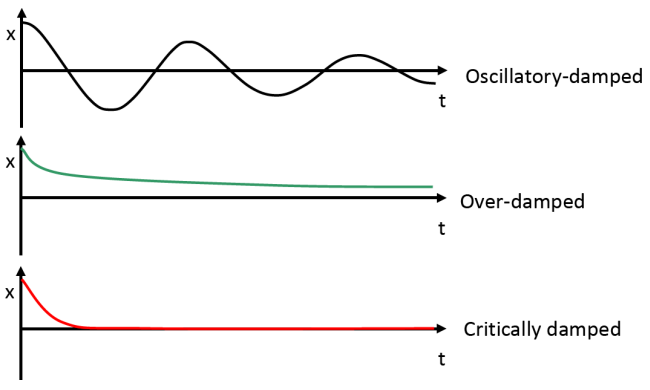


illustration from O. Brock's lecture

5:13

1D point mass – derivative feedback

Alternative parameterization:

Instead of the *gains* K_p and K_d it is sometimes more intuitive to set the

- wave length $\lambda = \frac{1}{\omega_0} = \frac{1}{\sqrt{K_p/m}}$, $K_p = m/\lambda^2$
- damping ratio $\xi = \frac{K_d}{\sqrt{4mK_p}} = \frac{\lambda K_d}{2m}$, $K_d = 2m\xi/\lambda$

$\xi > 1$: over-damped

$\xi = 1$: critically dampled

$\xi < 1$: oscillatory-damped

$$q(t) = q^* + be^{-\xi\ t/\lambda}\ e^{i\sqrt{1-\xi^2}\ t/\lambda}$$

5:14

1D point mass – integral feedback

- **Idea 3**

"Pull if the position error accumulated large in the past:"

$$u = K_p(q^* - q) + K_d(\dot{q}^* - \dot{q}) + K_i \int_{s=0}^{t} (q^*(s) - q(s))\ ds$$

- This is not a linear ODE w.r.t. $x = (q, \dot{q})$.
  However, when we extend the state to $x = (q, \dot{q}, e)$ we have the ODE

$$\dot{q} = \dot{q}$$
$$\ddot{q} = u/m = K_p/m(q^* - q) + K_d/m(\dot{q}^* - \dot{q}) + K_i/m\ e$$
$$\dot{e} = q^* - q$$

(no explicit discussion here)

5:15

1D point mass – PID control

$$u = K_p(q^* - q) + K_d(\dot{q}^* - \dot{q}) + K_i \int_{s=0}^{t} (q^* - q(s))\ ds$$

- **PID control**

– Proportional Control ("Position Control")

$f \propto K_p(q^* - q)$

– Derivative Control ("Damping")

$f \propto K_d(\dot{q}^* - \dot{q})$ ($\dot{x}^* = 0 \rightarrow$ damping)

– Integral Control ("Steady State Error")

$f \propto K_i \int_{s=0}^{t}(q^*(s) - q(s))\ ds$

5:16

Controlling a 1D point mass – lessons learnt

- Proportional and derivative feedback (PD control) are like adding a spring and damper to the point mass
- PD control is a *linear control law*

$$(q, \dot{q}) \mapsto u = K_p(q^* - q) + K_d(\dot{q}^* - \dot{q})$$

(linear in the *dynamic system state* $x = (q, \dot{q})$)

- With such linear control laws we can design approach trajectories (by tuning the gains)

– but no optimality principle behind such motions

5:17

Dynamics of mechanical systems

5:18

Two ways to derive dynamics equations for mechanical systems

- The Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = u$$

Used when you want to derive analytic equations of motion ("on paper")

- The Newton-Euler recursion (and related algorithms)

$$f_i = m\dot{v}_i\ , \quad u_i = I_i\dot{w} + w \times Iw$$

Algorithms that "propagate" forces through a kinematic tree and numerically compute the *inverse* dynamics $u = \text{NE}(q, \dot{q}, \ddot{q})$ or *forward* dynamics $\ddot{q} = f(q, \dot{q}, u)$.

5:19

The Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = u$$

- $L(q, \dot{q})$ is called **Lagrangian** and defined as

$$L = T - U$$

where T=kinetic energy and U=potential energy.

- q is called generalized coordinate – any coordinates such that $(q, \dot{q})$ describes the state of the system. Joint angles in our case.
- u are external forces

5:20

The Euler-Lagrange equation

- How is this typically done?
- **First**, describe the *kinematics and Jacobians* for every link i:

$$(q, \dot{q}) \mapsto \{T_{W\to i}(q), v_i, w_i\}$$

Recall $T_{W\to i}(q) = T_{W\to A}\, T_{A\to A'}(q)\, T_{A'\to B}\, T_{B\to B'}(q)\, \cdots$
Further, we know that a link's velocity $v_i = J_i \dot{q}$ can be described via its position Jacobian.
Similarly we can describe the link's *angular velocity* $w_i = J_i^w \dot{q}$ as linear in $\dot{q}$.

- **Second**, formulate the kinetic energy

$$T = \sum_i \frac{1}{2} m_i v_i^2 + \frac{1}{2} w_i^\top I_i w_i = \sum_i \frac{1}{2} \dot{q}^\top M_i \dot{q}\,, \quad M_i = \begin{pmatrix} J_i \\ J_i^w \end{pmatrix}^\top \begin{pmatrix} m_i \mathbf{I}_3 & 0 \\ 0 & I_i \end{pmatrix} \begin{pmatrix} J_i \\ J_i^w \end{pmatrix}$$

where $I_i = R_i \bar{I}_i R_i^\top$ and $\bar{I}_i$ the inertia tensor in link coordinates

- **Third**, formulate the potential energies (typically independent of $\dot{q}$)

$$U = g m_i \text{height}(i)$$

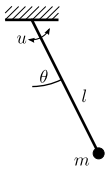
- **Fourth**, compute the partial derivatives analytically to get something like

$$\underbrace{u}_{\text{control}} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \underbrace{M}_{\text{inertia}}\, \ddot{q} + \underbrace{\dot{M}\dot{q} - \frac{\partial T}{\partial q}}_{\text{Coriolis}} + \underbrace{\frac{\partial U}{\partial q}}_{\text{gravity}}$$

which relates accelerations $\ddot{q}$ to the forces

5:21

Example: A pendulum



- Generalized coordinates: angle $q = (\theta)$
- Kinematics:
  - velocity of the mass: $v = (l\dot{\theta}\cos\theta, 0, l\dot{\theta}\sin\theta)$
  - angular velocity of the mass: $w = (0, -\dot{\theta}, 0)$
- Energies:

$$T = \frac{1}{2} m v^2 + \frac{1}{2} w^\top I w = \frac{1}{2}(m l^2 + I_2)\dot{\theta}^2\,, \quad U = -mgl\cos\theta$$

- Euler-Lagrange equation:

$$\begin{aligned} u &= \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \\ &= \frac{d}{dt}(m l^2 + I_2)\dot{\theta} + mgl\sin\theta = (m l^2 + I_2)\ddot{\theta} + mgl\sin\theta \end{aligned}$$

5:22

Newton-Euler recursion

- An algorithm that computes the *inverse dynamics*

$$u = \text{NE}(q, \dot{q}, \ddot{q}^*)$$

by recursively computing force balance at each joint:

  - **Newton's equation** expresses the force acting at the center of mass for an accelerated body:

$$f_i = m\dot{v}_i$$

  - **Euler's equation** expresses the torque (=control!) acting on a rigid body given an angular velocity and angular acceleration:

$$u_i = I_i \dot{w} + w \times I w$$

- **Forward recursion:** ($\approx$ kinematics)

  Compute (angular) velocities (v_i, w_i) *and* accelerations $(\dot{v}_i, \dot{w}_i)$ for every link (via forward propagation; see geometry notes for details)

- **Backward recursion:**

  For the leaf links, we now know the desired accelerations q^* and can compute the necessary joint torques. Recurse backward.

5:23

Numeric algorithms for forward and inverse dynamics

- **Newton-Euler recursion**: very fast ($O(n)$) method to compute *inverse* dynamics

$$u = \text{NE}(q, \dot{q}, \ddot{q}^*)$$

Note that we can use this algorithm to also compute
  - gravity terms: $u = \text{NE}(q, 0, 0) = G(q)$
  - Coriolis terms: $u = \text{NE}(q, \dot{q}, 0) = C(q, \dot{q})\, \dot{q} + G(q)$
  - column of Intertia matrix: $u = \text{NE}(q, 0, e_i) = M(q)\, e_i$

- **Articulated-Body-Dynamics**: fast method ($O(n)$) to compute *forward* dynamics $\ddot{q} = f(q, \dot{q}, u)$

5:24

Some last practical comments

- `[demo]`
- Use *energy conservation* to measure dynamic of physical simulation
- Physical simulation engines (developed for games):
  - ODE (Open Dynamics Engine)
  - Bullet (originally focussed on collision only)
  - Physx (Nvidia)

  Differences of these engines to Lagrange, NE or ABD:
  - Game engine can model much more: Contacts, tissues, particles, fog, etc
  - (The way they model contacts looks ok but is somewhat fictional)
  - On kinematic trees, NE or ABD are much more precise than game engines
  - Game engines do not provide *inverse* dynamics, $u = \text{NE}(q, \dot{q}, \ddot{q}$
- Proper modelling of contacts is really really hard

5:25

Dynamic control of a robot

5:26

- We previously learnt the effect of PID control on a 1D point mass
- Robots are not a 1D point mass
  - Neither is each joint a 1D point mass
  - Applying separate PD control in each joint neglects force coupling
    (Poor solution: Apply very high gains separately in each joint $\leftrightarrow$ make joints stiff, as with gears.)
- However, knowing the robot dynamics we can transfer our understanding of PID control of a point mass to general systems

5:27

General robot dynamics

- Let $(q, \dot{q})$ be the dynamic state and $u \in \mathbb{R}^n$ the controls (typically joint torques in each motor) of a robot
- Robot dynamics can generally be written as:

$$M(q)\ \ddot{q} + C(q, \dot{q})\ \dot{q} + G(q) = u$$

$M(q) \in \mathbb{R}^{n \times n}$	is positive definite intertia matrix (can be inverted $\rightarrow$ forward simulation of dynamics)
$C(q, \dot{q}) \in \mathbb{R}^n$	are the centripetal and coriolis forces
$G(q) \in \mathbb{R}^n$	are the gravitational forces
u	are the joint torques

(cf. to the Euler-Lagrange equation on slide 22)

- We often write more compactly:

$$M(q)\ \ddot{q} + F(q, \dot{q}) = u$$

5:28

Controlling a general robot

- From now on we jsut assume that we have algorithms to efficiently compute $M(q)$ and $F(q, \dot{q})$ for any $(q, \dot{q})$
- **Inverse dynamics:** If we know the desired $\ddot{q}^*$ for each joint,

$$u = M(q)\ \ddot{q}^* + F(q, \dot{q})$$

gives the necessary torques

- **Forward dynamics:** If we know which torques u we apply, use

$$\ddot{q}^* = M(q)^{-1}(u - F(q, \dot{q}))$$

to simulate the dynamics of the system (e.g., using Runge-Kutta)

5:29

Controlling a general robot – joint space approach

- Where could we get the desired $\ddot{q}^*$ from?

  Assume we have a nice smooth **reference trajectory** $q^{\text{ref}}_{0:T}$ (generated with some motion profile or alike), we can at each t read off the desired acceleration as

$$\ddot{q}^{\text{ref}}_t := \frac{1}{\tau}[(q_{t+1} - q_t)/\tau - (q_t - q_{t-1})/\tau] = (q_{t-1} + q_{t+1} - 2q_t)/\tau^2$$

  However, tiny errors in acceleration will accumulate greatly over time! This is Instable!!

- Choose a desired acceleration $\ddot{q}^*_t$ that implies a *PD-like behavior around the reference trajectory*!

$$\ddot{q}^*_t = \ddot{q}^{\text{ref}}_t + K_p(q^{\text{ref}}_t - q_t) + K_d(\dot{q}^{\text{ref}}_t - \dot{q}_t)$$

  This is a standard and very convenient heuristic to track a reference trajectory when the robot dynamics are known: *All joints will exactly behave like a 1D point particle around the reference trajectory!*

5:30

Controlling a robot – operational space approach

- Recall the inverse kinematics problem:
  - We know the desired step δy^* (or velocity $\dot{y}^*$) of the *endeffector*.
  - Which step δq (or velocities $\dot{q}$) should we make in the joints?
- Equivalent dynamic problem:
  - We know how the desired acceleration $\ddot{y}^*$ of the *endeffector*.
  - What controls u should we apply?

5:31

Operational space control

- Inverse kinematics:

$$q^* = \underset{q}{\text{argmin}} \|\phi(q) - y^*\|^2_C + \|q - q_0\|^2_W$$

- Operational space control (one might call it "Inverse task space dynamics"):

$$u^* = \underset{u}{\text{argmin}} \|\ddot{\phi}(q) - \ddot{y}^*\|^2_C + \|u\|^2_H$$

5:32

Operational space control

- We can derive the optimum perfectly analogous to inverse kinematics
  We identify the minimum of a locally squared potential, using the local linearization (and approx. $\ddot{J} = 0$)

$$\ddot{\phi}(q) = \frac{d}{dt}\dot{\phi}(q) \approx \frac{d}{dt}(J\dot{q} + \dot{J}q) \approx J\ddot{q} + 2\dot{J}\dot{q} = JM^{-1}(u - F) + 2\dot{J}\dot{q}$$

  We get

$$u^* = T^\sharp(\ddot{y}^* - 2\dot{J}\dot{q} + TF)$$
$$\text{with } T = JM^{-1}\ , \quad T^\sharp = (T^\top C T + H)^{-1} T^\top C$$

$$(C \to \infty\ \Rightarrow\ T^\sharp = H^{-1}T^\top(TH^{-1}T^\top)^{-1})$$

5:33

Controlling a robot – operational space approach

- Where could we get the desired $\ddot{y}^*$ from?
  - **Reference trajectory** $y^{\text{ref}}_{0:T}$ in operational space
  - **PD-like behavior** in each operational space:

$$\ddot{y}^*_t = \ddot{y}^{\text{ref}}_t + K_p(y^{\text{ref}}_t - y_t) + K_d(\dot{y}^{\text{ref}}_t - \dot{y}_t)$$

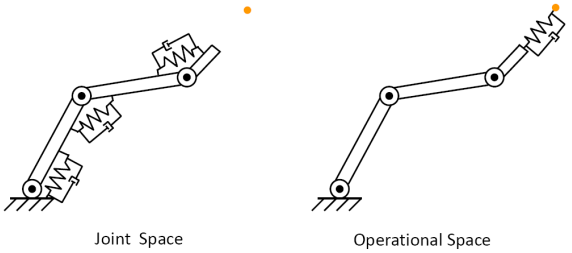


illustration from O. Brock's lecture

- Operational space control: *Let the system behave as if we could directly "apply a 1D point mass behavior" to the endeffector*

5:34

Multiple tasks

- Recall trick last time: we defined a "big kinematic map" $\Phi(q)$ such that

$$q^* = \underset{q}{\text{argmin}} \|q - q_0\|^2_W + \|\Phi(q)\|^2$$

- Works analogously in the dynamic case:

$$u^* = \underset{u}{\text{argmin}} \|u\|^2_H + \|\Phi(q)\|^2$$

5:35

6 Probability Basics

Probability Theory

- Why do we need probabilities?

  – Obvious: to express inherent stochasticity of the world (data)

- But beyond this: (also in a "deterministic world"):
  – lack of knowledge!
  – hidden (latent) variables
  – expressing *uncertainty*
  – expressing *information* (and lack of information)

- Probability Theory: an information calculus

6:1

Probability: Frequentist and Bayesian

- Frequentist probabilities are defined in the limit of an infinite number of trials
  Example: "The probability of a particular coin landing heads up is 0.43"

- Bayesian (subjective) probabilities quantify degrees of belief
  Example: "The probability of it raining tomorrow is 0.3"
  – Not possible to repeat "tomorrow"

6:2

Probabilities & Sets

- **Sample Space/domain** Ω, e.g. $\Omega = \{1,2,3,4,5,6\}$

- **Probability** $P:\ A \subset \Omega \mapsto [0,1]$
  e.g., $P(\{1\}) = \frac{1}{6},\quad P(\{4\}) = \frac{1}{6},\quad P(\{2,5\}) = \frac{1}{3},$

- Axioms: $\forall A, B \subseteq \Omega$
  – Nonnegativity $P(A) \geq 0$
  – Additivity $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$
  – Normalization $P(\Omega) = 1$

- Implications
  $0 \leq P(A) \leq 1$
  $P(\emptyset) = 0$
  $A \subseteq B \Rightarrow P(A) \leq P(B)$
  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  $P(\Omega \setminus A) = 1 - P(A)$

6:3

Probabilities & Random Variables

- For a random variable X with discrete domain $\text{dom}(X) = \Omega$ we write:
  $\forall_{x\in\Omega}:\ 0 \leq P(X{=}x) \leq 1$
  $\sum_{x\in\Omega} P(X{=}x) = 1$

  Example: A dice can take values $\Omega = \{1,..,6\}$.
  X is the random variable of a dice throw.
  $P(X{=}1) \in [0,1]$ is the probability that X takes value 1.

- A bit more formally: a random variable relates a measureable space with a domain (sample space) and thereby introduces a probability measure on the domain ("assigns a probability to each possible value")

6:4

Probabilty Distributions

- $P(X{=}1) \in \mathbb{R}$ denotes a specific probability
  $P(X)$ denotes the probability distribution (function over Ω)

  Example: A dice can take values $\Omega = \{1,2,3,4,5,6\}$.
  By $P(X)$ we discribe the full distribution over possible values $\{1,..,6\}$. These are 6 numbers that sum to one, usually stored in a *table*, e.g.: $[\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}]$

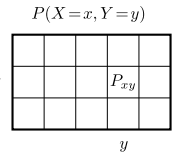
- In implementations we typically represent distributions over discrete random variables as tables (arrays) of numbers

- Notation for summing over a RV:
  In equation we often need to sum over RVs. We then write
  $$\textstyle\sum_X P(X)\ \cdots$$
  as shorthand for the explicit notation $\sum_{x\in\text{dom}(X)} P(X{=}x)\ \cdots$

6:5

Joint distributions

Assume we have *two* random variables X and Y



- Definitions:
  Joint: $P(X,Y)$
  Marginal: $P(X) = \sum_Y P(X,Y)$
  Conditional: $P(X|Y) = \frac{P(X,Y)}{P(Y)}$

  The conditional is normalized: $\forall_Y:\ \sum_X P(X|Y) = 1$

- X is *independent* of Y iff: $P(X|Y) = P(X)$
  (table thinking: all columns of $P(X|Y)$ are equal)

- The same for n random variables $X_{1:n}$ (stored as a rank n tensor)
  Joint: $P(x_{1:n})$, Marginal: $P(X_1) = \sum_{X_{2:n}} P(X_{1:n})$,
  Conditional: $P(X_1|X_{2:n}) = \frac{P(X_{1:n})}{P(X_{2:n})}$

6:6

Joint distributions

joint: $P(X,Y)$

marginal: $P(X) = \sum_Y P(X,Y)$

conditional: $P(X|Y) = \frac{P(X,Y)}{P(Y)}$

- Implications of these definitions:

Product rule: $P(X,Y) = P(X|Y)\,P(Y) = P(Y|X)\,P(X)$

Bayes' Theorem $P(X|Y) = \frac{P(Y|X)\,P(X)}{P(Y)}$

- The same for n variables, e.g., (X,Y,Z):

$P(X_{1:n}) = \prod_{i=1}^n P(X_i|X_{i+1:n})$

$P(X_1|X_{2:n}) = \frac{P(X_2|X_1,X_{3:n})\,P(X_1|X_{3:n})}{P(X_2|X_{3:n})}$

$P(X,Z,Y) = P(X|Y,Z)\,P(Y|Z)\,P(Z)$

$P(X|Y,Z) = \frac{P(Y|X,Z)\,P(X|Z)}{P(Y|Z)}$

$P(X,Y|Z) = \frac{P(X,Z|Y)\,P(Y)}{P(Z)}$

6:7

Bayes' Theorem

$$P(X|Y) = \frac{P(Y|X)\,P(X)}{P(Y)}$$

$$\text{posterior} = \frac{\text{likelihood} \cdot \text{prior}}{\text{normalization}}$$

6:8

Distributions over continuous domain

- Let X be a continuous RV. The **probability density function (pdf)** $p(x) \in [0,\infty)$ defines the probability

$$P(a \le X \le b) = \int_a^b p(x)\,dx \in [0,1]$$

The **(cumulative) probability distribution** $F(x) = P(X \le x) = \int_{-\infty}^x dx\, p(x) \in [0,1]$ is the cumulative integral with $\lim_{x\to\infty} F(x) = 1$.

(In discrete domain: *probability distribution* and *probability mass function* $P(X) \in [0,1]$ are used synonymously.)

- Two basic examples:

Gaussian: $\mathcal{N}(x\,|\,a,A) = \frac{1}{|2\pi A|^{1/2}}\, e^{-\frac{1}{2}(x-a)^\top A^{-1}(x-a)}$

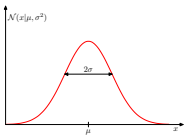
Dirac or δ ("point particle") $\delta(x) = 0$ except at $x=0$, $\int \delta(x)\,dx = 1$

$\delta(x) = \frac{\partial}{\partial x} H(x)$ where $H(x) = [x \ge 0]$ = Heavyside step function.

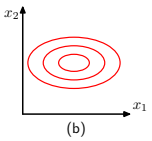
6:9

Gaussian distribution

- 1-dim: $\mathcal{N}(x\,|\,\mu,\sigma^2) = \frac{1}{|2\pi\sigma^2|^{1/2}}\, e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}$



- n-dim: $\mathcal{N}(x\,|\,a,A) = \frac{1}{|2\pi A|^{1/2}}\, e^{-\frac{1}{2}(x-a)^\top A^{-1}(x-a)}$



(b)

Useful identities:

Symmetry: $\mathcal{N}(x|a,A) = \mathcal{N}(a|x,A) = \mathcal{N}(x-a|0,A)$

Product:
$\mathcal{N}(x\,|\,a,A)\,\mathcal{N}(x\,|\,b,B) = \mathcal{N}(x\,|\,B(A+B)^{-1}a + A(A+B)^{-1}b,\, A(A+B)^{-1}B)\;\mathcal{N}(a\,|\,b,A+B)$

"Propagation":
$\int_y \mathcal{N}(x\,|\,a+Fy,A)\,\mathcal{N}(y\,|\,b,B)\,dy = \mathcal{N}(x\,|\,a+Fb, A+FBF^\top)$

Transformation:
$\mathcal{N}(Fx+f\,|\,a,A) = \frac{1}{|F|}\,\mathcal{N}(x\,|\,F^{-1}(a-f),\, F^{-1}AF^{-\top})$

Mre identities: see "Gaussian identities" http://userpage.fu-berlin.de/~mtoussai/notes/gaussians.pdf

6:10

Particle Approximation of a Distribution

- We approximate a distribution $p(x)$ over a continuous domain $\mathbb{R}^n$.

- A particle distribution $q(x)$ is a weighed set of N particles $\{(x^i, w^i)\}_{i=1}^N$
  - each particle has a location $x^i \in \mathbb{R}^n$ and a weight $w^i \in \mathbb{R}$
  - weights are normalized $\sum_i w^i = 1$

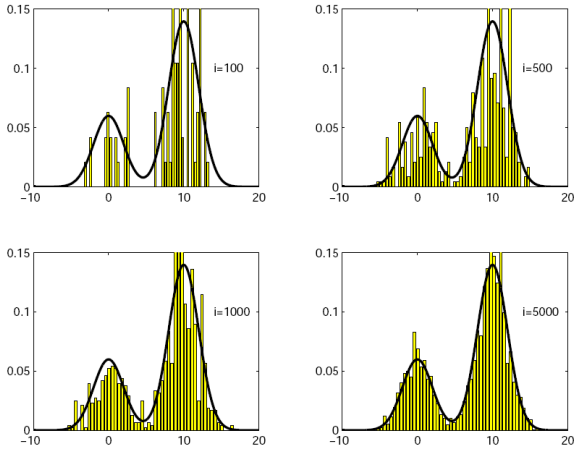
$$q(x) := \sum_{i=1}^N w^i \delta(x - x^i)$$

where $\delta(x-x^i)$ is the δ-distribution.

6:11

Particle Approximation of a Distribution

Histogram of a particle representation:



6:12

Particle Approximation of a Distribution

- For $q(x)$ to approximate a given $p(x)$ we want to choose particles and weights such that for any (smooth) f:

$$\lim_{N\to\infty} \langle f(x)\rangle_q = \lim_{N\to\infty} \sum_{i=1}^N w^i f(x^i) = \int_x f(x)p(x)dx = \langle f(x)\rangle_p$$

- How to do this? See *An Introduction to MCMC for Machine Learning* www.cs.ubc.ca/~nando/papers/mlintro.pdf

6:13

Some continuous distributions

Gaussian	$\mathcal{N}(x \mid a, A) = \frac{1}{\lvert 2\pi A\rvert^{1/2}}\, e^{-\frac{1}{2}(x-a)^\top A^{-1}(x-a)}$
Dirac or δ	$\delta(x) = \frac{\partial}{\partial x} H(x)$
Student's t (=Gaussian for $\nu \to \infty$, otherwise heavy tails)	$p(x;\nu) \propto [1 + \frac{x^2}{\nu}]^{-\frac{\nu+1}{2}}$
Exponential (distribution over single event time)	$p(x;\lambda) = [x \geq 0]\, \lambda e^{-\lambda x}$
Laplace ("double exponential")	$p(x;\mu,b) = \frac{1}{2b} e^{-\lvert x-\mu\rvert/b}$
Chi-squared	$p(x;k) \propto [x \geq 0]\, x^{k/2-1} e^{-x/2}$
Gamma	$p(x;k,\theta) \propto [x \geq 0]\, x^{k-1} e^{-x/\theta}$

6:14

7 Mobile Robotics

State estimation, Bayes filter, odometry, particle filter, Kalman filter, SLAM, joint Bayes filter, EKF SLAM, particle SLAM, graph-based SLAM

http://www.darpa.mil/grandchallenge05/

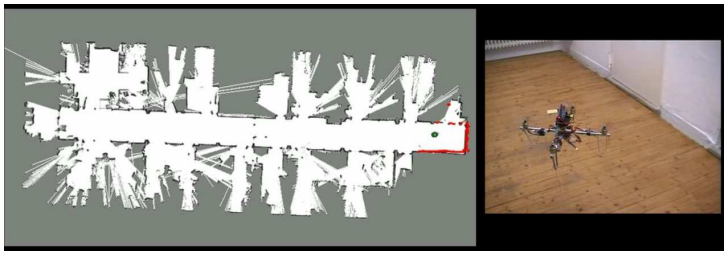
DARPA Grand Challenge 2005

7:1

http://www.darpa.mil/grandchallenge/index.asp

DARPA Grand Urban Challenge 2007

7:2



http://www.slawomir.de/publications/grzonka09icra/grzonka09icra.pdf

Quadcopter Indoor Localization

7:3

http://stair.stanford.edu/multimedia.php

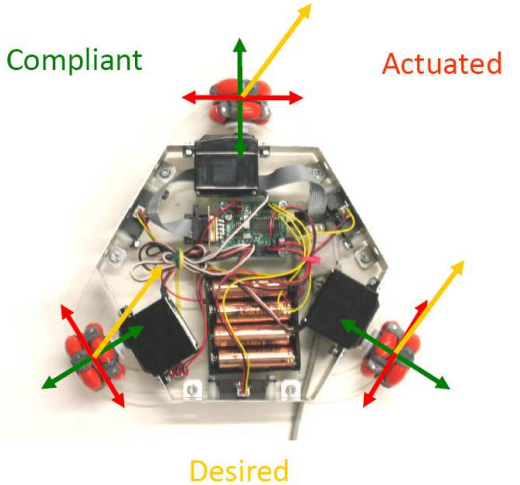
STAIR: STanford Artificial Intelligence Robot

7:4

Types of Robot Mobility

7:5

Types of Robot Mobility



7:6

- Each type of robot mobility corresponds to a system equation $x_{t+1} = x_t + \tau f(x_t, u_t)$

or, if the dynamics are stochastic,

$$P(x_{t+1} \,|\, u_t, x_t) = \mathcal{N}(x_{t+1} \,|\, x_t + \tau f(x_t, u_t), \Sigma)$$

- We considered control, path finding, and trajectory optimization

  For this we always assumed to know the state x_t of the robot (e.g., its posture/position)!

7:7

Outline

- PART I:

  A core challenge in mobile robotics is **state estimation**

  → Bayesian filtering & smoothing

  particles, Kalman

- PART II:

  Another challenge is to **build a map** while exploring

  → SLAM (simultaneous localization and mapping)

7:8

PART I: State Estimation Problem

- Our sensory data does not provide sufficient information to determine our location.

- Given the local sensor readings y_t, the current state x_t (location, position) is *uncertain*.

  – which hallway?

  – which door exactly?

  – which heading direction?

7:9

State Estimation Problem

- What is the probability of being in front of room 154, given we see what is shown in the image?

- What is the probability given that we were just in front of room 156?

- What is the probability given that we were in front of room 156 and moved 15 meters?

7:10

Recall Bayes' theorem

$$P(X|Y) = \frac{P(Y|X)\ P(X)}{P(Y)}$$

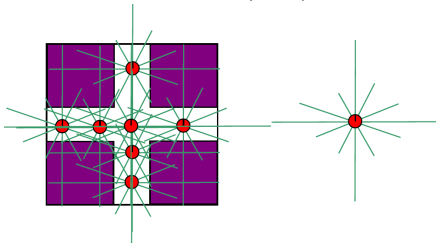
$$\text{posterior} = \frac{\text{likelihood} \cdot \text{prior}}{\text{(normalization)}}$$

7:11

- How can we apply this to the State Estimation Problem?

Using Bayes Rule:

$$P(\text{location} \,|\, \text{sensor}) = \frac{P(\text{sensor} \,|\, \text{location})\, P(\text{location})}{P(\text{sensor})}$$



7:12

Bayes Filter

x_t = state (location) at time t

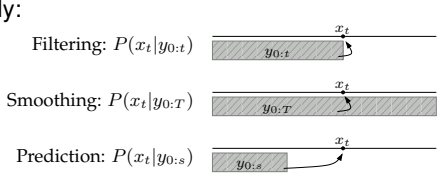
y_t = sensor readings at time t

$u_{t\text{-}1}$ = control command (action, steering, velocity) at time t-1

- Given the history $y_{0:t}$ and $u_{0:t\text{-}1}$, we want to compute the probability distribution over the state at time t

$$p_t(x_t) := P(x_t \,|\, y_{0:t}, u_{0:t\text{-}1})$$

- Generally:



7:13

Bayes Filter

$$\begin{aligned}
p_t(x_t) &:= P(x_t \,|\, y_{0:t}, u_{0:t\text{-}1}) \\
&= c_t\ P(y_t \,|\, x_t, y_{0:t\text{-}1}, u_{0:t\text{-}1})\ P(x_t \,|\, y_{0:t\text{-}1}, u_{0:t\text{-}1}) \\
&= c_t\ P(y_t \,|\, x_t)\ P(x_t \,|\, y_{0:t\text{-}1}, u_{0:t\text{-}1}) \\
&= c_t\ P(y_t \,|\, x_t) \int_{x_{t\text{-}1}} P(x_t, x_{t\text{-}1} \,|\, y_{0:t\text{-}1}, u_{0:t\text{-}1})\ dx_{t\text{-}1}
\end{aligned}$$

$$= c_t\ P(y_t \,|\, x_t) \int_{x_{t-1}} P(x_t \,|\, x_{t-1}, y_{0:t-1}, u_{0:t-1})\ P(x_{t-1} \,|\, y_{0:t-1}, u_{0:t-1})\ dx_{t-1}$$
$$= c_t\ P(y_t \,|\, x_t) \int_{x_{t-1}} P(x_t \,|\, x_{t-1}, u_{t-1})\ P(x_{t-1} \,|\, y_{0:t-1}, u_{0:t-1})\ dx_{t-1}$$
$$= c_t\ P(y_t \,|\, x_t) \int_{x_{t-1}} P(x_t \,|\, u_{t-1}, x_{t-1})\ p_{t-1}(x_{t-1})\ dx_{t-1}$$

using Bayes rule $P(X|Y,Z) = c\ P(Y|X,Z)\ P(X|Z)$ with some normalization constant c_t

uses conditional independence of the observation on past observations and controls

by definition of the marginal

by definition of a conditional

given x_{t-1}, x_t depends only on the controls u_{t-1} (Markov Property)

- A Bayes filter updates the posterior belief $p_t(x_t)$ in each time step using the:

  observation model $P(y_t \,|\, x_t)$

  transition model $P(x_t \,|\, u_{t-1}, x_{t-1})$

7:14

Bayes Filter

$$p_t(x_t) \propto \underbrace{P(y_t \,|\, x_t)}_{\text{new information}} \underbrace{\int_{x_{t-1}} P(x_t \,|\, u_{t-1}, x_{t-1})\ \underbrace{p_{t-1}(x_{t-1})}_{\text{old estimate}}\ dx_{t-1}}_{\text{predictive estimate } \hat{p}_t(x_t)}$$

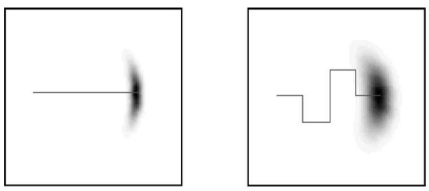
1. We have a belief $p_{t-1}(x_{t-1})$ of our previous position

2. We use the motion model to predict the current position
   $\hat{p}_t(x_t) \propto \int_{x_{t-1}} P(x_t \,|\, u_{t-1}, x_{t-1})\ p_{t-1}(x_{t-1})\ dx_{t-1}$

3. We integetrate this with the current observation to get a better belief
   $p_t(x_t) \propto P(y_t \,|\, x_t)\ \hat{p}_t(x_t)$

7:15

- Typical transition model $P(x_t \,|\, u_{t-1}, x_{t-1})$ in robotics:

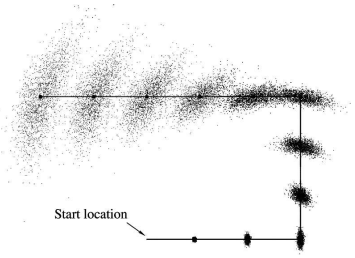


(from *Robust Monte Carlo localization for mobile robots* Sebastian Thrun, Dieter Fox, Wolfram Burgard, Frank Dellaert)

7:16

Odometry ("Dead Reckoning")

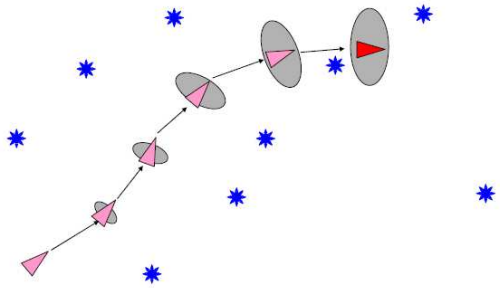
- The predictive distributions $\hat{p}_t(x_t)$ *without integrating observations* (removing the $P(y_t|x_t)$ part from the Bayesian filter)



(from *Robust Monte Carlo localization for mobile robots* Sebastian Thrun, Dieter Fox, Wolfram Burgard, Frank Dellaert)

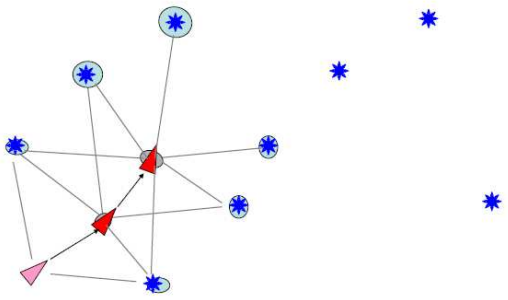
7:17

Again, predictive distributions $\hat{p}_t(x_t)$ *without integrating landmark observations*



7:18

The Bayes-filtered distributions $p_t(x_t)$ integrating landmark observations



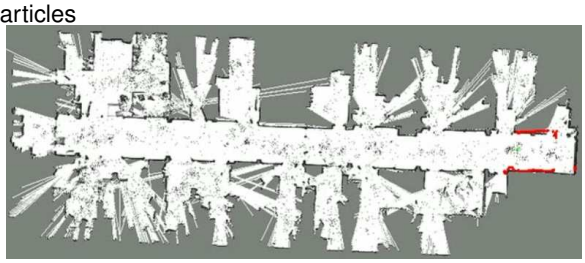
7:19

Bayesian Filters

- How to represent the belief $p_t(x_t)$:

- Gaussian

- Particles

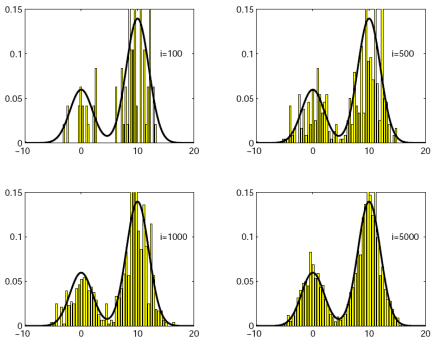


7:20

Recall: Particle Representation of a Distribution

- Weighed set of N particles $\{(x^i, w^i)\}_{i=1}^N$

$$p(x) \approx q(x) := \sum_{i=1}^N w^i \delta(x, x^i)$$



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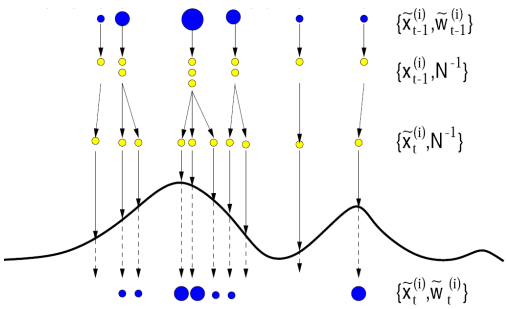
Particle Filter := Bayesian Filtering with Particles

(Bayes Filter: $p_t(x_t) \propto P(y_t \,|\, x_t) \int_{x_{t-1}} P(x_t \,|\, u_{t-1}, x_{t-1})\, p_{t-1}(x_{t-1})\, dx_{t-1}$)



1. Start with N particles $\{(x_{t-1}^i, w_{t-1}^i)\}_{i=1}^N$
2. Resample particles to get N weight-1-particles: $\{\hat{x}_{t-1}^i\}_{i=1}^N$
3. Use motion model to get new "predictive" particles $\{x_t^i\}_{i=1}^N$
   each $x_t^i \sim P(x_t \,|\, u_{t-1}, \hat{x}_{t-1}^i)$
4. Use observation model to assign new weights $w_t^i \propto P(y_t \,|\, x_t^i)$

7:22

- "Particle Filter"

aka *Monte Carlo Localization* in the mobile robotics community

Condensation Algorithm in the vision community

- Efficient resampling is important:

  Typically "Residual Resampling":

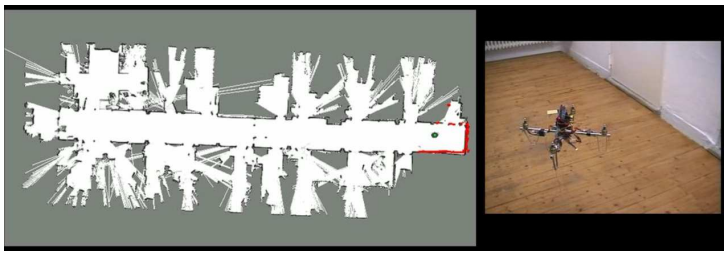
  Instead of sampling directly $\hat{n}^i \sim \text{Multi}(\{Nw_i\})$ set $\hat{n}^i = \lfloor Nw_i \rfloor + \bar{n}_i$ with $\bar{n}_i \sim \text{Multi}(\{Nw_i - \lfloor Nw_i \rfloor\})$

  Liu & Chen (1998): Sequential Monte Carlo Methods for Dynamic Systems.

  Douc, Cappé & Moulines: Comparison of Resampling Schemes for Particle Filtering.

7:23

Example: Quadcopter Localization



http://www.slawomir.de/publications/grzonka09icra/grzonka09icra.pdf

Quadcopter Indoor Localization

7:24

Typical Pitfall in Particle Filtering

- Predicted particles $\{x_t^i\}_{i=1}^N$ have very low observation likelihood $P(y_t \,|\, x_t^i) \approx 0$

  ("particles die over time")

- Classical solution: generate particles also with other than purely forward proposal $P(x_t \,|\, u_{t-1}, x_{t-1})$:

  – Choose a proposal that depends on the new observation y_t, ideally approximating $P(x_t \,|\, y_t, u_{t-1}, x_{t-1})$

  – Or mix particles sampled directly from $P(y_t \,|\, x_t)$ and from $P(x_t \,|\, u_{t-1}, x_{t-1})$.

  (*Robust Monte Carlo localization for mobile robots.* Sebastian Thrun, Dieter Fox, Wolfram Burgard, Frank Dellaert)

7:25

Kalman filter := Bayesian Filtering with Gaussians

Bayes Filter: $p_t(x_t) \propto P(y_t \,|\, x_t) \int_{x_{t-1}} P(x_t \,|\, u_{t-1}, x_{t-1})\, p_{t-1}(x_{t-1})\, dx_{t-1}$

- Can be computed analytically for linear-Gaussian observations and transitions:

$$P(y_t \,|\, x_t) = \mathcal{N}(y_t \,|\, Cx_t + c, W)$$

$$P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1}) = \mathcal{N}(x_t \,|\, A(u_{t\text{-}1})\, x_{t\text{-}1} + a(u_{t\text{-}1}), Q)$$

Defition:
$\mathcal{N}(x \,|\, a, A) = \frac{1}{|2\pi A|^{1/2}} \exp\{-\frac{1}{2}(x\text{-}a)^\top A^{\text{-}1} (x\text{-}a)\}$
Product:
$\mathcal{N}(x \,|\, a, A)\, \mathcal{N}(x \,|\, b, B) = \mathcal{N}(x \,|\, B(A{+}B)^{\text{-}1}a{+}A(A{+}B)^{\text{-}1}b, A(A{+}B)^{\text{-}1}B)\ \mathcal{N}(a \,|\, b, A{+}B)$
"Propagation":
$\int_y \mathcal{N}(x \,|\, a + Fy, A)\, \mathcal{N}(y \,|\, b, B)\, dy = \mathcal{N}(x \,|\, a + Fb, A + FBF^\top)$
Transformation:
$\mathcal{N}(Fx + f \,|\, a, A) = \frac{1}{|F|}\, \mathcal{N}(x \,|\, F^{\text{-}1}(a - f),\ F^{\text{-}1}AF^{\text{-}\top})$

(more identities: see "Gaussian identities" http://ipvs.informatik.uni-stuttgart.de/mlr/marc/notes/gaussians.pdf)

7:26

Kalman filter derivation

$$p_t(x_t) = \mathcal{N}(x_t \,|\, s_t, S_t)$$
$$P(y_t \,|\, x_t) = \mathcal{N}(y_t \,|\, Cx_t + c, W)$$
$$P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1}) = \mathcal{N}(x_t \,|\, Ax_{t\text{-}1} + a, Q)$$

$$p_t(x_t) \propto P(y_t \,|\, x_t) \int_{x_{t\text{-}1}} P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1})\, p_{t\text{-}1}(x_{t\text{-}1})\, dx_{t\text{-}1}$$
$$= \mathcal{N}(y_t \,|\, Cx_t + c, W) \int_{x_{t\text{-}1}} \mathcal{N}(x_t \,|\, Ax_{t\text{-}1} + a, Q)\, \mathcal{N}(x_{t\text{-}1} \,|\, s_{t\text{-}1}, S_{t\text{-}1})\, dx_{t\text{-}1}$$
$$= \mathcal{N}(y_t \,|\, Cx_t + c, W)\, \mathcal{N}(x_t \,|\, \underbrace{As_{t\text{-}1} + a}_{=:\hat{s}_t},\ \underbrace{Q + AS_{t\text{-}1}A^\top}_{=:\hat{S}_t})$$
$$= \mathcal{N}(Cx_t + c \,|\, y_t, W)\, \mathcal{N}(x_t \,|\, \hat{s}_t, \hat{S}_t)$$
$$= \mathcal{N}[x_t \,|\, C^\top W^{\text{-}1}(y_t - c),\ C^\top W^{\text{-}1}C]\, \mathcal{N}(x_t \,|\, \hat{s}_t, \hat{S}_t)$$
$$= \mathcal{N}(x_t \,|\, s_t, S_t) \cdot \langle \text{terms indep. of } x_t \rangle$$
$$S_t = (C^\top W^{\text{-}1}C + \hat{S}_t^{\text{-}1})^{\text{-}1} = \hat{S}_t - \underbrace{\hat{S}_t C^\top (W + C\hat{S}_t C^\top)^{\text{-}1}}_{\text{"Kalman gain" } K} C\hat{S}_t$$
$$s_t = S_t[C^\top W^{\text{-}1}(y_t - c) + \hat{S}_t^{\text{-}1}\hat{s}_t] = \hat{s}_t + K(y_t - C\hat{s}_t - c)$$

The second to last line uses the general Woodbury identity.
The last line uses $S_t C^\top W^{\text{-}1} = K$ and $S_t \hat{S}_t^{\text{-}1} = \mathbf{I} - KC$

7:27

Extended Kalman filter (EKF) and Unscented Transform

Bayes Filter: $p_t(x_t) \propto P(y_t \,|\, x_t) \int_{x_{t\text{-}1}} P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1})\, p_{t\text{-}1}(x_{t\text{-}1})\, dx_{t\text{-}1}$

- Can be computed analytically for linear-Gaussian observations and transitions:

$$P(y_t \,|\, x_t) = \mathcal{N}(y_t \,|\, Cx_t + c, W)$$

$$P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1}) = \mathcal{N}(x_t \,|\, A(u_{t\text{-}1})x_{t\text{-}1} + a(u_{t\text{-}1}), Q)$$

- If $P(y_t \,|\, x_t)$ or $P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1})$ are not linear:

$$P(y_t \,|\, x_t) = \mathcal{N}(y_t \,|\, g(x_t), W)$$

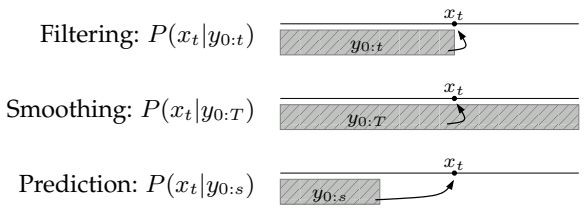
$$P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1}) = \mathcal{N}(x_t \,|\, f(x_{t\text{-}1}, u_{t\text{-}1}), Q)$$

– approximate f and g as locally linear (*Extended Kalman Filter*)

– or sample locally from them and reapproximate as Gaussian (*Unscented Transform*)

7:28

Bayes smoothing



7:29

Bayes smoothing

- Let $\mathcal{P} = y_{0:t}$ past observations, $\mathcal{F} = y_{t+1:T}$ future observations

$$P(x_t \,|\, \mathcal{P}, \mathcal{F}, u_{0:T}) \propto P(\mathcal{F} \,|\, x_t, \mathcal{P}, u_{0:T})\, P(x_t \,|\, \mathcal{P}, u_{0:T})$$
$$= \underbrace{P(\mathcal{F} \,|\, x_t, u_{t:T})}_{=:\beta_t(x_t)}\ \underbrace{P(x_t \,|\, \mathcal{P}, u_{0:t\text{-}1})}_{=:p(x_t)}$$

Bayesian smoothing fuses a forward filter $p_t(x_t)$ with a backward "filter" $\beta_t(x_t)$

- Backward recursion (derivation analogous to the Bayesian filter)

$$\beta_t(x_t) := P(y_{t+1:T} \,|\, x_t, u_{t:T})$$
$$= \int_{x_{t+1}} \beta_{t+1}(x_{t+1})\, P(y_{t+1} \,|\, x_{t+1})\, P(x_{t+1} \,|\, x_t, u_t)\, dx_{t+1}$$

7:30

PART II: Localization and Mapping

- The Bayesian filter requires an observation model $P(y_t \,|\, x_t)$

- A **map** is something that provides the observation model:

A map tells us for each x_t what the sensor readings y_t might look like

7:31

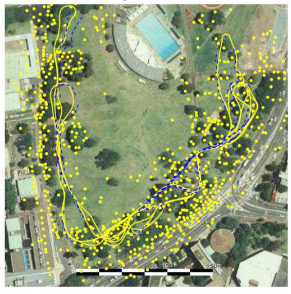
Types of maps

Grid map

K. Murphy (1999): *Bayesian map learning in dynamic environments.*
Grisetti, Tipaldi, Stachniss, Burgard, Nardi: *Fast and Accurate SLAM with Rao-Blackwellized Particle Filters*

Laser scan map

Landmark map



Victoria Park data set
M. Montemerlo, S. Thrun, D. Koller, & B. Wegbreit (2003): *FastSLAM 2.0: An improved particle filtering algorithm for simultaneous localization and mapping that provably converges.* IJCAI, 1151–1156.

7:32

Simultaneous Localization and Mapping Problem

- Notation:

  x_t = state (location) at time t

  y_t = sensor readings at time t

  $u_{t\text{-}1}$ = control command (action, steering, velocity) at time t-1

  m = the map

- Given the history $y_{0:t}$ and $u_{0:t\text{-}1}$, we want to compute the belief over the pose AND THE MAP m at time t

$$p_t(x_t, m) := P(x_t, m \,|\, y_{0:t}, u_{0:t\text{-}1})$$

- We assume to know:

  – transition model $P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1})$

  – observation model $P(y_t \,|\, x_t, m)$

7:33

SLAM: classical "chicken or egg problem"

- If we knew the state trajectory $x_{0:t}$ we could efficiently compute the belief over the map

$$P(m \,|\, x_{0:t}, y_{0:t}, u_{0:t\text{-}1})$$

- If we knew the map we could use a Bayes filter to compute the belief over the state

$$P(x_t \,|\, m, y_{0:t}, u_{0:t\text{-}1})$$

- SLAM requires to tie state estimation and map building together:

  1) Joint inference on x_t and m ($\rightarrow$ Kalman-SLAM)

  2) Tie a state hypothesis (=particle) to a map hypothesis ($\rightarrow$ particle SLAM)

  3) Frame everything as a graph optimization problem ($\rightarrow$ graph SLAM)

7:34

Joint Bayesian Filter over x and m

- A (formally) straight-forward approach is the joint Bayesian filter

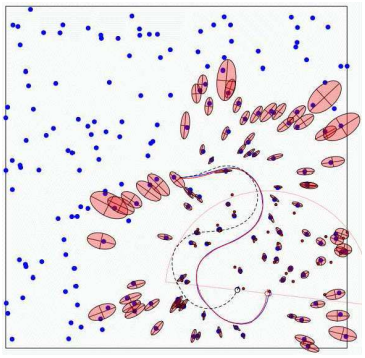
$$p_t(x_t, m) \;\propto\; P(y_t \,|\, x_t, m) \int_{x_{t\text{-}1}} P(x_t \,|\, u_{t\text{-}1}, x_{t\text{-}1})\; p_{t\text{-}1}(x_{t\text{-}1}, m)\; dx_{t\text{-}1}$$

But: How represent a belief over high-dimensional x_t, m?

7:35

Map uncertainty

- In the case the map $m = (\theta_1, .., \theta_N)$ is a set of N landmarks, $\theta_j \in \mathbb{R}^2$



- Use Gaussians to represent the uncertainty of landmark positions

7:36

(Extended) Kalman Filter SLAM

- Analogous to Localization with Gaussian for the pose belief $p_t(x_t)$

  – But now: joint belief $p_t(x_t, \theta_{1:N})$ is $3 + 2N$-dimensional Gaussian

  – Assumes the map $m = (\theta_1, .., \theta_N)$ is a set of N landmarks, $\theta_j \in \mathbb{R}^2$

  – Exact update equations (under the Gaussian assumption)

  – Conceptually very simple

- Drawbacks:

  – Scaling (full covariance matrix is $O(N^2)$)

  – Sometimes non-robust (uni-modal, "data association problem")

  – Lacks advantages of Particle Filter
  (multiple hypothesis, more robust to non-linearities)

7:37

SLAM with particles

Core idea: Each particle carries its own map belief

- Use a conditional representation "$p_t(x_t, m) = p_t(x_t)\; p_t(m \,|\, x_t)$"
  (This notation is flaky... the below is more precise)

- As for the Localization Problem use particles to represent the *pose belief* $p_t(x_t)$
  Note: Each particle actually "has a history $x^i_{0:t}$" – a whole trajectory!

- For each particle separately, estimate the *map belief* $p^i_t(m)$ conditioned on the particle history $x^i_{0:t}$.

The conditional beliefs $p_t^i(m)$ may be factorized over grid points or landmarks of the map

K. Murphy (1999): *Bayesian map learning in dynamic environments.*

http://www.cs.ubc.ca/~murphyk/Papers/map_nips99.pdf

7:38

Map estimation for a *given* particle history

- Given $x_{0:t}$ (e.g. a trajectory of a particle), what is the posterior over the map m?

  $\rightarrow$ simplified Bayes Filter:

  $$p_t(m) := P(m \,|\, x_{0:t}, y_{0:t}) \propto P(y_t \,|\, m, x_t)\; p_{t\text{-}1}(m)$$

  (no transtion model: assumption that map is constant)

- In the case of landmarks (FastSLAM):

  $m = (\theta_1, .., \theta_N)$

  θ_j = position of the jth landmark, $j \in \{1, .., N\}$

  n_t = which landmark we observe at time t, $\quad n_t \in \{1, .., N\}$

  We can use a separate (Gaussian) Bayes Filter for each θ_j conditioned on $x_{0:t}$, each θ_j is independent from each θ_k:

  $$P(\theta_{1:N} \,|\, x_{0:t}, y_{0:n}, n_{0:t}) = \prod_j P(\theta_j \,|\, x_{0:t}, y_{0:n}, n_{0:t})$$

7:39

Particle likelihood in SLAM

- Particle likelihood for Localization Problem:

  $$w_t^i = P(y_t \,|\, x_t^i)$$

  (determins the new importance weight w_t^i

- In SLAM the map is uncertain $\rightarrow$ each particle is weighted with the *expected* likelihood:

  $$w_t^i = \int P(y_t \,|\, x_t^i, m)\; p_{t-1}(m)\; dm$$

- In case of landmarks (FastSLAM):

  $$w_t^i = \int P(y_t \,|\, x_t^i, \theta_{n_t}, n_t)\; p_{t-1}(\theta_{n_t})\; d\theta_{n_t}$$

- Data association problem (actually we don't know n_t):

  For each particle separately choose $n_t^i = \text{argmax}_{n_t}\, w_t^i(n_t)$

7:40

Particle-based SLAM summary

- We have a set of N particles $\{(x^i, w^i)\}_{i=1}^N$ to represent the pose belief $p_t(x_t)$
- For each particle we have a separate map belief $p_t^i(m)$; in the case of landmarks, this factorizes in N separate 2D-Gaussians
- Iterate
  1. Resample particles to get N weight-1-particles: $\{\hat{x}_{t\text{-}1}^i\}_{i=1}^N$
  2. Use motion model to get new "predictive" particles $\{x_t^i\}_{i=1}^N$
  3. Update the map belief $p_m^i(m) \propto P(y_t \,|\, m, x_t)\; p_{t\text{-}1}^i(m)$ for each particle
  4. Compute new importance weights $w_t^i \propto \int P(y_t \,|\, x_t^i, m)\; p_{t-1}(m)\; d$ using the observation model and the map belief

7:41

Demo: Visual SLAM

- Map building from a freely moving camera

7:42

Demo: Visual SLAM

- Map building from a freely moving camera

  – SLAM has become a big topic in the vision community..

  – features are typically landmarks $\theta_{1:N}$ with SURF/SIFT features

  – PTAM (Parallel Tracking and Mapping) parallelizes computations...

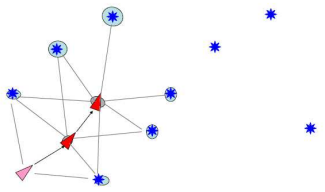
  PTAM1 PTAM2 DTAM KinectFusion

  G Klein, D Murray: *Parallel Tracking and Mapping for Small AR Workspaces* http://www.robots.ox.ac.uk/~gk/PTAM/

  Newcombe, Lovegrove & Davison: *DTAM: Dense Tracking and Mapping in Real-Time* ICCV 2011. http://www.doc.ic.ac.uk/~rnewcomb/

7:43

Alternative SLAM approach: Graph-based

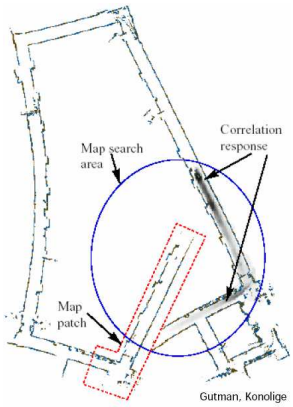


- Represent the previous trajectory as a graph
  – nodes = estimated positions & observations
  – edges = transition & step estimation based on scan matching
- Loop Closing: check if some nodes might coincide → new edges
- Classical Optimization:
  The whole graph defines an optimization problem: Find poses that minimize sum of edge & node errors

7:44

Loop Closing Problem

(Doesn't explicitly exist in Particle Filter methods: If particles cover the belief, then "data association" solves the "loop closing problem")



7:45

Graph-based SLAM

Life-long Map Learning for Graph-based SLAM Approaches in Static Environments Kretzschmar, Grisetti, Stachniss

7:46

SLAM code

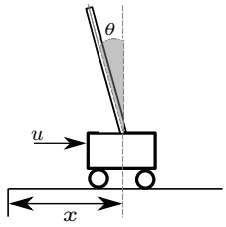
- Graph-based and grid map methods:
  http://openslam.org/
- Visual SLAM
  e.g. http://ewokrampage.wordpress.com/

7:47

8 Control Theory

Topics in control theory, optimal control, HJB equation, infinite horizon case, Linear-Quadratic optimal control, Riccati equations (differential, algebraic, discrete-time), controllability, stability, eigenvalue analysis, Lyapunov function

Cart pole example



state $\boldsymbol{x} = (x, \dot{x}, \theta, \dot{\theta})$

$$\ddot{\theta} = \frac{g\sin(\theta) + \cos(\theta)\left[-c_1 u - c_2 \dot{\theta}^2 \sin(\theta)\right]}{\frac{4}{3}l - c_2\cos^2(\theta)}$$

$$\ddot{x} = c_1 u + c_2\left[\dot{\theta}^2\sin(\theta) - \ddot{\theta}\cos(\theta)\right]$$

8:1

Control Theory

- Concerns controlled systems of the form

$$\dot{x} = f(x,u) + \text{noise}(x,u)$$

and a controller of the form

$$\pi : (x,t) \mapsto u$$

- We'll neglect stochasticity here
- When analyzing a given controller π, one analyzes **closed-loop system** as described by the differential equation

$$\dot{x} = f(x, \pi(x,t))$$

(E.g., analysis for convergence & stability)

8:2

Core topics in Control Theory

- **Stability***
  Analyze the stability of a closed-loop system
  → Eigenvalue analysis or Lyapunov function method
- **Controllability***
  Analyze which dimensions (DoFs) of the system can actually in principle be controlled
- **Transfer function**
  Analyze the closed-loop transfer function, i.e., "how frequencies are transmitted through the system". (→ Laplace transformation)
- **Controller design**
  Find a controller with desired stability and/or transfer function properties
- **Optimal control***
  Define a cost function on the system behavior. Optimize a controller to minimize costs

8:3

Control Theory references

- Robert F. Stengel: *Optimal control and estimation*
  Online lectures: http://www.princeton.edu/~stengel/MAE546Lectures.html (esp. lectures 3,4 and 7-9)
- From robotics lectures:
  Stefan Schaal's lecture Introduction to Robotics: http://www-clmc.usc.edu/Teaching/TeachingIntroductionToRoboticsSy
  Drew Bagnell's lecture on Adaptive Control and Reinforcement Learning http://robotwhisperer.org/acrls11/

8:4

Outline

- We'll first consider *optimal control*
  Goal: understand Algebraic Riccati equation significance for local neighborhood control
- Then controllability & stability

8:5

Optimal control (discrete time)

Given a controlled dynamic system

$$x_{t+1} = f(x_t, u_t)$$

we define a cost function

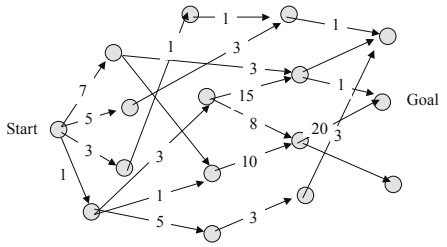
$$J^\pi = \sum_{t=0}^{T} c(x_t, u_t) + \phi(x_T)$$

where x_0 and the controller $\pi : (x,t) \mapsto u$ are given, which determines $x_{1:T}$ and $u_{0:T}$

8:6

Dynamic Programming & Bellman principle

An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.



"$V(\text{state}) = \min_{\text{edge}}[c(\text{edge}) + V(\text{next-state})]$"

8:7

Bellman equation (discrete time)

- Define the **value function** or optimal **cost-to-go function**

$$V_t(x) = \min_\pi \Big[\sum_{s=t}^{T} c(x_s, u_s) + \phi(x_T) \Big]_{x_t = x}$$

- Bellman equation

$$V_t(x) = \min_u \Big[c(x,u) + V_{t+1}(f(x,u)) \Big]$$

The argmin gives the optimal control signal: $\pi_t^*(x) = \text{argmin}_u \Big[\cdots \Big]$

Derivation:

$$\begin{aligned} V_t(x) &= \min_\pi \Big[\sum_{s=t}^{T} c(x_s, u_s) + \phi(x_T) \Big] \\ &= \min_{u_t} \Big[c(x,u_t) + \min_\pi [\sum_{s=t+1}^{T} c(x_s, u_s) + \phi(x_T)] \Big] \\ &= \min_{u_t} \Big[c(x,u_t) + V_{t+1}(f(x,u_t)) \Big] \end{aligned}$$

8:8

Optimal Control (continuous time)

Given a controlled dynamic system

$$\dot{x} = f(x,u)$$

we define a cost function with horizon T

$$J^\pi = \int_0^T c(x(t), u(t))\, dt + \phi(x(T))$$

where the start state $x(0)$ and the controller $\pi : (x,t) \mapsto u$ are given, which determine the closed-loop system trajectory $x(t), u(t)$ via $\dot{x} = f(x, \pi(x,t))$ and $u(t) = \pi(x(t), t)$

8:9

Hamilton-Jacobi-Bellman equation (continuous time)

- Define the **value function** or optimal **cost-to-go function**

$$V(x,t) = \min_\pi \Big[\int_t^T c(x(s), u(s))\, ds + \phi(x(T)) \Big]_{x(t)=x}$$

- Hamilton-Jacobi-Bellman equation

$$-\tfrac{\partial}{\partial t} V(x,t) = \min_u \Big[c(x,u) + \tfrac{\partial V}{\partial x} f(x,u) \Big]$$

The argmin gives the optimal control signal: $\pi^*(x) = \text{argmin}_u \Big[\cdots \Big]$

Derivation:

$$\begin{aligned} \frac{dV(x,t)}{dt} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\dot{x} \\ c(x,u^*) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,u^*) \\ -\frac{\partial V}{\partial t} &= c(x,u^*) + \frac{\partial V}{\partial x} f(x,u^*) \end{aligned}$$

8:10

Infinite horizon case

$$J^\pi = \int_0^\infty c(x(t), u(t))\, dt$$

- This cost function is stationary (time-invariant)!

  $\to$ the optimal value function is stationary ($V(x,t) = V(x)$)

  $\to$ the optimal control signal depends on x but not on t

  $\to$ the optimal controller π^* is stationary

- The HBJ and Bellman equations remain "the same" but with the same (stationary) value function independent of t:

$$\begin{aligned} 0 &= \min_u \Big[c(x,u) + \frac{\partial V}{\partial x} f(x,u) \Big] && \text{(cont. time)} \\ V(x) &= \min_u \Big[c(x,u) + V(f(x,u)) \Big] && \text{(discrete time)} \end{aligned}$$

The argmin gives the optimal control signal: $\pi^*(x) = \text{argmin}_u \Big[\cdots \Big]$

8:11

Infinite horizon examples

- Cart-pole balancing:
  – You always want the pole to be upright ($\theta \approx 0$)
  – You always want the car to be close to zero ($x \approx 0$)
  – You want to spare energy (apply low torques) ($u \approx 0$)
  You might define a cost

$$J^\pi = \int_0^\infty \|\theta\|^2 + \epsilon\|x\|^2 + \rho\|u\|^2$$

- Reference following:
  – You always want to stay close to a reference trajectory $r(t)$
  Define $\tilde{x}(t) = x(t) - r(t)$ with dynamics $\dot{\tilde{x}}(t) = f(\tilde{x}(t) + r(t), u) - \dot{r}(t)$
  Define a cost

$$J^\pi = \int_0^\infty \|\tilde{x}\|^2 + \rho\|u\|^2$$

- Many many problems in control can be framed this way

8:12

Comments

- The Bellman equation is fundamental in optimal control theory, but also Reinforcement Learning
- The HJB eq. is a differential equation for $V(x,t)$ which is in general hard to solve
- The (time-discretized) Bellman equation can be solved by Dynamic Programming starting backward:

$$V_T(x) = \phi(x)\ , \quad V_{T\text{-}1}(x) = \min_u \Big[c(x,u) + V_T(f(x,u)) \Big] \quad \text{etc.}$$

But it might still be hard or infeasible to represent the functions $V_t(x)$ over continuous x!

- Both become significantly simpler under linear dynamics and quadratic costs:

$$\rightarrow \text{Riccati equation}$$

8:13

Linear-Quadratic Optimal Control

linear dynamics

$$\dot{x} = f(x,u) = Ax + Bu$$

quadratic costs

$$c(x,u) = x^\top Q x + u^\top R u \;, \quad \phi(x_T) = x_T^\top F x_T$$

- Note: Dynamics neglects constant term; costs neglect linear and constant terms. This is because
  - – constant costs are irrelevant
  - – linear cost terms can be made away by redefining x or u
  - – constant dynamic term only introduces a constant drift

8:14

Linear-Quadratic Control as Local Approximation

- LQ control is important also to control non-LQ systems in the **neighborhood** of a desired state!

Let x^* be such a desired state (e.g., cart-pole: $x^* = (0,0,0,0)$)

- – linearize the dynamics around x^*
- – use 2nd order approximation of the costs around x^*
- – control the system *locally* as if it was LQ
- – pray that the system will never leave this neighborhood!

8:15

Riccati differential equation = HJB eq. in LQ case

- In the Linear-Quadratic (LQ) case, the value function always is a quadratic function of x!

Let $V(x,t) = x^\top P(t) x$, then the HBJ equation becomes

$$\begin{aligned}
-\frac{\partial}{\partial t} V(x,t) &= \min_u \Big[c(x,u) + \frac{\partial V}{\partial x} f(x,u) \Big] \\
-x^\top \dot{P}(t) x &= \min_u \Big[x^\top Q x + u^\top R u + 2 x^\top P(t)(Ax + Bu) \Big] \\
0 &= \frac{\partial}{\partial u} \Big[x^\top Q x + u^\top R u + 2 x^\top P(t)(Ax + Bu) \Big] \\
&= 2u^\top R + 2x^\top P(t) B \\
u^* &= -R^{-1} B^\top P x
\end{aligned}$$

$\Rightarrow$ **Riccati differential equation**

$$-\dot{P} = A^\top P + PA - PBR^{-1}B^\top P + Q$$

8:16

Riccati differential equation

$$-\dot{P} = A^\top P + PA - PBR^{-1}B^\top P + Q$$

- This is a differential equation for the matrix $P(t)$ describing the quadratic value function. If we solve it with the finite horizon constraint $P(T) = F$ we solved the optimal control problem

- The optimal control $u^* = -R^{-1}B^\top P x$ is called **Linear Quadratic Regulator**

Note: If the state is dynamic (e.g., $x = (q, \dot{q})$) this control is linear in the positions and linear in the velocities and is an instance of **PD control**

The matrix $K = R^{-1}B^\top P$ is therefore also called **gain matrix**
For instance, if $x(t) = (q(t) - r(t), \dot{q}(t) - \dot{r}(t))$ for a reference $r(t)$ and $K = [K_p \quad K_d]$ then

$$u^* = K_p(r(t) - q(t)) + K_d(\dot{r}(t) - \dot{q}(t))$$

8:17

Riccati equations

- Finite horizon continuous time

Riccati differential equation

$$-\dot{P} = A^\top P + PA - PBR^{-1}B^\top P + Q \;, \quad P(T) = F$$

- Infinite horizon continuous time

Algebraic Riccati equation (ARE)

$$0 = A^\top P + PA - PBR^{-1}B^\top P + Q$$

- Finite horizon discrete time $(J^\pi = \sum_{t=0}^T \|x_t\|_Q^2 + \|u_t\|_R^2 + \|x_T\|_F^2)$

$$P_{t-1} = Q + A^\top [P_t - P_t B (R + B^\top P_t B)^{-1} B^\top P_t] A \;, \quad P_T = F$$

- Infinite horizon discrete time $(J^\pi = \sum_{t=0}^\infty \|x_t\|_Q^2 + \|u_t\|_R^2)$

$$P = Q + A^\top [P - PB(R + B^\top PB)^{-1} B^\top P] A$$

8:18

Example: 1D point mass

- Dynamics:

$$\ddot{q}(t) = u(t)/m$$

$$\begin{aligned}
x = \begin{pmatrix} q \\ \dot{q} \end{pmatrix} \;, \quad \dot{x} = \begin{pmatrix} \dot{q} \\ \ddot{q} \end{pmatrix} = \begin{pmatrix} \dot{q} \\ u(t)/m \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1/m \end{pmatrix} u \\
&= Ax + Bu \;, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \;, \quad B = \begin{pmatrix} 0 \\ 1/m \end{pmatrix}
\end{aligned}$$

- Costs:

$$c(x,u) = \epsilon\|x\|^2 + \varrho\|u\|^2\,, \quad Q = \epsilon\mathbf{I}\,, \quad R = \varrho\mathbf{I}$$

- Algebraic Riccati equation:

$$P = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \quad u^* = -R^{-1}B^\top Px = \frac{-1}{\varrho m}[cq + b\dot{q}]$$
$$0 = A^\top P + PA - PBR^{-1}B^\top P + Q$$
$$= \begin{pmatrix} c & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} - \frac{1}{\varrho m^2}\begin{pmatrix} c^2 & bc \\ bc & b^2 \end{pmatrix} + \epsilon\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

8:19

Example: 1D point mass (cont.)

- Algebraic Riccati equation:

$$P = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \quad u^* = -R^{-1}B^\top Px = \frac{-1}{\varrho m}[cq + b\dot{q}]$$
$$0 = \begin{pmatrix} c & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} - \frac{1}{\varrho m^2}\begin{pmatrix} c^2 & bc \\ bc & b^2 \end{pmatrix} + \epsilon\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

First solve for c, then for $b = m\sqrt{\varrho}\sqrt{c+\epsilon}$. Whether the damping ration $\xi = \frac{b}{\sqrt{4mc}}$ depends on the choices of ϱ and ϵ.

- The Algebraic Riccati equation is usually solved numerically. (E.g. `are`, `care` or `dare` in Octave)

8:20

Optimal control comments

- HJB or Bellman equation are very powerful concepts

- Even if we can solve the HJB eq. and have the optimal control, we still don't know how the system really behaves?
  – Will it actually reach a "desired state"?
  – Will it be stable?
  – It is actually "controllable" at all?

- Last note on optimal control:
  Formulate as a constrainted optimization problem with objective function J^π and constraint $\dot{x} = f(x,u)$. $\lambda(t)$ are the Langrange multipliers. It turns out that $\frac{\partial}{\partial x}V(x,t) = \lambda(t)$. (See Stengel.)

8:21

Relation to other topics

- Optimal Control:

$$\min_\pi\ J^\pi = \int_0^T c(x(t),u(t))\,dt + \phi(x(T))$$

- Inverse Kinematics:

$$\min_q\ f(q) = \|q - q_0\|_W^2 + \|\phi(q) - y^*\|_C^2$$

- Operational space control:

$$\min_u\ f(u) = \|u\|_H^2 + \|\ddot{\phi}(q) - \ddot{y}^*\|_C^2$$

- Trajectory Optimization: (control hard constraints could be included)

$$\min_{q_{0:T}}\ f(q_{0:T}) = \sum_{t=0}^T \|\Psi_t(q_{t\text{-}k},..,q_t)\|^2 + \sum_{t=0}^T \|\Phi_t(q_t)\|^2$$

- Reinforcement Learning:
  – Markov Decision Processes $\leftrightarrow$ discrete time stochastic controlled system $P(x_{t+1}\,|\,u_t, x_t)$
  – Bellman equation $\rightarrow$ Basic RL methods (Q-learning, etc)

8:22

Controllability

8:23

Controllability

- As a starting point, consider the claim:
  "Intelligence means to gain maximal controllability over all degrees of freedom over the environment."

  Note:
  – controllability (ability to control) $\neq$ control
  – What does controllability mean exactly?

- I think the general idea of *controllability* is really interesting
  – Linear control theory provides one specific definition of controllability, which we introduce next..

8:24

Controllability

- Consider a linear controlled system

$$\dot{x} = Ax + Bu$$

How can we tell from the matrices A and B *whether we can control x to eventually reach any desired state?*

- Example: x is 2-dim, u is 1-dim:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}u$$

Is x "controllable"?

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u$$

Is x "controllable"?

8:25

Controllability

We consider a linear stationary (=time-invariant) controlled system

$$\dot{x} = Ax + Bu$$

- **Complete controllability:** All elements of the state can be brought from arbitrary initial conditions to zero in finite time
- A system is completely controllable iff the **controllability matrix**

$$C := \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

has full rank $\dim(x)$ (that is, all rows are linearly independent)

- *Meaning of C:*
  The ith row describes how the ith element x_i can be influenced by u
  "B": $\dot{x}_i$ is directly influenced via B
  "AB": $\ddot{x}_i$ is "indirectly" influenced via AB (u directly influences some $\dot{x}_j$ via B; the dynamics A then influence $\ddot{x}_i$ depending on $\dot{x}_j$)
  "A^2B": $\dddot{x}_i$ is "double-indirectly" influenced
  etc...

  Note: $\ddot{x} = A\dot{x} + B\dot{u} = AAx + ABu + B\dot{u}$
  $\dddot{x} = A^3x + A^2Bu + AB\dot{u} + B\ddot{u}$

8:26

Controllability

- When all rows of the controllability matrix are linearly independent $\Rightarrow$ $(u, \dot{u}, \ddot{u}, ...)$ can influence all elements of x independently
- If a row is zero $\rightarrow$ this element of x cannot be controlled at all
- If 2 rows are linearly dependent $\rightarrow$ these two elements of x will remain coupled forever

8:27

Controllability examples

$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}1\\ 1\end{pmatrix}u \qquad C = \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$ rows linearly dependent

$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}1\\ 0\end{pmatrix}u \qquad C = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ 2nd row zero

$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}u \qquad C = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ good!

8:28

Controllability

Why is it important/interesting to analyze controllability?

- The Algebraic Riccati Equation will always return an "optimal" controller – but controllability tells us whether such a controller even has a chance to control x
- *"Intelligence means to gain maximal controllability over all degrees of freedom over the environment."*
  – real environments are non-linear
  – "to have the ability to *gain* controllability over the environment's DoFs"

8:29

Stability

8:30

Stability

- One of the most central topics in control theory
- Instead of designing a controller by first designing a cost function and then applying Riccati,
  design a controller such that the desired state is provably a stable equilibrium point of the closed loop system

8:31

Stability

- Stability is an analysis of the *closed loop* system. That is: for this analysis we don't need to distinguish between system and controller explicitly. Both together define the dynamics

$$\dot{x} = f(x, \pi(x,t)) =: f(x)$$

- The following will therefore discuss stability analysis of general differential equations $\dot{x} = f(x)$
- What follows:
  – 3 basic definitions of stability
  – 2 basic methods for analysis by Lyapunov

8:32

Aleksandr Lyapunov (1857–1918)

8:33

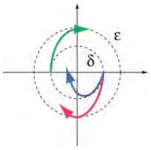
Stability – 3 definitions

$\dot{x} = F(x)$ with equilibrium point $x = 0$

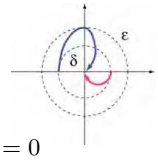
- x_0 is an **equilibrium point** $\iff f(x_0) = 0$

- **Lyapunov stable** or **uniformly stable** $\iff$

$$\forall \epsilon:\ \exists \delta \text{ s.t. } \|x(0)\| \le \delta \Rightarrow \|x(t)\| \le \epsilon$$

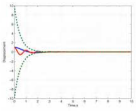


(when it starts off δ-near to x_0, it will remain ϵ-near forever)



- **asymtotically stable** $\iff$

Lyapunov stable and $\lim_{t\to\infty} x(t) = 0$



- **exponentially stable** $\iff$

asymtotically stable and $\exists \alpha, a$ s.t. $\|x(t)\| \le a e^{-\alpha t}\|x(0)\|$

($\rightarrow$ the "error" time integral $\int_0^\infty \|x(t)\|dt \le \frac{a}{\alpha}\|x(0)\|$ is bounded!)

8:34

Linear Stability Analysis

("Linear" $\leftrightarrow$ "local" for a system linearized at the equilibrium point.)

- Given a linear system

$$\dot{x} = Ax$$

Let λ_i be the **eigenvalues** of A

– The system is *asymptotically stable* $\iff \forall_i:\ \text{real}(\lambda_i) < 0$

– The system is *unstable stable* $\iff \exists_i:\ \text{real}(\lambda_i) > 0$

– The system is *marginally stable* $\iff \forall_i:\ \text{real}(\lambda_i) \le 0$

- Meaning: An eigenvalue describes how the system behaves along one state dimension (along the eigenvector):

$$\dot{x}_i = \lambda_i x_i$$

As for the 1D point mass the solution is $x_i(t) = a e^{\lambda_i t}$ and
– imaginary $\lambda_i \rightarrow$ oscillation
– negative $\text{real}(\lambda_i) \rightarrow$ exponential decay $\propto e^{-|\lambda_i| t}$
– positive $\text{real}(\lambda_i) \rightarrow$ exponential explosion $\propto e^{|\lambda_i| t}$

8:35

Linear Stability Analysis: Example

- Let's take the 1D point mass $\ddot{q} = u/m$ in *closed loop* with a PD $u = -K_p q - K_d \dot{q}$
- Dynamics:

$$\dot{x} = \begin{pmatrix} \dot{q} \\ \ddot{q} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + 1/m \begin{pmatrix} 0 & 0 \\ -K_p & -K_d \end{pmatrix} x$$

$$A = \begin{pmatrix} 0 & 1 \\ -K_p/m & -K_d/m \end{pmatrix}$$

- Eigenvalues:

The equation $\lambda \begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -K_p/m & -K_d/m \end{pmatrix} \begin{pmatrix} q \\ \dot{q} \end{pmatrix}$ leads to the equation $\lambda \dot{q} = \lambda^2 q = -K_p/m q - K_d/m \lambda q$ or $m\lambda^2 + K_d \lambda + K_p = 0$ with solution (compare slide 05:10)

$$\lambda = \frac{-K_d \pm \sqrt{K_d^2 - 4mK_p}}{2m}$$

For $K_d^2 - 4mK_p$ negative, the $\text{real}(\lambda) = -K_d/2m$

$\Rightarrow$ Positive derivative gain K_d makes the system stable.

8:36

Side note: Stability for discrete time systems

- Given a discrete time linear system

$$x_{t+1} = Ax_t$$

Let λ_i be the **eigenvalues** of A

– The system is *asymptotically stable* $\iff \forall_i:\ |\lambda_i| < 1$

– The system is *unstable stable* $\iff \exists_i:\ |\lambda_i| > 1$

– The system is *marginally stable* $\iff \forall_i:\ |\lambda_i| \le 1$

8:37

Linear Stability Analysis comments

- The same type of analysis can be done locally for non-linear systems, as we will do for the cart-pole in the exercises
- We can design a controller that minimizes the (negative) eigenvalues of A:

$\leftrightarrow$ controller with fastest asymtopic convergence

This is a real alternative to optimal control!

8:38

Lyapunov function method

- A method to analyze/prove stability for general non-linear systems is the famous "Lyapunov's second method"

Let D be a region around the equilibrium point x_0

- A **Lyaponov function** $V(x)$ for a system dynamics $\dot{x} = f(x)$ is
  – positive, $V(x) > 0$, everywhere in D except...
  at the equilibrium point where $V(x_0) = 0$
  – always decreases, $\dot{V}(x) = \frac{\partial V(x)}{\partial x}\dot{x} < 0$, in D except...
  at the equilibrium point where $f(x) = 0$ and therefore $\dot{V}(x) = 0$
- If there exists a D and a Lyapunov function $\Rightarrow$ the system is *asymtotically stable*

If D is the whole state space, the system is *globally stable*

8:39

Lyapunov function method

- The Lyapunov function method is very general. $V(x)$ could be "anything" (energy, cost-to-go, whatever). Whenever one finds some $V(x)$ that decreases, this proves stability

- The problem though is to think of some $V(x)$ given a dynamics!

  (In that sense, the Lyapunov function method is rather a method of proof than a concrete method for stability analysis.)

- In standard cases, a good guess for the Lyapunov function is either the energy or the value function

8:40

Lyapunov function method – Energy Example

- Let's take the 1D point mass $\ddot{q} = u/m$ in *closed loop* with a PD $u = -K_p q - K_d \dot{q}$, which has the solution (slide 05:14):

$$q(t) = b e^{-\xi/\lambda\, t}\, e^{i\omega_0\sqrt{1-\xi^2}\, t}$$

- Energy of the 1D point mass: $V(q,\dot{q}) := \frac{1}{2} m \dot{q}^2$

$$\dot{V}(x) = e^{-2\xi/\lambda\, t} V(x(0))$$

  (using that the energy of an undamped oscillator is conserved)

- $V(x) < 0 \iff \xi > 0 \iff K_d > 0$

  Same result as for the eigenvalue analysis

8:41

Lyapunov function method – value function Example

- Consider infinite horizon linear-quadratic optimal control. The solution of the Algebraic Riccati equation gives the optimal controller.
- The value function satisfies

$$\begin{aligned}
V(x) &= x^\top P x \\
\dot{V}(x) &= [Ax + Bu^*]^\top P x + x^\top P [Ax + Bu^*] \\
u^* &= -R^{-1} B^\top P x = Kx \\
\dot{V}(x) &= x^\top [(A+BK)^\top P + P(A+BK)] x \\
&= x^\top [A^\top P + PA + (BK)^\top P + P(BK)] x \\
0 &= A^\top P + PA - PBR^{-1}B^\top P + Q \\
\dot{V}(x) &= x^\top [PBR^{-1}B^\top P - Q + (PBK)^\top + PBK] x \\
&= -x^\top [Q + K^\top R K] x
\end{aligned}$$

  (We could have derived this easier! $x^\top Q x$ are the immediate state costs, and $x^\top K^\top R K x = u^\top R u$ are the immediate control costs—and $\dot{V}(x) = -c(x,u^*)$! See slide 11 bottom.)

- That is: V is a Lyapunov function if $Q + K^\top R K$ is positive definite!

8:42

Observability & Adaptive Control

- When some state dimensions are not directly observable: analyzing higher order derivatives to *infer* them.

  Very closely related to controllability: Just like the controllability matrix tells whether state dimensions can (indirectly) be controlled; an observation matrix tells whether state dimensions can (indirectly) be inferred.

- Adaptive Control: When system dynamics $\dot{x} = f(x,u,\beta)$ has unknown parameters β.

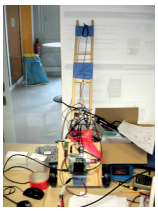
  – One approach is to estimate β from the data so far and use optimal control.

  – Another is to design a controller that has an additional internal update equation for an estimate $\hat{\beta}$ and is provably stable. (See Schaal's lecture, for instance.)

8:43

9 Practical: A 2-wheeled *Racer*

A 2-wheeled racer



- Educational ideas:
  - have a really **dynamic** system
  - have a system which, in the "racing" limit, is hard to control
  - learn about hardware, communication, etc
  - challenges connecting theory with practise:
- Real world issues:
  - control interface ("setting velocities") is adventurous
  - PARTIAL OBSERVABILITY: we only have a noisy accelerometer & gyroscope
  - unknown time delays
  - unknown system parameters (masses, geometry, etc)

9:1

Intro

9:2

[demo]

9:3

Components

- Odroid: on-board PC running xubuntu
- Motor unit: motors, motor driver, motor controller, Hall sensor
- IMU (inertial measurement unit): 3D accelerometer, 3D gyroscope, (magnetic)
- Communication: USB-to-I2C communicates with both, motors and IMU

- See Marcel's thesis

9:4

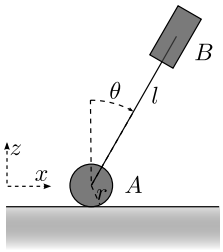
Code

- From Marcel's thesis:
  - Control loop (around 36 msec)

- Kalman filter tests on the accelerometer:
  - Caroline

9:5

2D Modelling



- See theoretical modelling notes

9:6

3D Modelling

- Account for centrifugal forces in a curve
- Generalized coordinates $q = (x, y, \phi, \theta)$, with steering angle ϕ

- Exercise: Derive general Euler-Lagrange equations

9:7

Clash of theory and real world

9:8

The control interface

- Theory assumed torque control
- In real, the motor controller "does things somehow". We can set:
  - a target velocities $v^*_{l,r}$
  - desired acceleration level $a^*_{l,r} \in \{-10, .., -1, 1, .., 10\}$
- The controller will then ramp velocity in 25msec steps depending on a^* until target v^* is reached
- Unknown: time delays, scaling of a^*?

- Potential approach:
  - Assume acceleration control interface
  - Consider constrained Euler-Lagrange equations

9:9

Coping with the partial observability

- Theoretical view: In LQG systems it is known that optimal control under partial observability is the same as optimal control assuming the Bayes estimated state as true current state. *Uncertainty principle.*
- Use a Bayes filter to estimate the state $(q, \dot{q})$ from all sensor information we have
- Sensor information:
  - Accelerometer readings $\tilde{a}_{x,y,z}$
  - Gyro readings $\tilde{g}_{x,y,z}$
  - Motor positions $\tilde{\theta}_{l,r}$. Note that $\tilde{\theta} \propto x/r - \theta$ desribes the relative angle between the pole and the wheels
- Open issue: time delays – relevant?

9:10

Coping with unknown system parameters

- **System identification**
- We derived the eqs of motion $Bu = M\ddot{q} + F$ (for 2D) – but don't know the parameters
  - m_A, I_A, m_B, I_B: masses and inertias of bodies A (=wheel) and B (=pendulum)
  - r: radius of the wheel
  - l: length of the pendulum (height of its COM)
- Focus on the local linearization around $(q, \dot{q}) = 0$
- OR: Use blackbox optimization to fit parameters to data

9:11

Data

- We need data to understand better what's going on!
- Lot's of data of full control cycles around $(q, \dot{q}) = 0$
  (sensor reading, control signals, cycle time)
- Data specifically on how motors accelerate when setting a desired acceleration level

9:12

Or completely different: Reinforcement Learning

- Alternatively one fully avoids modelling → Reinforcement Learning
- Roughly: blackbox optimization (e.g., EA) of PD parameters

9:13

Modelling

9:14

Modelling overview I

We have exact analytical models (and implemented) for the following:

- Euler-Lagange equations

$$M(q)\ \ddot{q} + F(q, \dot{q}) = B(q)\ u$$
$$\ddot{q} = M^{-1}(Bu - F)$$

→ **energy check**

→ **physical simulation**

- Local linearization $(x = (q, \dot{q}))$

$$\ddot{q} = Ax + a + \bar{B}u$$
$$A = \frac{\partial}{\partial x} M^{-1}(Bu - F)\ , \quad \bar{B} = M^{-1}B$$

→ **gradient check**

→ **Riccati eqn** → nice controller [demo]

9:15

Modelling overview II

- Sensor model

$$y^{\text{acc}} = c_1\ R\ [\ddot{p}_B - (0, g)^\top]\ , \quad R = \begin{pmatrix} \cos(\theta + c_2) & -\sin(\theta + c_2) \\ \sin(\theta + c_2) & \cos(\theta + c_2) \end{pmatrix}$$
$$y^{\text{gyro}} = c_3(\dot{\theta} + c_4)$$
$$y^{\text{enc}} = c_5(x/r - \theta)$$
$$y = (y^{\text{acc}}, y^{\text{gyro}}, y^{\text{enc}}) \in \mathbb{R}^4$$

- Local linearization

$$C = \frac{\partial y}{\partial (q, \dot{q})} = \begin{pmatrix} \frac{\partial y}{\partial q} & \frac{\partial y}{\partial \dot{q}} \end{pmatrix} + \frac{\partial y}{\partial \ddot{q}}\ \frac{\partial \ddot{q}}{\partial (q, \dot{q})}$$

→ **gradient check**

→ **Kalman filtering** [demo]

9:16

Modelling overview III

- Constrained Euler-Lagange equations for acceleration control
  - Our motors actually don't allow to set torques – but rather set accelerations. Setting accelerations implies the constraint
    $$B'\ddot{q} = u'$$
  - Using $\ddot{q} = M^{-1}(Bu - F)$ we can retrieve the torque
    $$u = (B'M^{-1}B)^{-1}[u' + B'M^{-1}F]$$
    that exactly generates this acceleration
  - Plugging this back into $\ddot{q} = M^{-1}(Bu - F)$ we get
    $$\ddot{q} = B'^{\#}u' - (\mathbf{I} - B'^{\#}B')M^{-1}F\ , \quad B'^{\#} = M^{-1}B(B'M^{-1}B)^{-1}$$

9:17

Modelling summary

- We now have all analytic models we need
- **In simulation** we have no problem to apply
  - Riccati to retrieve a (locally) optimal linear regulator
  - Kalman to optimally (subject to linearizations) estimate the state
- The crux: we have 12 unknown parameters
  $$m_A, I_A, m_B, I_B, \quad r, l, l_C, \quad c_1, .., c_5$$
  (plus sensor noise parameters $\sigma_a, \sigma_g, \sigma_e$)

9:18

System Identification

9:19

System Identification

- Given data $D = \{(x, u, y)_t\}_{t=1}^T$, learn

$$(x, u) \mapsto x' \quad \text{or} \quad P(x'|x, u)$$
$$(x, u) \mapsto y \quad \text{or} \quad P(y|x, u)$$

9:20

Regression options for system identification

- Linear: (linear in finite number of parameters)

$$f(x; \theta) = \phi(x)^\top \theta$$

- Blackbox parameteric:
  - Given some blackbox parameteric model $f(x; \theta)$ with finite parameters θ; use blackbox optimization
- Non-parameteric:
  - Kernel methods
  - Gaussian processes
  - Are closely related to linear models

- In all cases one typically minimizes the squared error

$$L^{\text{ls}}(\theta) = \sum_{i=1}^n (y_i - f(x_i; \theta))^2$$

- We can use the mean $\frac{1}{n} L^{\text{ls}}(\theta)$ as estimate of the output variance σ^2 to define

$$P(y|x; \theta) = \mathcal{N}(y|f(x; \theta), \sigma^2)$$

9:21

System Id examples: Kinematics

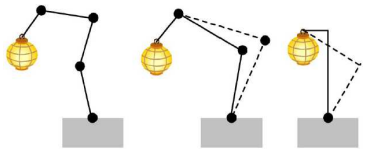
- If the kinematics ϕ are unknown, learn them from data!

Literature:

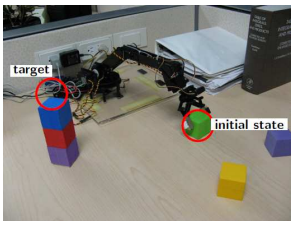
Todorov: Probabilistic inference of multi-joint movements, skeletal parameters and marker attachments from diverse sensor data. (IEEE Transactions on Biomedical Engineering 2007)

Deisenroth, Rasmussen & Fox: Learning to Control a Low-Cost Manipulator using Data-Efficient Reinforcement Learning (RSS 2011)

9:22



Todorov: Probabilistic inference of multi-joint movements, skeletal parameters and marker attachments from diverse sensor data. (IEEE Transactions on Biomedical Engineering 2007)



Deisenroth, Rasmussen & Fox: Learning to Control a Low-Cost Manipulator using Data-Efficient Reinforcement Learning (RSS 2011)

9:23

System Id examples: Dynamics

- If the dynamics $\dot{x} = f(x, u)$ are unknown, learn them from data!

Literature:

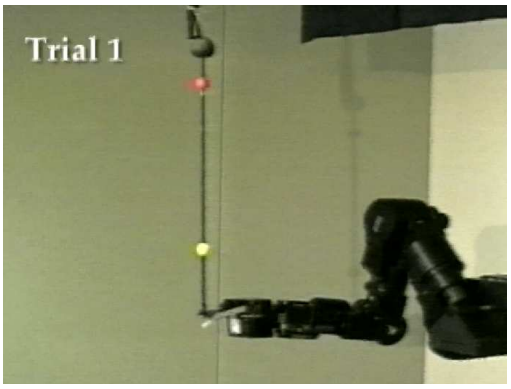
Moore: Acquisition of Dynamic Control Knowledge for a Robotic Manipulator (ICML 1990)

Atkeson, Moore & Schaal: Locally weighted learning for control. Artificial Intelligence Review, 1997.

Schaal, Atkeson & Vijayakumar: Real-Time Robot Learning with Locally Weighted Statistical Learning. (ICRA 2000)

Vijayakumar et al: Statistical learning for humanoid robots, Autonomous Robots, 2002.

9:24



(Schaal, Atkeson, Vijayakumar)

- Use a simple regression method (locally weighted Linear Regression) to estimate $\dot{x} = f(x, u)$

9:25

Regression basics

[ML slides]

9:26

Applying System Id to the racer?

- Core problem:

We have no ground truth data!

- We can record data (u, y) (controls & observations), but not x!

- Try an EM like approach:
  - Hand-estimate the parameters as good as possible

  - Use a Kalman filter (better: smoother!!) to estimate the unobserved x during
  - Option (a): Learn local linear models $\ddot{q} = Ax + a + Bu$ and $y = Cx + c + Du$
    Option (b): Improve the parameters $\theta = (m_A, I_A, m_B, I_B, r, l, l_C, c_1, .., c_5)$
  - Repeat with Kalman smoothing

- I have no idea whether/how well this'll work

9:27

Data

9:28

We've collected data

- Motor responses
  - Free running wheels (no load..)
  - Setting extreme target velocities v^* and different acceleration levels $a^* \in \{-10, .., -1, 1, .., 10\}$ we can generate well-defined accelerations

- Balancing trials
  - the gyroscope picks up some oscillations
  - the accelerometer is very noisy, perhaps correlated with jerky controls
  - only 30Hz!

9:29

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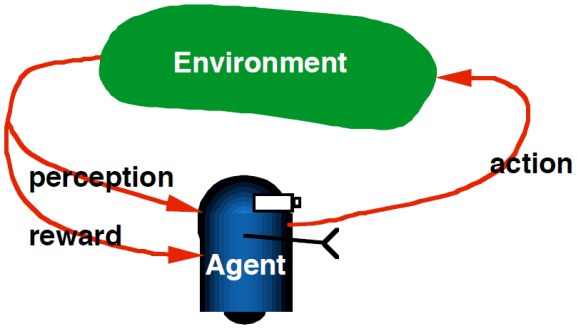
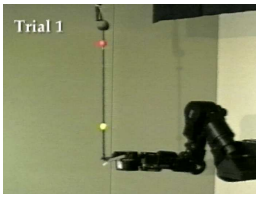
10 Reinforcement Learning in Robotics

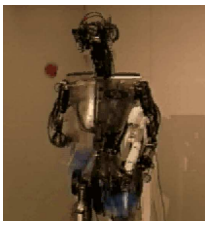
RL = Learning to Act



from Satinder Singh's *Introduction to RL*, videolectures.com

10:1





(around 2000, by Schaal, Atkeson, Vijayakumar)

(2007, Andrew Ng et al.)

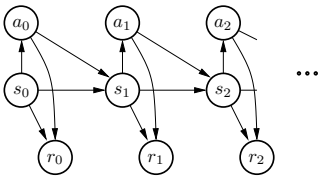
10:2

Applications of RL

- Robotics
  – Navigation, walking, juggling, helicopters, grasping, etc...
- Games
  – Backgammon, Chess, Othello, Tetris, ...
- Control
  – factory processes, resource control in multimedia networks, elevators, ...
- Operations Research
  – Warehousing, transportation, scheduling, ...

10:3

Markov Decision Process



$$P(s_{0:T+1}, a_{0:T}, r_{0:T}; \pi) = P(s_0) \prod_{t=0}^{T} P(a_t|s_t;\pi)\, P(r_t|s_t,a_t)\, P(s_{t+1}|s_t,a_t)$$

– world's initial state distribution $P(s_0)$

– world's transition probabilities $P(s_{t+1} \mid s_t, a_t)$

– world's reward probabilities $P(r_t \mid s_t, a_t)$

– agent's *policy* $\pi(a_t \mid s_t) = P(a_0|s_0;\pi)$ (or deterministic $a_t = \pi(s_t)$)

- **Stationary MDP:**
  – We assume $P(s' \mid s, a)$ and $P(r|s,a)$ independent of time
  – We also define $R(s,a) := \mathsf{E}\{r|s,a\} = \int r\, P(r|s,a)\, dr$

10:4

... in the notation this Robotic's lecture

- We have a (potentially stochastic) controlled system

$$\dot{x} = f(x,u) + \text{noise}(x,u)$$

- We have costs (neg-rewards), e.g. in the finite horizon case:

$$J^\pi = \int_0^T c(x(t),u(t))\, dt + \phi(x(T))$$

- We want a policy ("controller")

$$\pi : (x,t) \mapsto u$$

10:5

Reinforcement Learning = the dynamics f and costs c are unknown

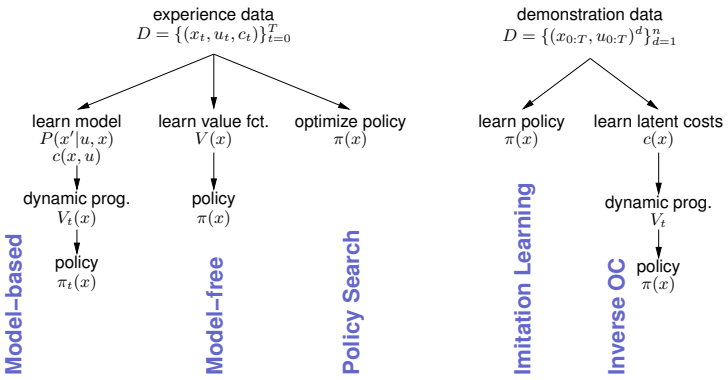
- All the agent can do is collect data

$$D = \{(x_t, u_t, c_t)\}_{t=0}^{T}$$

What can we do with this data?

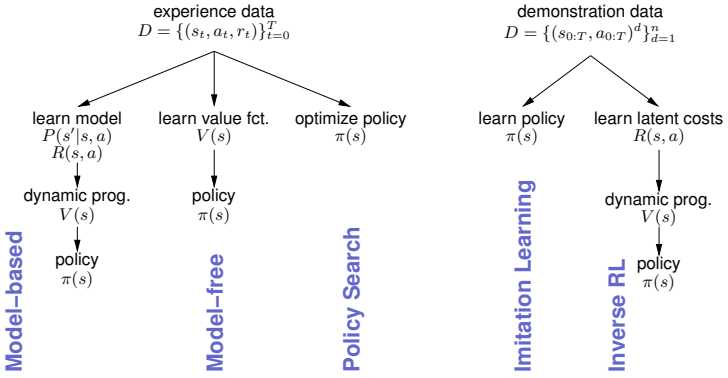
10:6

Five approaches to RL



10:7

Five approaches to RL



10:8

Imitation Learning

$$D = \{(s_{0:T}, a_{0:T})^d\}_{d=1}^n \quad \overset{\text{learn/copy}}{\rightarrow} \quad \pi(s)$$

- Use ML to imitate demonstrated state trajectories $x_{0:T}$

Literature:

Atkeson & Schaal: Robot learning from demonstration (ICML 1997)

Schaal, Ijspeert & Billard: Computational approaches to motor learning by imitation (Philosophical Transactions of the Royal Society of London. Series B: Biological Sciences 2003)

Grimes, Chalodhorn & Rao: Dynamic Imitation in a Humanoid Robot through Nonparametric Probabilistic Inference. (RSS 2006)

Rüdiger Dillmann: Teaching and learning of robot tasks via observation of human performance (Robotics and Autonomous Systems, 2004)

10:9

Imitation Learning

- There a many ways to imitate/copy the oberved policy:

Learn a density model $P(a_t \mid s_t)P(s_t)$ (e.g., with mixture of Gaussians) from the observed data and use it as policy (Billard et al.)

Or trace observed trajectories by minimizing perturbation costs (Atkeson & Schaal 1997)

10:10

Imitation Learning

Atkeson & Schaal

10:11

Inverse RL

$$D = \{(s_{0:T}, a_{0:T})^d\}_{d=1}^n \quad \overset{\text{learn}}{\rightarrow} \quad R(s,a) \quad \overset{\text{DP}}{\rightarrow} \quad V(s) \quad \rightarrow \quad \pi(s)$$

- Use ML to "uncover" the latent reward function in observed behavior

Literature:

Pieter Abbeel & Andrew Ng: Apprenticeship learning via inverse reinforcement learning (ICML 2004)

Andrew Ng & Stuart Russell: Algorithms for Inverse Reinforcement Learning (ICML 2000)

Nikolay Jetchev & Marc Toussaint: Task Space Retrieval Using Inverse Feedback Control (ICML 2011).

10:12

Inverse RL (Apprenticeship Learning)

- Given: demonstrations $D = \{x_{0:T}^d\}_{d=1}^n$
- Try to find a reward function that **discriminates demonstrations from other policies**

  – Assume the reward function is linear in some features $R(x) = w^\top \phi(x)$

  – Iterate:

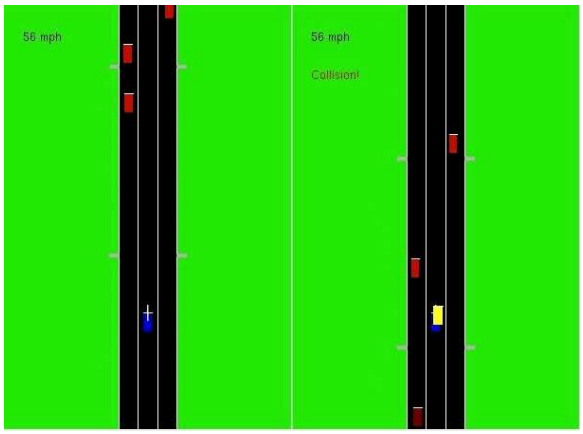
  1. Given a set of candidate policies $\{\pi_0, \pi_1, ..\}$
  2. Find weights w that maximize the value margin between teacher and all other candidates

$$\max_{w,\xi} \ \xi$$
$$\text{s.t. } \forall_{\pi_i}: \quad \underbrace{w^\top \langle \phi \rangle_D}_{\text{value of demonstrations}} \geq \underbrace{w^\top \langle \phi \rangle_{\pi_i}}_{\text{value of } \pi_i} + \xi$$
$$\|w\|^2 \leq 1$$

3. Compute a new candidate policy π_i that optimizes $R(x) = w^\top \phi(x)$ and add to candidate list.

(Abbeel & Ng, ICML 2004)

10:13



10:14

Policy Search with Policy Gradients

10:15

Policy gradients

- In continuous state/action case, represent the policy as linear in arbitrary state features:

$$\pi(s) = \sum_{j=1}^{k} \phi_j(s)\beta_j = \phi(s)^\top \beta \qquad \text{(deterministic)}$$
$$\pi(a \,|\, s) = \mathcal{N}(a \,|\, \phi(s)^\top \beta, \Sigma) \qquad \text{(stochastic)}$$

with k features ϕ_j.

- Given an episode $\xi = (s_t, a_t, r_t)_{t=0}^H$, we want to estimate

$$\frac{\partial V(\beta)}{\partial \beta}$$

10:16

Policy Gradients

- One approach is called REINFORCE:

$$\frac{\partial V(\beta)}{\partial \beta} = \frac{\partial}{\partial \beta} \int P(\xi|\beta)\, R(\xi)\, d\xi = \int P(\xi|\beta) \frac{\partial}{\partial \beta} \log P(\xi|\beta) R(\xi) d\xi$$
$$= \mathsf{E}\{\xi|\beta\} \frac{\partial}{\partial \beta} \log P(\xi|\beta) R(\xi) = \mathsf{E}\{\xi|\beta\} \sum_{t=0}^{H} \gamma^t \frac{\partial \log \pi(a_t|s_t)}{\partial \beta} \underbrace{\sum_{t'=t}^{H} \gamma^{t'-t} r_{t'}}_{Q^\pi(s_t, a_t, t)}$$

- Another is Natural Policy Gradient
  - Estimate the Q-function as linear in the basis functions $\frac{\partial}{\partial \beta} \log \pi(a|s)$:

$$Q(x, u) \approx \Big[\frac{\partial \log \pi(a|s)}{\partial \beta}\Big]^\top w$$

  - Then the natural gradient ($\frac{\partial V(\beta)}{\partial \beta}$ multiplied with inv. Fisher metric) updates

$$\beta^{\text{new}} = \beta + \alpha w$$

- Another is PoWER, which requires $\frac{\partial V(\beta)}{\partial \beta} = 0$

$$\beta \leftarrow \beta + \frac{\mathsf{E}\{\xi|\beta\} \sum_{t=0}^{H} \epsilon_t Q^\pi(s_t, a_t, t)}{\mathsf{E}\{\xi|\beta\} \sum_{t=0}^{H} Q^\pi(s_t, a_t, t)}$$

10:17

Kober & Peters: *Policy Search for Motor Primitives in Robotics*, NIPS 2008.

(serious reward shaping!)

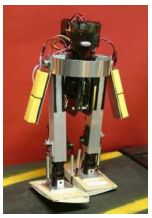
10:18

Learning to walk in 20 Minutes

- Policy Gradient method (Reinforcement Learning)

  Stationary policy parameterized as linear in features $u = \sum_i w_i \phi_i(q,$
- Problem: find parameters w_i to minimize expected costs

  cost = mimick $(q, \dot{q})$ of the passive down-hill walker at "certain point in cycle"



Learning To Walk

Tedrake, Zhang & Seung: *Stochastic policy gradient reinforcement learning on a simple 3D biped.* IROS, 2849-2854, 2004. http://groups.csail.mit.edu/robotics-center/public_papers/Tedrake04a.pdf

10:19

Policy Gradients – references

Peters & Schaal (2008): *Reinforcement learning of motor skills with policy gradients*, Neural Networks.

Kober & Peters: *Policy Search for Motor Primitives in Robotics*, NIPS 2008.

Vlassis, Toussaint (2009): *Learning Model-free Robot Control by a Monte Carlo EM Algorithm.* Autonomous Robots 27, 123-130.

Rawlik, Toussaint, Vijayakumar(2012): *On Stochastic Optimal Control and Reinforcement Learning by Approximate Inference.* RSS 2012. *(ψ-learning)*

- These methods are sometimes called **white-box optimization**: They optimize the policy parameters β for the total reward $R = \sum \gamma^t r_t$ while tying to exploit knowledge of how the process is actually parameterized

10:20

Black-Box Optimization

10:21

"Black-Box Optimization"

- The term is not really well defined
  – I use it to express that *only* $f(x)$ can be evaluated
  – $\nabla f(x)$ or $\nabla^2 f(x)$ are not (directly) accessible
  More common terms:
- **Global optimization**
  – This usually emphasizes that methods should not get stuck in local optima
  – Very very interesting domain – close analogies to (active) Machine Learning, bandits, POMDPs, optimal decision making/planning, optimal experimental design
  – Usually mathematically well founded methods
- **Stochastic search** or **Evolutionary Algorithms** or **Local Search**
  – Usually these are local methods (extensions trying to be "more" global)
  – Various interesting heuristics
  – Some of them (implicitly or explicitly) locally approximating gradients or 2nd order models

10:22

Black-Box Optimization

- Problem: Let $x \in \mathbb{R}^n$, $f:\ \mathbb{R}^n \to \mathbb{R}$, find

$$\min_x\ f(x)$$

where we can only evaluate $f(x)$ for any $x \in \mathbb{R}^n$

- A constrained version: Let $x \in \mathbb{R}^n$, $f:\ \mathbb{R}^n \to \mathbb{R}$, $g:\ \mathbb{R}^n \to \{0,1\}$, find

$$\min_x\ f(x) \quad \text{s.t.} \quad g(x) = 1$$

where we can only evaluate $f(x)$ and $g(x)$ for any $x \in \mathbb{R}^n$
I haven't seen much work on this. Would be interesting to consider this more rigorously.

10:23

A zoo of approaches

- People with many different backgrounds drawn into this
  Ranging from heuristics and Evolutionary Algorithms to heavy mathematics
  – Evolutionary Algorithms, esp. Evolution Strategies, Covariance Matrix Adaptation, Estimation of Distribution Algorithms
  – Simulated Annealing, Hill Climing, Downhill Simplex
  – local modelling (gradient/Hessian), global modelling

10:24

Optimizing and Learning

- Black-Box optimization is strongly related to learning:
- When we have local a gradient or Hessian, we can take that local information and run – no need to keep track of the history or learn (exception: BFGS)
- In the black-box case we have no local information directly accessible
  $\to$ one needs to account for the history in some way or another to have an idea where to continue search
- "Accounting for the history" very often means learning: Learning a local or global model of f itself, learning which steps have been successful recently (gradient estimation), or which step directions, or other heuristics

10:25

Stochastic Search

10:26

Stochastic Search

- The general recipe:
  – The algorithm maintains a probability distribution $p_\theta(x)$
  – In each iteration it takes n samples $\{x_i\}_{i=1}^n \sim p_\theta(x)$
  – Each x_i is evaluated $\to$ data $\{(x_i, f(x_i))\}_{i=1}^n$
  – That data is used to update θ
- Stochastic Search:

Input: initial parameter θ, function $f(x)$, distribution model $p_\theta(x)$, update heuristic $h(\theta, D)$
Output: final θ and best point x
1: **repeat**
2: Sample $\{x_i\}_{i=1}^n \sim p_\theta(x)$
3: Evaluate samples, $D = \{(x_i, f(x_i))\}_{i=1}^n$
4: Update $\theta \leftarrow h(\theta, D)$
5: **until** θ converges

10:27

Stochastic Search

- The parameter θ is the only "knowledge/information" that is being propagated between iterations
  θ encodes what has been learned from the history
  θ defines where to search in the future
- Evolutionary Algorithms: θ is a parent population
  Evolution Strategies: θ defines a Gaussian with mean & variance
  Estimation of Distribution Algorithms: θ are parameters of some distribution model, e.g. Bayesian Network
  Simulated Annealing: θ is the "current point" and a temperature

10:28

Example: Gaussian search distribution (μ, λ)-ES

From 1960s/70s. Rechenberg/Schwefel

- Perhaps the simplest type of distribution model

$$\theta = (\hat{x}) \ , \quad p_t(x) = \mathcal{N}(x|\hat{x}, \sigma^2)$$

a n-dimenstional isotropic Gaussian with fixed deviation σ

- Update heuristic:
  - Given $D = \{(x_i, f(x_i))\}_{i=1}^{\lambda}$, select μ best: $D' = \text{bestOf}_\mu(D)$
  - Compute the new mean $\hat{x}$ from D'

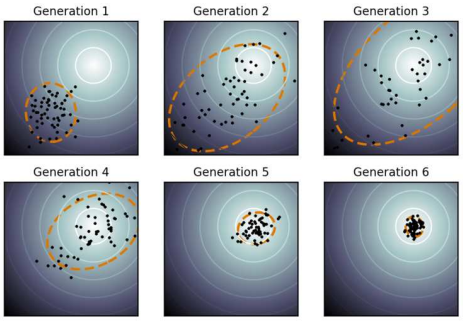
- This algorithm is called "Evolution Strategy (μ, λ)-ES"
  - The Gaussian is meant to represent a "species"
  - λ offspring are generated
  - the best μ selected

10:29

Covariance Matrix Adaptation (CMA-ES)

- An obvious critique of the simple Evolution Strategies:
  - The search distribution $\mathcal{N}(x|\hat{x}, \sigma^2)$ is isotropic
    (no going *forward*, no preferred direction)
  - The variance σ is fixed!

- Covariance Matrix Adaptation Evolution Strategy (CMA-ES)



10:30

Covariance Matrix Adaptation (CMA-ES)

- In Covariance Matrix Adaptation

$$\theta = (\hat{x}, \sigma, C, p_\sigma, p_C) \ , \quad p_\theta(x) = \mathcal{N}(x|\hat{x}, \sigma^2 C)$$

where C is the covariance matrix of the search distribution

- The θ maintains two more pieces of information: p_σ and p_C capture the "path" (motion) of the mean $\hat{x}$ in recent iterations
- Rough outline of the θ-update:
  - Let $D' = \text{bestOf}_\mu(D)$ be the set of selected points
  - Compute the new mean $\hat{x}$ from D'
  - Update p_σ and p_C proportional to $\hat{x}_{k+1} - \hat{x}_k$
  - Update σ depending on $|p_\sigma|$
  - Update C depending on $p_c p_c^\top$ (rank-1-update) and $\text{Var}(D')$

10:31

CMA references

Hansen, N. (2006), "The CMA evolution strategy: a comparing review"

Hansen et al.: Evaluating the CMA Evolution Strategy on Multimodal Test Functions, PPSN 2004.

Function	f_{stop}	init	n	CMA-ES	DE	RES	LOS
$f_{\text{Ackley}}(x)$	1e-3	$[-30, 30]^n$	20	**2667**	.	.	6.0e4
			30	**3701**	12481	1.1e5	9.3e4
			100	**11900**	36801	.	.
$f_{\text{Griewank}}(x)$	1e-3	$[-600, 600]^n$	20	**3111**	8691	.	.
			30	**4455**	11410 *	*8.5e-3/2e5*	.
			100	**12796**	31796	.	.
$f_{\text{Rastrigin}}(x)$	0.9	$[-5.12, 5.12]^n$	20	68586	**12971**	.	9.2e4
		DE: $[-600, 600]^n$	30	147416	**20150** *	1.0e5	2.3e5
			100	1010989	**73620**	.	.
$f_{\text{Rastrigin}}(\boldsymbol{A}x)$	0.9	$[-5.12, 5.12]^n$	30	**152000**	*171/1.25e6* *	.	.
			100	**1011556**	*944/1.25e6* *	.	.
$f_{\text{Schwefel}}(x)$	1e-3	$[-500, 500]^n$	5	43810	**2567** *	.	7.4e4
			10	240899	**5522** *	.	5.6e5

- For "large enough" populations local minima are avoided

- An interesting variant:

  Igel et al.: A Computational Efficient Covariance Matrix Update and a $(1+1)$-CMA for Evolution Strategies, GECCO 2006.

10:32

CMA conclusions

- It is a good starting point for an off-the-shelf black-box algorithm
- It includes components like estimating the local gradient (p_σ, p_C), the local "Hessian" ($\text{Var}(D')$), smoothing out local minima (large populations)

10:33

Stochastic search conclusions

Input: initial parameter θ, function $f(x)$, distribution model $p_\theta(x)$, update heuristic $h(\theta, D)$
Output: final θ and best point x

1: **repeat**
2: Sample $\{x_i\}_{i=1}^n \sim p_\theta(x)$
3: Evaluate samples, $D = \{(x_i, f(x_i))\}_{i=1}^n$
4: Update $\theta \leftarrow h(\theta, D)$
5: **until** θ converges

- The framework is very general
- The crucial difference between algorithms is their choice of $p_\theta(x)$

10:34

RL under Partial Observability

- Data:

$$D = \{(u_t, c_t, y_t)_t\}_{t=0}^T$$

$\rightarrow$ state x_t not observable

- Model-based RL is dauting: Learning $P(x'|u, x)$ and $P(y|u, x)$ with latent x is very hard

- Model-free: The policy needs to map the **history** to a new control

$$\pi : (y_{t-h,..,t\text{-}1}, u_{t-h,..,t-1}) \mapsto u_t$$

or any **features of the history**

$$u_t = \phi(y_{t-h,..,t\text{-}1}, u_{t-h,..,t-1})^\top w$$

10:35

Features for the racer?

- Potential features might be:

$$\Big(y_t, \dot{y}_t, \langle y\rangle_{0.5}, \langle \dot{y}\rangle_{0.5}, \langle y\rangle_{0.9}, \langle \dot{y}\rangle_{0.9}, u_t, u_{t-1}\Big)$$

where $\dot{y} = \frac{y_t - y_{t-1}}{\tau_t}$ and $\langle y\rangle_\alpha$ is a low-pass filter

10:36

11 SKIPPED THIS TERM – Grasping (brief intro)

Force closure, alternative/bio-inspired views

Grasping

- The most elementary type of interaction with (manipulation of) the environment.

  – Basis for intelligent behavior.

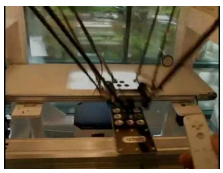
- In industrial settings with high precision sensors and actuators: very fast and precise.

- In general real world with uncertain actuators and perception, still a great research challenge, despite all the theory that has been developed.

11:1

Pick-and-place in industry





(This type of kinematics is called “Delta Robot”, which is a “parallel robot” just as the Stewart platform.)

11:2

Research

- Using ultra high speed and precise cameras and localization: High speed robot hand from the Ishikawa Komuro’s “Sensor Fusion” Lab

http://www.k2.t.u-tokyo.ac.jp/fusion/index-e.html

- Asimo’s grasping:

11:3

Outline

- Introduce to the basic classical concepts for grasping (force closure)

- Discussion and alternative views

- References:

  Craig’s *Introduction to robotics: mechanics and control* – chapter 3.

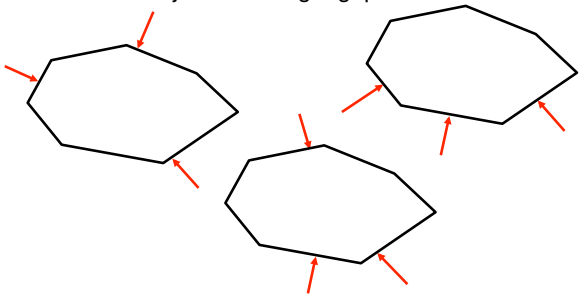
  Matt Mason’s lecture: *Static and Quasistatic Manipulation*
  www.cs.cmu.edu/afs/cs/academic/class/16741-s07/www/lecture18.pdf

  Daniela Rus and Seth Teller’s lecture: *Grasping and Manipulation*
  courses.csail.mit.edu/6.141/spring2011/pub/lectures/Lec13-Manip
  pdf

11:4

Force Closure

- Which of these objects is in “tight grip”?

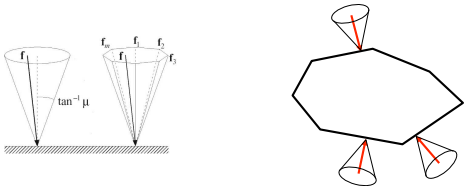


Defining “tight grip”: Assume fingers (vectors) have no friction – but can exert arbitrary normal forces. *Can we generate (or counter-act) arbitrary forces on the object?*

11:5

Force Closure – more rigorously

- Assume that each “finger” is a point that can apply forces on the object as decribed by the **friction cone**



- Each finger is described by a point p_i and a force $f_i \in F_i$ in the fingers friction cone. Together they can exert the the force an torque:

$$f^{\text{total}} = \sum_i f_i \,, \quad \tau^{\text{total}} = \sum_i f_i \times (p_i - c)$$

- **Force closure** $\iff$ we can generate (counter-act) arbitrary f^{total} and τ^{total} by choosing $f_i \in F_i$ appropriately.

  $\leftrightarrow$ Check whether the *positive linear span of the fiction cones* covers the whole space.

11:6

Form & Force Closure

- Force closure: The contacts can apply an arbitrary wrench (=force-torque) to the object.
- Form closure: The object is at an isolated point in configuration space. Note: form closure $\iff$ frictionless force closure
- Equilibrium: The contact forces can balance the object's weight and other external forces.

11:7

Traditional research into force closure

- Theorem (Mishra, Schwartz and Sharir, 1987):

  For any bounded shape that is not a surface of revolution, a force closure (or first order form closure) grasp exists.

- Guaranteed synthesis:

  1) put fingers "everywhere"

  2) while redundant finger exists delete any redundant finger

  (A finger is redundant if it can be deleted without reducing the positive linear span.)

  Theorem (Mishra, Schwartz, and Sharir, 1987):

  For any surface not a surface of revolution, [the above method] yields a grasp with at most 6 fingers in the plane, at most 12 fingers in three space.

11:8

Traditional research into force closure

- Force closure turn into a continuous optimization criterion:

  – Constrain the absolute forces each finger can apply (cut the friction cones)

  – The friction cones define a *finite* convex polygon in 6D wrench space

  $\rightarrow$ What is the inner radius of this convex wrench polygon?

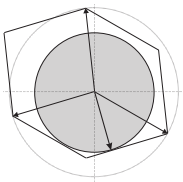


Illustration from Suárez, Roa, Cornellà (2006): *Grasp Quality Measures*

11:9

The "mitten thought experiment"

- From Oliver Brock's research website:

  "Our approach to grasping is motivated by the "mitten thought experiment". This experiment illustrates that a sensory information-deprived subject (blindfolded, wearing a thick mitten to eliminate tactile feedback) is able to grasp a large variety of objects reliably by simply closing the hand, provided that a second experimenter appropriately positioned the object relative to the hand. This thought experiments illustrates that an appropriate perceptual strategy (the experimenter) in conjunction with a simple compliance-based control strategy (the mitten hand) can lead to outstanding grasping performance."

Illustration from O. Brock's page

11:10

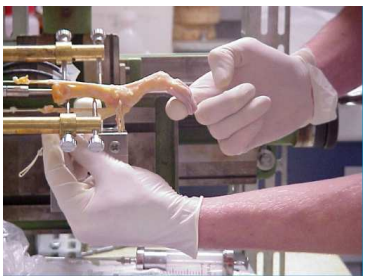
Food for thought

- Are point contact a good model?
- Is the whole idea of "arranging friction cones" the right approach?
- What about biomechanics?

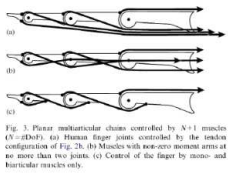
11:11

Biomechanics of the human hand

- Finger tendons:



De Bruijne et al. 1999



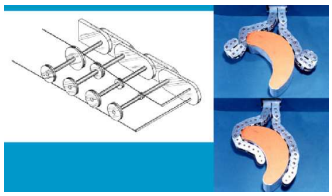
J.N.A.L. Leijnse, 2005

See Just Herder's lecture on "Biograsping"

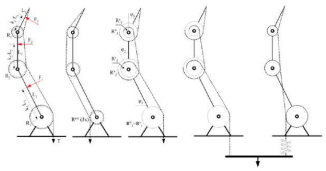
http://www.slideshare.net/DelftOpenEr/bio-inspired-design-lectu

11:12

Tendon-based hand mechanisms



Shape Gripper, Shigeo Hirose, TITECH



Jaster Schuurmans, 2004

DLR hand

11:13

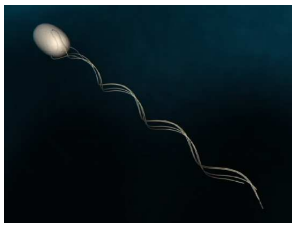
12 SKIPPED THIS TERM – Legged Locomotion (brief intro)

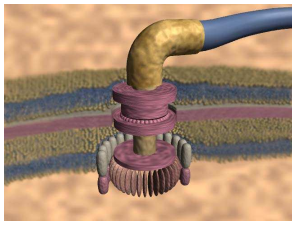
Why legs, Raibert hopper, statically stable walking, zero moment point, human walking, compass gait, passive walker

Legged Locomotion

- Why legs?

Bacterial Flagellum: (rotational "motors" in Biology?)





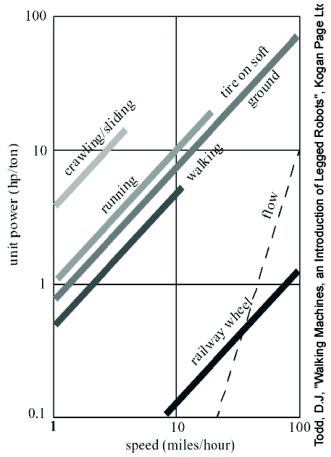
12:1

Legged Locomotion

- Why legs?
  - Human/Animal Locomotion Research
  - Potentially less weight
  - Better handling of rough terrains (climbing, isolated footholds, ladders, stairs)

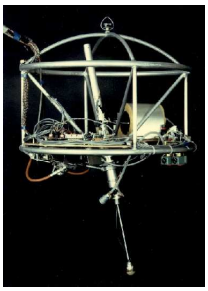
12:2

Rolling vs walking



12:3

One-legged locomotion



- Three separate controllers for:
  - hopping height
  - horizontal velocity (foot placement)
  - attitude (hip torques during stance)
- Each a simple (PD-like) controller

Raibert et al.: *Dynamically Stable Legged Locomotion.* 1985 http://dspace.mit.edu/handle/1721.1/6820

Tedrake: *Applied Optimal Control for Dynamically Stable Legged Locomotion.* PhD thesis (2004). http://groups.csail.mit.edu/robotics-center/public_papers/Tedrake04b.pdf

12:4

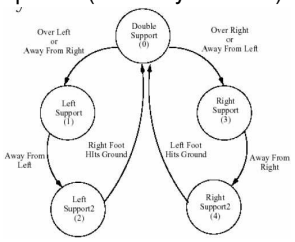
Biped locomotion

- Walking vs Running

  Walking := in all instances at least one foot is on ground

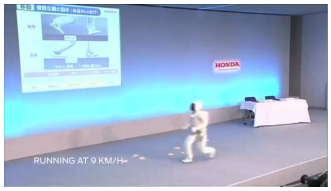
  Running := otherwise

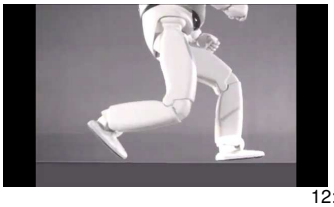
- 2 phases of Walking
  - double-support phase (in Robotics often statically stable)
  - single-support phase (statically instable)



12:5

Asimo





12:6

Statically Stable Walk

- You could rest (hold pose) at any point in time and not fall over

  $\Longleftrightarrow$ *CoG projected on ground is within support polygon*

  CoG = center of gravity of all body masses

  support polygon = hull of foot contact points

- Try yourself: Move as slow as you can but make normal length steps...

12:7

Zero Moment Point

- Vukobratovic's view on walking, leading to ZMP ideas:

  "Basic characteristics of all biped locomotion systems are:

  (i) the possibility of rotation of the overall system about one of the foot edges caused by strong disturbances, which is equivalent to the appearance of an unpowered (passive) DOF,

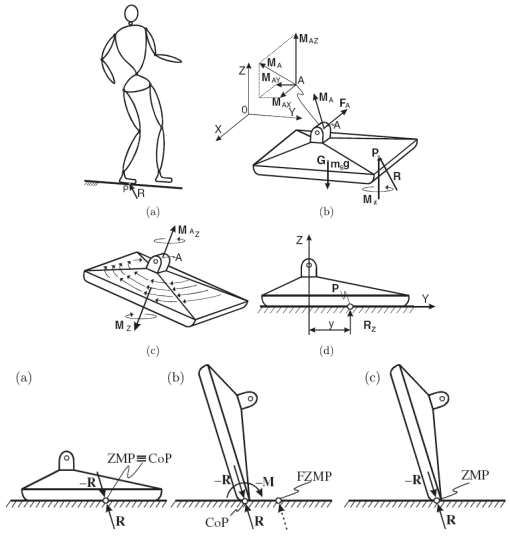
  (ii) gait repeatability (symmetry), which is related to regular gait only

  (iii) regular interchangeability of single- and double-support phases"

Vukobratovic & Borovac: *Zero-moment point—Thirty five years of its life.* International Journal of Humanoid Robotics 1, 157-173, 2004. http://www.cs.cmu.edu/~cga/legs/vukobratovic.pdf

12:8

Zero Moment Point



12:9

Zero Moment Point

- "ZMP is defined as that point on the ground at which the net moment of the inertial forces and the gravity forces has no component along the horizontal axes." (Vukobratovic & Borovac, 2004)

- f_i = force vector acting on body i (gravity plus external)

  w_i = angular vel vector of body i

  I_i = interia tensor ($\in \mathbb{R}^{3\times3}$) of body i

  $r_i = p_i - p_{\text{ZMP}}$ = relative position of body i to ZMP

- Definition: p_{ZMP} is the point on the ground for which

$$\sum_i r_i \times f_i + I_i \dot{w}_i + w_i \times I_i w_i \stackrel{!}{=} (0, 0, *)^\top$$

See also: Popovic, Goswami & Herr: *Ground Reference Points in Legged Locomotion: Definitions, Biological Trajectories and Control Implications.* International Journal of Robotics Research 24(12), 2005. http://www.cs.cmu.edu/~cga/legs/Popovic_Goswami_Herr_IJRR_Dec_2005.pdf

12:10

Zero Moment Point

- If the ZMP is outside the support polygon $\rightarrow$ foot rolls over the edge

  (presumes foot is a rigid body, and non-compliant control)

- Locomotion is *dynamically stable* if the ZMP remains within the foot-print polygons.

  ($\leftrightarrow$ the support can always apply some momentum, if necessary)

- Computing the ZMP in practise:

  – either exact robot model

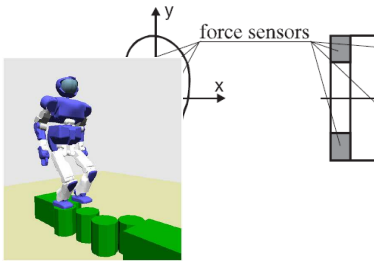
  – or foot pressure sensors

12:11

Zero Moment Point – example

- combine ZMP with 3D linear inverted pendulum model and model-predictive control





HRP-2 stair climbing

Kajita et al.: *Biped Walking Pattern Generation by using Preview Control of Zero-Moment Point.* ICRA 2003. http://eref.uqu.edu.sa/files/eref2/folder1/biped_walking_pattern_generation_by_usin_53925.pdf

12:12

ZMP Summary

- ZMP is the "rescue" for conventional methods:

  – ZMP-stability $\rightarrow$ the robot can be controlled as if foot is "glued" (virtually) to the ground!

  – The whole body behaves like a "conventional arm"

  – Can accellerate $\ddot{q}$ any DoF $\rightarrow$ conventional dynamic control

  fully actuated system

- Limitations:

  – Humans don't use ZMP stability, we allow our feet to roll (toe-off, heel-strike: ZMP at edge of support polygon)

  – Can't describe robots with point feet (walking on stilts)

12:13

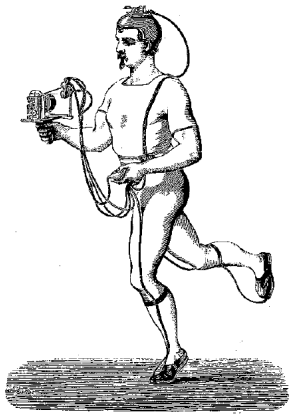
Models of human bipedal locomotion

The following illustrations are from:

McMahon: *Mechanics of Locomotion.* IJRR 3:4-28, 1984
http://www.cs.cmu.edu/~cga/legs/mcmahon1.pdf
http://www.cs.cmu.edu/~cga/legs/mcmahon2.pdf

12:14

Walking research from Marey 1874:

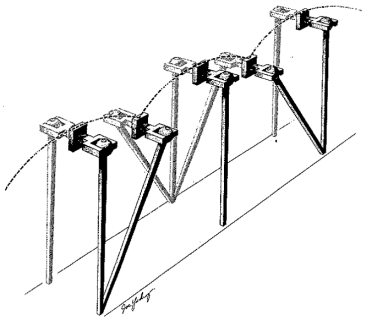


12:15

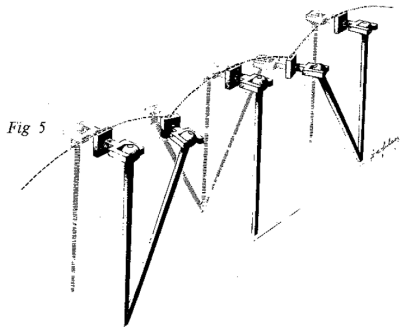
Six determinants of gait

following Saunders, Inman & Eberhart (1953)

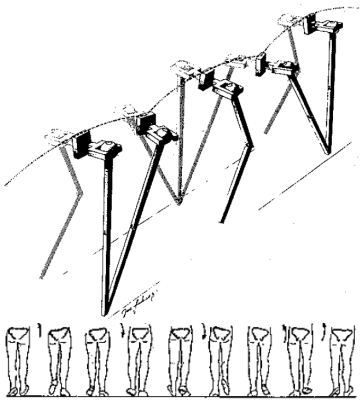
1. Compass Gait:



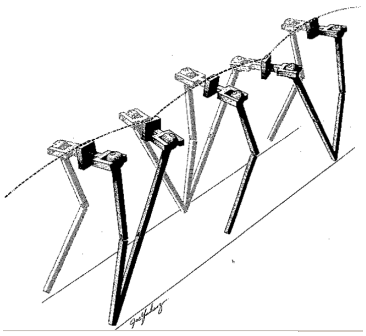
2. Pelvic Rotation:



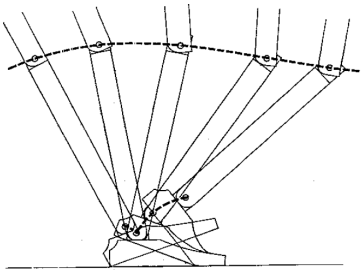
3. Pelvic Tilt:



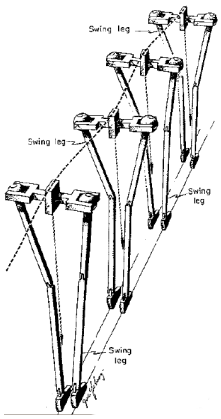
4. Stance Knee Flexion:



5. Stance Ankle Flexion:



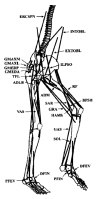
6. Pelvis Lateral Displacement:

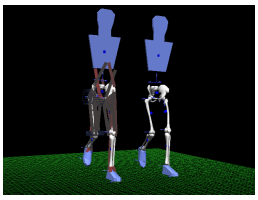


12:16

Models of human bipedal locomotion

- Human model with 23 DoFs, 54 muscles
  - Compare human walking data with model
  - Model: optimize energy-per-distance
  - Energy estimated based on metabolism and muscle heat rate models





Anderson & Pandy: *Dynamic Optimization of Human Walking.* Journal of Biomechanical Engineering 123:381-390, 2001. http://e.guigon.free.fr/rsc/article/AndersonPandy01.pdf

Anderson & Pandy: *Static and dynamic optimization solutions for gait are practically equivalent.* Journal of Biomechanics 34 (2001) 153-161. http://www.bme.utexas.edu/faculty/pandy/StaticOptWalking2001.pdf

12:17

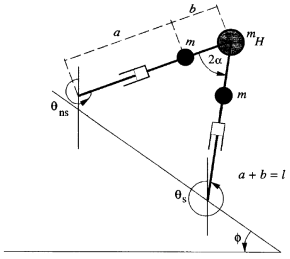
Models of human bipedal locomotion

- Suggest different principles of human motion:
  - passive dynamics (Compass Gait) $\leftrightarrow$ underactuated system
  - modulation of basic passive dynamics
  - Energy minimization

12:18

Passive dynamic walking: Compass Gait

- Basic 2D planar model of the Compass Gait:



The pose is described by $q = (\theta_s, \theta_{ns})$, the state by $(q, \dot{q})$

Goswami, Thuilot & Espiau: *A study of the passive gait of a compass-like biped robot: symmetry and chaos.* International Journal of Robotics Research 17, 1998. http://www.ambarish.com/paper/COMPASS_IJRR_Goswami.pdf

12:19

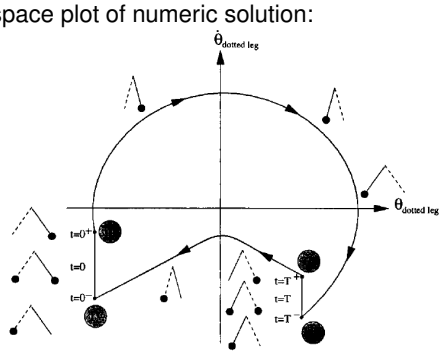
Passive dynamic walking: Compass Gait

- Swing phase has analytic equations of motions

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = 0$$

but can't be solved analytically...

- Phase space plot of numeric solution:



12:20

Passive walker examples

compass gait simulation

controlled on a circle

passive walker

- Minimally actuated: Minimal Control on rough terrain

12:21

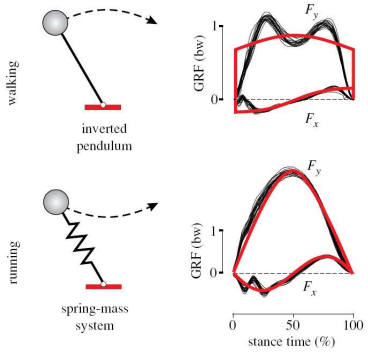
Impact Models in the Compass Gait

- Switch between two consecutive swing phases: depends on slope!
- Typical assumptions made in simulation models:
  - The contact of the swing leg with the ground results in no rebound and no slipping of the swing leg.
  - At the moment of impact, the stance leg lifts from the ground without interaction.
  - The impact is instantaneous.
  - The external forces during the impact can be represented by impulses.
  - The impulsive forces may result in an instantaneous change in the velocities, but there is no instantaneous change in the configuration.
  - The actuators cannot generate impulses and, hence, can be ignored during impact.

Westervelt, Grizzle & Koditschek: *Hybrid Zero Dynamics of Planar Biped Walkers.* IEEE Trans. on Automatic Control 48(1), 2003.
http://repository.upenn.edu/cgi/viewcontent.cgi?article=1124&cor ese_papers

12:22

Implausibility of the stiff Compass Gait leg



Geyer, Seyfarth & Blickhan: *Compliant leg behaviour explains basic dynamics of walking and running.* Proc. Roy. Soc. Lond. B, 273(1603): 2861-2867, 2006. http://www.cs.cmu.edu/~cga/legs/GeyerEA06RoySocBiolSci.pdf

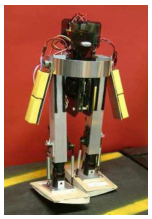
12:23

Learning to walk in 20 Minutes

- Policy Gradient method (Reinforcement Learning)

  Stationary policy parameterized as linear in features $u = \sum_i w_i \phi_i(q,$
- Problem: find parameters w_i to minimize expected costs

  cost = mimick $(q, \dot{q})$ of the passive down-hill walker at "certain point in cycle"



Learning To Walk

Tedrake, Zhang & Seung: *Stochastic policy gradient reinforcement learning on a simple 3D biped.* IROS, 2849-2854, 2004. http://groups.csail.mit.edu/robotics-center/public_papers/Tedrake04a.pdf

12:24

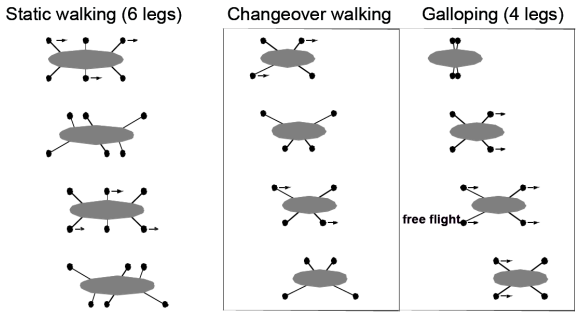
Summary

- ZMP type walking was successful (ASIMO, HRP-2, etc), but limited
- Future types of walking:
  - Exploit passive dynamics, cope with *underactuation*
  - Follow some general optimiality principle (but real-time!)
  - Learn (esp. Reinforcement Learning)
  - Compliant hardware! (controllable elasticity & damping)
- Recommended reading: Tedrake: *Underactuated Robotics: Learning, Planning, and Control for Effcient and Agile Machines.* Course Notes for MIT 6.832

www.cs.berkeley.edu/~pabbeel/cs287-fa09/readings/Tedrake-Aug09.pdf

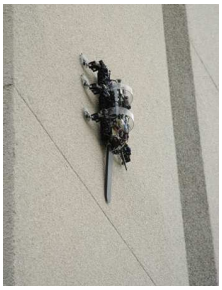
12:25

Finally, multi-legged locomotion



12:26

Finally, multi-legged locomotion



http://www.bostondynamics.com/robot_rise.html

12:27

13 Exercises

13.1 Exercise 1

13.1.1 Geometry

Read the notes on basic 3D geometry at http://ipvs.informatik.uni-stuttgart.de/mlr/marc/notes/3d-geometry.pdf at least until section 2. We will recap this briefly also in the lecture. Prepare questions for the exercises if you have any.

a) You have a book (coordinate frame B) lying on the table (world frame W). You move the book 1 unit to the right, then rotate it by 45° counter-clock-wise. Given a dot p marked on the book at position $p^B = (1, 1)$ in the book coordinate frame, what are the coordinates p^W of that dot with respect to the world frame? Given a point x with coordinates $x^W = (0, 1)$ in world frame, what are its coordinates x^B in the book frame? What is the *coordinate* transformation from world frame to book frame, and from book frame to world frame?

13.1.2 Vector derivatives

Let $x \in \mathbb{R}^n,\ y \in \mathbb{R}^d,\ f, g : \mathbb{R}^n \to \mathbb{R}^d,\ A \in \mathbb{R}^{d\times n},\ C \in \mathbb{R}^{d\times d}$.

a) What is $\frac{\partial}{\partial x} x$?

b) What is $\frac{\partial}{\partial x}[x^\top x]$?

c) What is $\frac{\partial}{\partial x}[f(x)^\top f(x)]$?

d) What is $\frac{\partial}{\partial x}[f(x)^\top C g(x)]$?

e) Let B and C be symmetric (and pos.def.). What is the minimum of $(Ax - y)^\top C(Ax - y) + x^\top Bx$?

13.1.3 Simulation software

Future exercises will require to code some examples in C/C++. Test if you can compile and run the lib that accompanies this lecture. Report on problems with installation.

On Ubuntu:

- install the packages

```
liblapack-dev freeglut3-dev libqhull-dev libf2c2-dev
libann-dev gnuplot doxygen
```

- get the code from
  http://ipvs.informatik.uni-stuttgart.de/mlr/marc/source-code/libRoboticsCourse.13.tgz

- ```
  tar xvzf libRoboticsCourse.13.tgz
  cd share/examples/Ors/ors
  make
  ./x.exe
  ```

13.2 Exercise 2

13.2.1 Task spaces and Jacobians

In the lecture we introduced the basic kinematic maps $\phi^{\text{pos}}_{\text{eff},v}(q)$ and $\phi^{\text{vec}}_{\text{eff},v}(q)$, and their Jacobians, $J^{\text{pos}}_{\text{eff},v}(q)$ and $J^{\text{vec}}_{\text{eff},v}(q)$. In the following you may assume that we know how to compute these for any q. The problem is to express other kinematic maps and their Jacobians in terms of these knowns.

a) Assume you would like to control the pointing direction of the robot's head (e.g., its eyes) to point to an external world point x^W. What task map can you define to achieve this? What is the Jacobian?

b) You would like the two hands or the robot to become parallel (e.g. for clapping). What task map can you define to achieve this? What is the Jacobian?

c) You would like to control a standard endeffector position p_{eff} to be at y^*, as usual. Can you define a 1-dimensional task map $\phi : \mathbb{R}^n \to \mathbb{R}$ to achieve this? What is its Jacobian?

13.2.2 IK in the simulator

Download the simulator code from http://userpage.fu-berlin.de/~mtoussai/source-code/libRoboticsCourse.13.tgz. (See last exercise for instructions.) The header `<src/Ors/roboticsCou` provides a very simple interface to the simulator—we will use only this header and some generic matrix functionalities.

Consider the example in `teaching/RoboticsCourse/01-kir` (rename `main.problem.cpp` to `main.cpp`). The goal is to reach the coordinates $y^* = (-0.2, -0.4, 1.1)$ with the right hand of the robot. Assume $W = \mathbf{I}$ and $\sigma = .01$.
  ```

a) The example solution generates a motion in one step using inverse kinematics $\delta q = J^\sharp \, \delta y$ with $J^\sharp = (J^\top J + \sigma^2 W)^{-1} J^\top$. Record the task error, that is, the deviation of hand position from y^* after the step. Why is it larger than expected?

b) Try to do 100 smaller steps $\delta q = \alpha J^\sharp \, \delta y$ with $\alpha = .1$ (each step starting with the outcome of the previous step). How does the task error evolve over time?

c) Generate a nice trajectory composed of $T = 100$ time steps. Interpolate the target linearly $\hat{y} \leftarrow y_0 + (t/T)(y^* - y_0)$ in each time step.

d) Generate a trajectory that moves the right hand in a circle centered at $(-0.2, -0.4, 1.1)$, aligned with the xz-plane, with radius 0.2.

13.3 Exercise 3

In the tutorial we will first discuss again the last week's 2nd exercise.

13.3.1 Motion profiles

Construct a motion profile that accelerates constantly in the first quarter of the trajectory, then moves with constant velocity, then decelerates constantly in the last quarter. Write down the equation MP : $[0, 1] \mapsto [0, 1]$.

13.3.2 Multiple task variables & Peg in a Hole

In our `libRoboticsCourse.12.tgz` in `teaching/Robotics` you find an example problem (rename `main.problem.cpp` to `main.cpp`), where the goal is to stick the green peg into the blue "hole".

The initial implementation fails: it does not find an appropriate path to insert the peg from the top; and it does not use `kinematicsVec(y,"peg")` with target `ARR(0,0,-1)` to force the peg to point down.

Implement a nice peg-in-a-hole movement. You may devide the whole movement in several sections and use task space or joint space interpolations with a smooth motion profiles within each. The motion should reach the final position with very high accuracy in finite time and without collisions.

Bonus: How can we apply joint space interpolation? How could one avoid zero velocities at the junction of sections?

13.4 Exercise 4

13.4.1 The Dijkstra algorithm

Write a proper pseudo code for the Dijkstra algorithm on a general undirected graph $G = (V, E)$. A graph is defined by the set V of nodes and the set E of edges; each edge $e \in E$ is a tuple $e = (v_1, v_2)$ of nodes.[1] Determine the computational complexity of the algorithm.

13.4.2 RRTs for path finding

In our `libRoboticsCourse.12.tgz` in `teaching/Robotics` you find an example problem (rename `main.problem.cpp` to `main.cpp`).

a) The code demonstrates an RRT exploration and displays the explored endeffector positions. What is the endeffector's exploration distribution in the limit $n \to \infty$? Specify such a distribution analytically for a planar 2 link arm.

b) First grow an RRT *backward* target configuration $q^* = (0.945499, 0.431195, -1.97155, 0.623969, 2.22355, -0.665206, -1.48356)$ that we computed in the last exercises. Stop when there exists a node close (<`stepSize`) to the $q = 0$ configuration. Read out the collision free path from the tree and display it. Why would it be more difficult to grow the tree *forward* from $q = 0$ to q^*?

c) Find a collision free path using bi-directional RRTs (that is, 2 RRTs growing together). Use q^* to root the backward tree and $q = 0$ to root the forward tree. Stop when a newly added node is close to the other tree. Read out the collision free path from the tree and display it.

d) (Bonus) Think of a method to make the found path smoother (while keeping it collision free). You're free to try anything. Implement the method and display the smooth trajectory.

[1] Ideally, use the LaTeX package `algpseudocode` to write the pseudo code (see http://en.wikibooks.org/wiki/LaTeX/Algorithms#Typesetting_using_the_algorithmicx_package)

e) (Bonus) Follow the smooth trajectory using a sinus motion profile using kinematic control.

13.5 Exercise 5

In the lecture we discussed PD force control on a 1D point mass, which leads to oscillatory behavior for high K_p and damped behavior for high K_d (slide 05:13). Slide 05:14 replaces the parameters K_p, K_d by two other, more intuitive parameters, λ and ξ: λ roughly denotes the time (or time steps) until the goal is reached, and ξ whether it is reached agressively ($\xi > 1$, which overshoots a bit) or by exponential decay ($\xi \leq 1$). Use this to solve the following exercise.

13.5.1 PD force control on a 1D mass point

a) Implement the system equation for a 1D point mass with mass $m = 3.456$. That is, implement the Euler intergration of the system dynamics that computes x_{t+1} given x_t and u_t in each iteration. (No need for the robot simulator—implement it directly.) Assume a step time of $\tau = 0.01$sec. Generate a trajectory from the start position $q_0 = 0$ that approaches the goal position $q^* = 1$ with high precision within about 1 second using PD force control. Find 3 different parameter sets for K_p and K_d to get oscillatory, overdamped and critical damped behaviors. Plot the point trajectory (e.g. using the routine `gnuplot(arr& q); MT::wait();` .)

b) Repeat for time horizon $t = 2$sec and $t = 5$sec. How should the values of K_p and K_d change when we have more time?

c) Implement a PID controller (including the integral (stationary error) term). How does the solution behave with only K_i turned on ($K_p = K_d = 0$); how with K_i and K_d non-zero?

13.5.2 A distance measure in phase space for kinodynamic RRTs

Consider the 1D point mass with mass $m = 1$ state $x = (q, \dot{q})$. The 2D space of $(q, \dot{q})$ combining position and velocity is also called phase space.

Draft an RRT algorithm for rapidly exploring the phase space of the point mass. Provide explicit descriptions of the subroutines needed in lines 4-6 of the algorithm on slide 03:58. (No need to implement it.)

Consider a current state $x_0 = (0, 1)$ (at position 0 with velocity 1). Pick *any* random phase state $x_{\text{target}} \in \mathbb{R}^2$. How would you connect x_0 with x_{target} in a way that fulfils the differential constraints of the point mass dynamics? Given this trajectory connecting x_0 with x_{target}, how would you quantify/meassure the distance? (If you defined the connecting trajectory appropriately, you should be able to give an analytic expression for this distance.) Given a set (tree) of states $x_{1:n}$ and you pick the closest to x_{target}, how would you "grow" the tree from this closest point towards x_{target}?

13.6 Exercise 6

13.6.1 Direct PD control to hold an arm steady

In our code, in `03-dynamics` you find an example (rename `main.problem.cpp` to `main.cpp`). Please change `../02-pegInAHole/pegInAHole.ors` to `pegArm.ors`. You will find an arm with three joints that is swinging freely under gravity.

a) Apply direct PD control (*without* using M and F) to each joint separately and try to find parameters K_p and K_d (potentially different for each joint) to hold the arm steady, i.e., $q^* = 0$ and $\dot{q}^* = 0$. If you are successful, try the same for the arm in `pegArm2.ors`.

b) (Bonus) Try to use a PID controller that also includes the integral error

$$u = K_p(q^* - q) + K_d(\dot{q}^* - \dot{q}) + K_i \int_{s=0}^{t} (q^* - q(s))\, ds \;.$$

13.6.2 PD acceleration control to hold an arm steady

As above, try to hold the arm steady at $q^* = 0$ and $\dot{q}^* = 0$. But now use the knowledge of M and F in each time step. For this, decide on a desired wavelength λ and damping behavior ξ and compute the respective K_p and K_d (assuming $m = 1$), the same for each joint. Use the PD equation to determine desired accelerations $\ddot{q}^*$ (slide 05:31) and use inverse dynamics to determine the necessary u.

Try this for both, `pegArm.ors` and `pegArm2.ors`.

13.6.3 The dynamic peg-in-a-hole problem

In the exercise 3 you generated nice collision-free trajectories for peg-in-a-hole using inverse kinematics.

a) Follow these reference trajectories using PD acceleration control (slide 05:31) and thereby solve the peg-in-a-hole problem with a noisy dynamic system.

b) Increase noise into the dynamic system (change to `setDynamicSimulationNoise(2.);`). Record the trajectory of the 3rd joint (`q(2)`) and plot it. Tune the PD parameters to get an oscillatory behavior.

13.7 Exercise 7

13.7.1 Particle Filtering the location of a car

Start from the code in `RoboticsCourse/05-car`.

The `CarSimulator` simulates a car exactly as described on slide 03:48 (using Euler integration with step size 1sec). At each time step a control signal $u = (v, \phi)$ moves the car a bit and Gaussian noise with standard deviation $\sigma_{\text{dynamics}} = .03$ is added to x, y and θ. Then, in each step, the car meassures the relative positions to some landmars, resulting in an observation $y_t \in \mathbb{R}^{m\times 2}$; these observations are Gaussian-noisy with standard deviation $\sigma_{\text{observation}} = .5$. In the current implementation the control signal $u_t = (.1, .2)$ is fixed (roughly driving circles).

a) Odometry (dead reckoning): First write a particle filter (with $N = 100$ particles) that ignores the observations. For this you need to use the cars system dynamics (described on 03:48) to propagate each particle, and add some noise σ_{dynamics} to each particle (step 3 on slide 07:23). Draw the particles (their x, y component) into the display. Expected is that the particle cloud becomes larger and larger.

b) Next implement the likelihood weights $w_i \propto P(y_t|x_t^i) = \mathcal{N}(y_t|y(x_t^i), \sigma) \propto e^{-\frac{1}{2}(y_t - y(x_t^i))^2/\sigma^2}$ where $y(x_t^i)$ is the (ideal) observation the car would have if it were in the particle possition x_t^i. Since $\sum_i w_i = 1$, normalize the weights after this computation.

c) Test the full particle filter including the likelihood weights (step 4) and resampling (step 2). Test using a larger ($10\sigma_{\text{observation}}$) and smaller ($\sigma_{\text{observation}}/10$) variance in the computation of the likelihood.

13.7.2 Gaussians

On slide 06:11 there is the definition of a multivariate (n-dim) Gaussian distribution. Proof the following using only the definition. (You may ignore terms independent of x.)

a) Proof that:

$$\mathcal{N}(x|a, A) = \mathcal{N}(a|x, A)$$

$$\mathcal{N}(x \,|\, a, A) = |F|\, \mathcal{N}(Fx \,|\, Fa,\ FAF^\top)$$

$$\mathcal{N}(Fx + f \,|\, a, A) = \tfrac{1}{|F|}\, \mathcal{N}(x \,|\, F^{-1}(a - f),\ F^{-1}AF^{-\top})$$

b) Prove:

$$\mathcal{N}(x \,|\, a, A)\ \mathcal{N}(x \,|\, b, B) \propto \mathcal{N}\big(x \,\big|\, (A^{-1} + B^{-1})^{-1}[A^{-1}a + B^{-1}b]\ ,\ (A^{-1} + B^{-1})^{-1}\big)$$

c) Prove:

$$\int_y \mathcal{N}(x \,|\, a + Fy, A)\ \mathcal{N}(y \,|\, b, B)\ dy = \mathcal{N}(x \,|\, a + Fb, A + FBF^\top)$$

13.8 Exercise 8

13.8.1 Kalman filter

We consider the same car example as for the last exercise, but track the car using a Kalman filter.

a) To apply a Kalman filter (slide 07:28) we need Gaussian models for $P(x_t \,|\, x_{t\text{-}1}, u_{t\text{-}1})$ as well as $P(y_t|x_t)$. We assume that the dynamics model is given as a local Gaussian of the form

$$P(x_{t+1} \,|\, x_t, u_t) = \mathcal{N}(x_{t+1}|x_t + B(x_t)u_t, \sigma_{\text{dynamics}})$$

where the matrix $B(x_t)$ gives the local linearization of the car dynamics (slide 05:27). What is $B(x_t)$ (the Jacobian of the state change w.r.t. u) for the car dynamics?

b) Concerning the observation likelihood $P(y_t|x_t)$ we assume

$$P(y_t|x_t, \theta_{1:N}) = \mathcal{N}(y_t \,|\, C(x_t)x_t + c(x_t), \sigma_{\text{observation}})$$

What is the matrix $C(x_t)$ (the Jacobian of the landmark positions w.r.t. the car state) in our example?

c) Start with the code in `RoboticsCourse/06-kalmanSLAM`.

Write a Kalman filter to track the car. You can use the routine `getObservationJacobianAtState` to access $C(x_t) = \frac{\partial y}{\partial x}$. Note that $c(x_t) = \hat{y}_t - C(x_t)x_t$, where $\hat{y}_t$ is the mean observation in state x_t (there is another routine for this).

13.8.2 Kalman SLAM

Slide 07:38 outlines how to use a high-dimensional Kalman filter to simultaneously estimate the robot position (localization) and the landmarks position (mapping).

a) Concerning $P(y_t|x_t, \theta_{1:N})$ we assume

$$P(y_t|x_t, \theta_{1:N}) = \mathcal{N}(y_t \,|\, D(x_t)\boldsymbol{\theta} + d(x_t), \sigma_{\text{observation}})$$

where $D(x_t) = \frac{\partial y}{\partial \boldsymbol{\theta}}$ is the observation Jacobian w.r.t. the unknown landmarks, and $\boldsymbol{\theta} \in \mathbb{R}^{2N}$ is the same as $\theta_{1:N}$ written as a $2N$-dim vector.

Write a pseudo-code for Kalman SLAM. (There is much freedom in how to organize the code, choices of notation and variables, etc. Try to write is as concise as possible.)

b) Try to implement Kalman SLAM, which tracks the car simultaneous to the landmarks. You should now access the routines `getMeanObservationAtStateAndLandmarks` and `getObservationJacobianAtStateAndLandmarks` to retrieve the mean observation and the necessary Jacobians given the current mean estimate $\boldsymbol{\theta}$ of the landmarks.

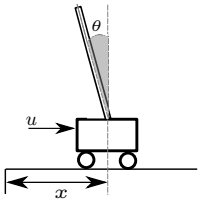
13.9 Exercise 9

13.9.1 Aggressive quadcopter maneuvers

In the article D. Mellinger, N. Michael and V. Kumar (2010): *Trajectory generation and control for precise aggressive maneuvers with quadrotors* `www.seas.upenn.edu/~dmel/mellingerISER2010.pdf` the methods used for aggressive quadcopter maneuvers are described (see also the videos at `http://www.youtube.com/watch?v=geqip_0Vjec`).

Read the essential parts of the paper to be able to explain how the quadcopter is controlled. (Neglect the part on parameter adaptation.)

13.9.2 Cart pole swing-up



The cart pole (as described, e.g., in the Sutton-Barto book) is a standard benchmark to test stable control strategies. We will assume the model known.

The state of the cart-pole is given by $\boldsymbol{x} = (x, \dot{x}, \theta, \dot{\theta})$, with $x \in \mathbb{R}$ the position of the cart, $\theta \in \mathbb{R}$ the pendulums angular deviation from the upright position and $\dot{x}, \dot{\theta}$ their respective temporal derivatives. The only control signal $u \in \mathbb{R}$ is the force applied on the cart. The analytic model of the cart pole is

$$\ddot{\theta} = \frac{g\sin(\theta) + \cos(\theta)\left[-c_1 u - c_2\dot{\theta}^2\sin(\theta)\right]}{\frac{4}{3}l - c_2\cos^2(\theta)} \tag{2}$$

$$\ddot{x} = c_1 u + c_2\left[\dot{\theta}^2\sin(\theta) - \ddot{\theta}\cos(\theta)\right] \tag{3}$$

with $g = 9.8ms^2$ the gravitational constant, $l = 1m$ the pendulum length and constants $c_1 = (M_p + M_c)^{-1}$ and $c_2 = lM_p(M_p + M_c)^{-1}$ where $M_p = M_c = 1kg$ are the pendulum and cart masses respectively.

a) Implement the system dynamics using the Euler integration with a time step of $\Delta = 1/60s$. Test the implementation by initializing the pole almost upright ($\theta = .1$) and watching the dynamics. To display the system, start from the code in `course/07-cartPole`. The state of the cart pole can be displayed using OpenGL with the `state.gl.update()` function.

b) Design a controller that stabilizes the pole in upright position *and* the cart in the zero position – any heuristic is allowed (we will use Ricatti methods later). You may want to assume that the range of θ and $\dot{\theta}$ are limited to some small interval around zero (theoretically the implication is that the local linearization of the system is a good approximation). Your controller then needs to ensure that the system does not escape such an interval. (It would be rather hard to design a general controller that can handle any initial state and return the system stably to the target state.)

Test your controller on two problems:

– When the dynamics are deterministic (as above) but the initial position is perturbed by $\theta = .1$.

– When additionally the dynamics are stochastic (add Gaussian noise with standard deviation $\sigma = .01$ to the

system state in each Euler integration step).

13.10 Exercise 10

13.10.1 Read the other exercise

On the webpage there is a 2nd exercise sheet *e10-riccati*. Please read this carefully. You don't need to do the exercise – the Octave solution was anyway in `07-cartPole/cartPole.m`. But you need to understand what's happening – we will do exactly the same for the Racer below.

13.10.2 Balance the Racer

Download the new `libRoboticsCourse.13.tgz` from the webpage. This now includes a folder `09-racer`, which simulates the racer using Runge Kutta. Note that $q = (x, \theta)$. Currently it applies a control signal $u = 0$. Design a controller $\pi : \ (q, \dot{q}) \mapsto u$ that balances the robot.

13.10.3 Use the local linearization and Algebraic Riccati equation

The code implements a routine `getDynamics` that, for the current state $(q, \dot{q})$, computes the local linear dynamics

$$M\ddot{q} + F = Bu$$

Use this to apply the Algebraic Riccati equation, as in the exercise *e10-riccati*, to compute a linear regulator using Octave. Test robustness w.r.t. system noise, that is, increasing `dynamicsNoise`.

13.11 Exercise 10

13.11.1 Local linearization and Algebraic Riccati equation

The state of the cart-pole is given by $x = (p, \dot{p}, \theta, \dot{\theta})$, with $p \in \mathbb{R}$ the position of the cart, $\theta \in \mathbb{R}$ the pendulums angular deviation from the upright position and $\dot{p}, \dot{\theta}$ their respective temporal derivatives. The only control signal $u \in \mathbb{R}$ is the force applied on the cart. The analytic model of the cart pole is

$$\ddot{\theta} = \frac{g \sin(\theta) + \cos(\theta)\left[-c_1 u - c_2 \dot{\theta}^2 \sin(\theta)\right]}{\frac{4}{3} l - c_2 \cos^2(\theta)} \tag{4}$$

$$\ddot{p} = c_1 u + c_2 \left[\dot{\theta}^2 \sin(\theta) - \ddot{\theta} \cos(\theta)\right] \tag{5}$$

with $g = 9.8ms^2$ the gravitational constant, $l = 1m$ the pendulum length and constants $c_1 = (M_p + M_c)^{-1}$ and $c_2 = l M_p (M_p + M_c)^{-1}$ where $M_p = M_c = 1kg$ are the pendulum and cart masses respectively.

a) Derive the local linearization of these dynamics around $x^* = (0, 0, 0, 0)$. The eventual dynamics should be in the form

$$\dot{x} = Ax + Bu$$

Note that

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\partial \ddot{p}}{\partial p} & \frac{\partial \ddot{p}}{\partial \dot{p}} & \frac{\partial \ddot{p}}{\partial \theta} & \frac{\partial \ddot{p}}{\partial \dot{\theta}} \\ 0 & 0 & 0 & 1 \\ \frac{\partial \ddot{\theta}}{\partial p} & \frac{\partial \ddot{\theta}}{\partial \dot{p}} & \frac{\partial \ddot{\theta}}{\partial \theta} & \frac{\partial \ddot{\theta}}{\partial \dot{\theta}} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \frac{\partial \ddot{p}}{\partial u} \\ 0 \\ \frac{\partial \ddot{\theta}}{\partial u} \end{pmatrix}$$

where all partial derivatives are taken at the point $p = \dot{p} = \theta = \dot{\theta} = 0$.

The solution (to continue with the other parts) is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{\frac{4}{3} l - c_2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ c_1 \\ 0 \\ \frac{-c_1}{\frac{4}{3} l - c_2} \end{pmatrix}$$

b) We assume a stationary infinite-horizon cost function of the form

$$J^\pi = \int_0^\infty c(x(t), u(t))\, dt$$

$$c(x, u) = x^\top Q x + u^\top R u$$

$$Q = \mathrm{diag}(c, 0, 1, 0)\ , \quad R = \mathbf{I}\ .$$

That is, we penalize position offset $c\|p\|^2$ and pole angle offset $\|\theta\|^2$. Choose $c = \varrho = 1$ to start with.

Solve the Algebraic Riccati equation

$$0 = A^\top P + P^\top A - P B R^{-1} B^\top P + Q$$

by initializing $P = Q$ and iterating using the following iteration:

$$P_{k+1} = P_k + \epsilon [A^\top P_k + P_k^\top A - P_k B R^{-1} B^\top P_k + Q]$$

Choose $\epsilon = 1/1000$ and iterate until convergence. Output the gains $K = -R^{-1}B^\top P$. (Why should this iteration converge to the solution of the ARE?)

c) Solve the same Algebraic Riccati equation by calling the `are` routine of the octave control package (or a similar method in Matlab). For Octave, install the Ubuntu packages `octave3.2`, `octave-control`, and `qtoctave`, perhaps use `pkg load control` and `help are` in octave to ensure everything is installed, use `P=are(A,B*inverse(R)*B',Q)` to solve the ARE. Output $K = -R^{-1}B^\top P$ and compare to b).

(I found the solution $K = (1.00000, 2.58375, 52.36463, 15.25927.)$

d) Implement the optimal Linear Quadratic Regulator $u^* = -R^{-1}B^\top P x$ on the cart pole. Increase ϱ (e.g. to 100) and observe how the control strategy changes.

13.12 Exercise 11

13.12.1 Kalman filtering

I collected new data from the racer's IMU, with higher frame rate. To collect this data I fixed the wheels (motors don't turn, the motors' encoder is constantly zero) and moved the racer by hand back and forth from lying on the ground to approximately balancing.

Please find the data files `01-imu.dat`, `02-imu.dat`, `01-times.dat`, `02-times.dat` on the course page. `01` and `02` refer to two different trials—start with `02`. The `imu` files contain the 4D IMU signal (the 4th entry is constantly zero: the motor encoders). The `times` files contain the real time in seconds that correspond to these readings (you will need these to determine the time interval τ between two steps).

Also, find on the webpage the two files `racer.h` and `racer.cpp`, which implement an updated model of the racer.

Implement a Kalman filter to estimate the state trajectory $q(t)$ from this data. For this,

- Initialize the state of the `Racer` model with `R.q(1)=MT_PI/2.;` (lying down)
- Assume the following simplified dynamic model:

$$A = \mathbf{I}_4 + \tau \begin{pmatrix} 0 & \mathbf{I}_2 \\ 0 & 0 \end{pmatrix}, \quad a = 0\,, \quad Q = \mathrm{diag}(10^{-6}, 10^{-6}, 1, 1) \tag{6}$$

This dynamics $\dot{x} = Ax + a$, with $x = (q, \dot{q})$ simply says that the velocities are "copied" with high precision to the next time slice, but the accelerations in the next time slice are $\mathcal{N}(0,1)$ distributed (very uncertain). Clearly this is a rough approximation – but fully sufficient for the current scenario.

- In the Kalman filter loop step as follows:
  - Retrieve the observation model (C, c, W) for the current `Racer` state using `Racer::getObservation`. Also retrieve y_{pred} here: the predicted sensor readings.
  - Use the true sensor readings (from the data file) and the dynamics and observation model for a Kalman step. Compute τ from the data files.
  - Set the state of the `Racer` model to the Kalman estimate using `R.q = ...` and `R.q_dot = ...`, and display the state using `R.gl.update()`
  - Output the Kalman's mean estimate x, the predicted y_{pred}, and the true sensor readings y_{true} in one line of a file
- Plot all curves of the output file. In particular, compare the predicted sensor outputs y_{pred} with the true ones y_{true}. Do they match?

13.12.2 Identification of the sensor model

Now that we have an estimated underlying state trajectory $q(t), \dot{q}(t)$, we can learn an even better sensor model. Usually this means to learn a mapping from the dynamic state to the sensor readings:

$$x(t) \mapsto y(t)$$

However, we exploit that we already have a sensor model implemented, with hand-tuned parameters, and want to learn a model that improves upon this (or corrects this). Therefore we learn a mapping

$$(x, y_{\text{pred}}) \mapsto y_{\text{true}}$$

where y_{pred} is the output of the implemented sensor model.

We take the output data file of the previous exercise as the basis to learn this mapping.

Use multivariate linear regression, to compute such a linear map. See `http://ipvs.informatik.uni-stuttgart`

de/mlr/marc/teaching/13-MachineLearning/02-regression.pdf if you need details. Use

$$\hat{\beta}^{\text{ridge}} = (\boldsymbol{X}^\top \boldsymbol{X} + \lambda I)^{-1} \boldsymbol{X}^\top \boldsymbol{y}$$

where $\boldsymbol{y}$ is a matrix, containing the multivariate output in each row; $\boldsymbol{X}$ is a matrix containing the multivariate input (and an appended 1) in each row, and $\lambda \approx 10^{-4}$ is some small number.

a) What is the mean squared error of $(y_{\text{pred}} - y_{\text{true}})^2$ (not using the learned mapping)

b) What is the mean squared error of $(f(x, y_{\text{pred}}) - y_{\text{true}})^2$ using the learned linear map f

c) (Bonus) Can you weave in this learned mapping into the Kalman filter of the first exercise?

13.13 Exercise 12

13.13.1 Controllability

Consider the local linearization of the cart-pole,

$$\dot{x} = Ax + Bu\,, \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g}{\frac{4}{3}l - c_2} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ c_1 \\ 0 \\ \frac{-c_1}{\frac{4}{3}l - c_2} \end{pmatrix}$$

Is the system controllable?

13.13.2 Stable control for the cart-pole

Consider a linear controller $u = w^\top x$ with 4 parameters $w \in \mathbb{R}^4$ for the cart-pole.

a) What is the closed-loop linear dynamics $\dot{x} = \hat{A}x$ of the system?

b) Test if the controller with $w = (1.00000, 2.58375, 52.36463, 15.25927)$ (computed using ARE) is asymtotically stable. What are the eigenvalues?

c) Come up with a method that finds parameters w such that the closed-loop system is "maximally stable" around $x^* = (0, 0, 0, 0)$ (e.g., asymptotically stable with fastest convergence rate).

Output the optimal parameters and test them on the cart-pole simulation you developed in exercise 9 (in `course/07-carte...`).

13.13.3 Lyapunov stability

Recall that a general controlled dynamic system can be described with the Euler-Lagrange equation as

$$\underbrace{Bu}_{\text{control}} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \underbrace{M}_{\text{inertia}} \ddot{q} + \underbrace{\dot{M}\dot{q} - \frac{\partial T}{\partial q}}_{\text{Coriolis}} + \underbrace{\frac{\partial U}{\partial q}}_{\text{gravity}}$$

Consider a dynamic *without* Coriolis forces and constant M (independent of a and $\dot{q}$).

Can you give sufficient conditions on $U(q)$ (potential energy) and $u(q)$ (control policy) such that energy is a Lyapunov function?

13.14 Exercise 13

On Wed. 29th we meet as usual for the exercise.

On Tue. 28th, 14:00 my office, interested students are invited to try whatever they like on the racer hardware, play around, etc.

13.14.1 Policy Search for the Racer

We consider again the simulation of the racer as given in `09-racer` in your code repo.

In this exercise the goal is to find a policy

$$\pi:\ \phi(y) \mapsto u = \phi(u)^\top w$$

that maps some features of the (history of the) direct sensor signals y to the control policy.

Use black-box optimization to find parameters w that robustly balance the racer.

Some notes:

- Features: In the lecture I suggested that a range of interesting features is:

  $$\left(y_t, \dot{y}_t, \langle y \rangle_{0.5}, \langle \dot{y} \rangle_{0.5}, \langle y \rangle_{0.9}, \langle \dot{y} \rangle_{0.9}, u_t, u_{t-1}\right)$$

  However, I noticed that a balancing policy can also be found for the direct sensor signals only, that is:

  $$\phi(y) = (1, y) \in \mathbb{R}^5$$

  (the augmentation by 1 is definitly necessary).

- Cost function: Realistically, the running costs c_t would have to be defined on the sensor signals only. (On the real robot we don't know the real state – if the robot is to reward itself it needs to rely on sensor signals only.) I tried

  ```
  costs += .1*MT::sqr(y(3)) + 1.*MT::sqr(y(2));
  ```

  which combines the wheel encoder `y(3)` and the gyroscope reading `y(2)`. That worked mediocre. For a start, cheat and directly use the state of the simulator to compute costs:

  ```
  costs += 1.*MT::sqr(R.q(0)) + 10.*MT::sqr(R.q(1));
  ```

- Episodes and duration costs: To compute the cost for a given w you need to simulate the racer for a couple of time steps. For the optimization it is really bad if an episode is so long that it includes a complete failure and wrapping around of the inverted pendulum. Therefore, abort an episode if `fabs(R.q(1))` too large and penalize an abortion with an extra cost, e.g., proportional to $T-t$. Try different episode horizons T up to 500; maybe increase this horizon stage-wise.

- Optimizer: You are free to use any optimizer you like. On the webpage you find a reference implementation of CMA by Niko Hansen (with wrapper using our `arr`), which you may use. In that case, add `cmaes.o` and `search_CMA.o` in the `Makefile`. The typical loop for optimization is

  ```
  SearchCMA cma;
  cma.init(5, -1, -1, -0.1, 0.1);
  arr samples, values;

  uint T=500;
  for(uint t=0;t<1000;t++){
    cma.step(samples, values);
    for(uint i=0;i<samples.d0;i++) values(i) = evaluateConfiguration(samples[i], T);
    uint i=values.minIndex();
    cout <<t <<' ' <<values(i) <<' ' <<samples[i] <<endl;
  }
  ```

13.15 Exercise 12 – SKIPPED THIS TERM

13.15.1 Balancing a pin in a hole with a torque-controlled arm

Discplaimer: To actually solve this problem (in the simulator) is hard work and beyond the scope of this exercise. Instead, describe precisely how you would solve the problem.

Consider a free swining pole (with a 2 DoF universal joint) at the tip of a 6 DoF arm. The arm joints can be articulated using torques (no direct position/velocity control). The pole itself cannot be articulated, only balanced using the arm. Above the arm (in the "ceiling") is a hole. Initially the pole is *hanging down*; the problem is to swing up the pole and then balance it to insert it in the ceiling's hole without collision. We assume to have full access to the system dynamics in the form $\ddot{q}M(q)+F(\dot{q},q)=u$, where $q=(q^{\text{arm}},q^{\text{pole}})$ contains the 6 arm DoFs $q^{\text{arm}}\in\mathbb{R}^6$ and the 2 pole DoFs $q^{\text{pole}}\in\mathbb{R}^2$. Since we cannot actuate the pole directly, $u^{\text{pole}}\equiv 0$.

If you would like to see the system, download the newest version of the http://userpage.fu-berlin.de/~mtoussai/source-code/libRoboticsCourse.11.1.tgz and have a look at `course/08-pin_ba`

For all questions: Write down the precise controller equations, or cost function, or constraints, or search/optimization strategy, or whatever is necessary to precisely define your solution.

a) Hold steady: Let $q^{\text{arm}}\in\mathbb{R}^6$ be the 6D arm joints and q_0^{arm} their initial position. How can you hold the arm steady around q_0 (counter-balance gravity and potential perturbations while the pole is hanging down)? Which methods from the lecture do you use?

b) Swing up (1): How can you find a rough plan of how to swing up the pole? Which methods from the lecture do you use? (What does *not* work: using the local linearization of the pole around standing-up, because this linearization is totally wrong if the pole is hanging down.)

c) Swing up (2): Given a rough plan of the swing up, is there a way to follow this rough plan directly? Alternatively, how would you refine the plan to become an "optimal" swing up? Which methods from the lecture do you use?

d) Balance steadily: When you managed to swing up the pole, how could you balance it straight-up, even under perturbations (Gaussian noise in the pole dynamics)?

e) Pin-in-a-hole (1): How can you find a rough plan of how to insert the balanced pole into the ceiling's hole?

f) Pin-in-a-hole (2): Given that rough plan, how can you follow this plan while balancing the pole in a stable way (robust under perturbations)? Alternatively, how can you refine the plan to become more optimal?

g) Is there a way to solve the whole problem in a holistic way, to find an "optimal" solution of the whole procedure? (The swing-up already targets towards the ceiling's hole etc.) How could this be done realistically?

h) Partial observability: Do you think the whole problem could be solved if q^{pole} is non-observable (the pole is invisible, but the q^{arm} are observed at every time step)? How?

14 Topic list

This list summarizes the lecture's content and is intended as a guide for preparation for the exam. (Going through all exercises is equally important!) References to the lectures slides are given in the format (lecture:slide).

14.1 Kinematics

- 3D geometry
  - Definition of an object pose, frame, transformations (2:5,6)
  - Homogeneous transformations (4×4 matrix) (2:8,9)
  - Composition of transformations, notation $x^W = T_{W\to A}\, T_{A\to B}\, T_{B\to C}\, x^C$ (2:10)
- Fwd kinematics & Jacobian
  - Fwd kinematics as composition of transformations (2:12)
  - Transformations of a rotational joint (2:13)
  - Kinematic maps $\phi^{\text{pos}}:\ q \mapsto y$ and ϕ^{vec} (2:15)
  - Definition of a Jacobian (2:17)
  - Derivation of the position and vector Jacobians J^{pos}, J^{vec} (2:18,19)
- Inverse kinematics (IK)
  - Optimality criterion for IK (2:23,24)
  - Using the local linearization to find the optimum (2:26)
  - Pseudo code of Inverse Kinematics control (2:28)
  - Definition & example for a singularity (3:24)
- Motion profiles & Interpolation
  - Motion profiles (esp. sine profile) (2:32)
  - Joint space vs. task space interpolation of a motion (2:34,35) [e.g. using a motion profile in one or the other space]
- Multiple Tasks
  - How to incorporate multiple tasks (2:40,41)
  - What are interesting task variables; know at leads about pos, vec, align, and limits (2:46-52)
- Further
  - Definition of a singularity (2:58) Be able to give example
  - Be able to explain the consequences of the local linearization in IK (big steps → errors)

14.2 Path planning

- Basics
  - Path finding vs. trajectroy optimization vs. feedback control (3:5,6)
  - Roughly: BUG algorithms (3:8-12)
  - Potential functions, and that they're nothing but IK with special task variables (3:17)
  - Dijkstra Algorithm (3:26,18-25)
- Probabilistic Road Maps (PRMs)
  - Definition (3:29,30) & Generation (4:31)
  - Importance of local planner (3:32)
  - Roughly: knowing about probabilistic completeness (3:34)
  - Roughly: alternative sampling strategies (3:35)
- Rapidly Exploring Random Trees (RRTs)
  - Algorithm (3:39)
  - Goal-directed (3:40) & bi-direction (3:42) extensions
- Non-holonomic Systems
  - Definition of non-holonomicity (3:36) Be able to give example
  - Path finding: control-based sampling (3:52)
  - RRT extension for control-based exploration (3:57)
  - Roughly: Intricacies with metrics for non-holonomic systems (3:58-60)

14.3 Dynamics

- 1D point mass & PID control
  - General form of a dynamic system (5:3)
  - Dynamics of a 1D point mass (5:6)
  - Position, derivative and integral feedback to control it to a desired state (5:7,10,15)
  - Solution to the *closed-loop* PD system equations (5:11)
  - Qualitative behaviors: oscillatory-damped, overdamped, critically damped (5:13,14)
- Euler-Lagrange equation
  - Definition (5:20)
  - Roughly: application to robotic systems (5:21) Understand at least $T = \frac{1}{2}\dot{q}^\top M \dot{q}$
  - Be able to apply on minimalistic system (5:22)

- Robot dynamics & joint/operational space control

  – General form of the dynamics equation (5:28)

  – Joint space control: given desired $\ddot{q}^*$, choose $u^* = M(q)\ \ddot{q}^* + F(q, \dot{q})$ (5:30)

  – Operational space control: given desired $\ddot{y}^*$, choose $u^* = T^\sharp(\ddot{y}^* - \dot{J}\dot{q} + TF)$ (5:31, 32, 34)

  – Joint space approach to follow a reference trajectory $q^{\text{ref}}_{0:T}$ (5:30)

14.4 Mobile Robotics

- Probability Basics

  – Definitions of random variable, probability distribution, joint, marginal, conditional distribution, independence (6:4-7)

  – Bayes' Theorem (6:7,8)

  – Continuous distributions, Gaussian, particles (6:9-13)

- State Estimation

  – Formalization of the state estimation problem (7:13)

  – The Bayes Filter as the general analytic solution (7:14,15)

  – Gaussians and particles to approximate the Bayes filter and make it computationally feasible:

  – Details of a Particle Filter (7:22)

  – Kalman filter (esp. assumptions made, not eq. or derivation) (7:26)

  – Extended KF (assumptions made) (7:28)

  – Odometry (dead reckoning) as "Bayes filter without observations" (7:17,18)

  – What is smoothing (7:29)

- SLAM

  – In what sense SLAM is a "chicken or egg problem" (7:34)

  – Joint inference over x and m: Extended Kalman SLAM (7:37)

  – Particle-based SLAM (map belief for each particle) (7:38-41)

  – Roughly: graph-based SLAM & loop closing (7:44-46)

14.5 Control Theory

- Generally

  – What we mean by "closed-loop system" (8:2)

  – Topics in Control Theory (8:3)

- Optimal Control

  – Definition of the (continuous time) optimal control problem (8:9)

  – Concept & definition of the value function (8:10)

  – HJB equation (8:10)

  – Infinite horizon $\rightarrow$ stationary solution (8:11)

  – Awareness that optimal control is not the only approach; it shifts the problem of designing a controller to designing a cost function.

- Linear-Quadratic Optimal Control

  – Definition of problem (esp. assumptions made) (8:14)

  – Be able to express system dynamics in (locally linearized) standard form $\dot{x} = Ax + Bu$ (8:19)

  – Fact that the value function is quadratic $V(x,t) = x^\top P(t) x$ (8:16)

  – Riccati differential equation (8:16)

  – How P gives the optimal Linear-Quadratic Regulator (8:17)

  – Algebraic (infinite horizon) Riccati equation (8:18)

- Controllability

  – Definition and understanding/interpretation of the controllability matrix C (8:27)

  – Definition of controllability (8:26,27)

  – Be able to apply on simple examples (8:28)

- Stability

  – Definitions of stability (8:34)

  – Eigenvalue analysis for linear systems (8:35)

  – Optimize controllers for negativity of eigenvalues

  – Definition of a Lyapunov function (8:39)

  – Lyanunov's theorem: $\exists$ Lyapunov fct. $\rightarrow$ stability (8:39)

  – Energy and value function as candidates for a Lyapunov fct. (8:41,42)