POLYNOMIAL EXPANSION AND SUBLINEAR SEPARATORS

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Abstract. Let $\mathcal{C}$ be a class of graphs that is closed under taking subgraphs. We prove that if for some fixed $0 < \delta \leq 1$, every n-vertex graph of $\mathcal{C}$ has a balanced separator of order $O(n^{1-\delta})$, then any depth-r minor (i.e. minor obtained by contracting disjoint subgraphs of radius at most r) of a graph in $\mathcal{C}$ has average degree $O\big((r \operatorname{polylog} r)^{1/\delta}\big)$. This confirms a conjecture of Dvořák and Norin.

1. Introduction

For an integer $r \geq 0$, a *depth-r minor* of a graph G is a subgraph of a graph that can be obtained from G by contracting pairwise vertex-disjoint subgraphs of radius at most r. Let $d(G)$ denote the average degree of a graph $G = (V, E)$, i.e. $d(G) = 2|E|/|V|$. For some function f, we say that a class $\mathcal{C}$ of graphs has *expansion bounded by f* if for any graph $G \in \mathcal{C}$ and any integer r, any depth-r minor of G has average degree at most $f(r)$. We say that a class has *bounded expansion* if it has expansion bounded by some function f, and *polynomial expansion* if f can be taken to be a polynomial.

Classes of bounded expansion play a central role in the study of sparse graphs [7]. From an algorithmic point of view, a very useful property of theses classes is that when their expansion is not too large (say subexponential), graphs in the class have sublinear separators. A *separator* in a graph $G = (V, E)$ is a pair of subsets (A, B) of vertices of G such that $A \cup B = V$ and no edge of G has one endpoint in $A \setminus B$ and the other in $B \setminus A$. The separator (A, B) is said to be *balanced* if both $|A \setminus B|$ and $|B \setminus A|$ contain at most $\frac{2}{3}|V|$ vertices. The *order* of the separator (A, B) is $|A \cap B|$.

A class $\mathcal{C}$ of graphs is *monotone* if for any graph $G \in \mathcal{C}$, any subgraph of G is in $\mathcal{C}$. Dvořák and Norin [5] observed that the following can be deduced from a result of Plotkin, Rao, and Smith [8].

Theorem 1 ([5])**.** *Let $\mathcal{C}$ be a monotone class of graphs with expansion bounded by $r \mapsto c(r+1)^{1/4\delta - 1}$, for some constant $c > 0$ and $0 < \delta \leq 1$. Then there is a constant C such that every n-vertex graph of $\mathcal{C}$ has a balanced separator of order $Cn^{1-\delta}$.*

Dvořák and Norin [5] also proved the following partial converse.

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Theorem 2 ([5]). *Let $\mathcal{C}$ be a monotone class of graphs such that for some fixed constants $C > 0$ and $0 < \delta \leq 1$, every n-vertex graph of $\mathcal{C}$ has a balanced separator of order $Cn^{1-\delta}$. Then the expansion of $\mathcal{C}$ is bounded by some function $f(r) = O(r^{5/\delta^2})$.*

They conjectured that the exponent $5/\delta^2$ of the polynomial expansion in Theorem 2 could be improved to match (asymptotically) that of Theorem 1.

Conjecture 3 ([5]). *There exists a real $c > 0$ such that the following holds. Let $\mathcal{C}$ be a monotone class of graphs such that for some fixed constants $C > 0$ and $0 < \delta \leq 1$, every n-vertex graph of $\mathcal{C}$ has a balanced separator of order $Cn^{1-\delta}$. Then the expansion of $\mathcal{C}$ is bounded by some function $f(r) = O(r^{c/\delta})$.*

In this short note, we prove this conjecture.

Theorem 4. *For any $C > 0$ and $0 < \delta \leq 1$, if a monotone class $\mathcal{C}$ has the property that every n-vertex graph in $\mathcal{C}$ has a balanced separator of order at most $Cn^{1-\delta}$, then $\mathcal{C}$ has expansion bounded by the function $f : r \mapsto c_1 \cdot (r+1)^{1/\delta}(\frac{1}{\delta}\log(r+3))^{c_2/\delta}$, for some constants c_1 and c_2 depending only on $\mathcal{C}$.*

In particular Conjecture 3 holds for any real number $c > 1$. The proof of Theorem 4 is given in the next section, and we conclude with some open problems in Section 3.

2. Proof of Theorem 4

We need the following results. The first is a classical connection between balanced separators and tree-width (see [5]).

Lemma 5. *Any graph G has a balanced separator of order at most $\mathrm{tw}(G) + 1$.*

Dvořák and Norin [4] proved that the following partial converse holds.

Theorem 6 ([4]). *If every subgraph of G has a balanced separator of order at most k, then G has tree-width at most $105k$.*

Note that in our proof of Theorem 4 we could also use the weaker (and easier) result of [1] that under the same hypothesis, G has tree-width at most $1 + k\log|V(G)|$, but the computation is somewhat less cumbersome if we use Theorem 6 instead.

For a set S of vertices in a graph G, we let $N(S)$ denote the set of vertices not in S with at least one neighbor in S. We will use the following result of Shapira and Sudakov [9].

Theorem 7 ([9]). *Any graph G contains a subgraph H of average degree $d(H) \geq \frac{255}{256}d(G)$ such that for any set S of at most $n/2$ vertices of H (where $n = |V(H)|$), $|N(S)| \geq \frac{1}{2^8 \log n(\log\log n)^2}|S|$.*

In fact, we will only need a much weaker version, where the vertex-expansion is of order $\Omega\left(\frac{1}{\mathrm{polylog}\, n}\right)$ instead of $\Omega\left(\frac{1}{\log n(\log\log n)^2}\right)$.

Finally, we need a result of Chekuri and Chuzhoy [2] on bounded-degree subgraphs of large tree-width in a graph of large tree-width.

Theorem 8 ([2])**.** *There are constants α, β such that for any integer $k \geq 2$, any graph G of tree-width at least k contains a subgraph H of tree-width at least $\alpha k/(\log k)^{\beta}$ and maximum degree 3.*

Let us remark that instead of Theorem 8, our proof of Theorem 4 could rely on an earlier result of Chekuri and Chuzhoy [3] which, under the same assumptions, merely guarantees the existence of a subgraph of G of treewidth $\Omega(k/(\log k)^6)$ and maximum degree $O((\log k)^3)$.

We are now ready to prove our main result.

Proof of Theorem 4. Let G be a graph of $\mathcal{C}$ and let F be a depth-r minor of G. Our goal is to prove that $d(F) \leq c_1 \cdot (r+1)^{1/\delta}(\frac{1}{\delta}\log(r+3))^{c_2/\delta}$, for some constants c_1 and c_2 depending only on C. Note that for any $r \geq 0$ and $0 < \delta \leq 1$,

$$c_1 \cdot (r+1)^{1/\delta}(\tfrac{1}{\delta}\log(r+3))^{c_2/\delta} \geq \max\left\{c_1(\log 3)^{c_2}, c_1\exp(\tfrac{c_2}{\delta}\log\tfrac{\log 3}{\delta})\right\},$$

so we can assume without loss of generality that

$$d(F) \geq \max\left\{10^8, \exp\left(4 \cdot \tfrac{\beta+3}{\delta}\log(2 \cdot \tfrac{\beta+3}{\delta})\right)\right\}$$

by choosing appropriate values of c_1, c_2. By Theorem 7, F has a subgraph H of average degree $d(H) \geq \frac{255}{256}d(F)$ such that for any set S of at most $|V(H)|/2$ vertices of H,

$$|N(S)| \geq \tfrac{1}{2^8\log|V(H)|(\log\log|V(H)|)^2}|S| \geq \tfrac{1}{2^8(\log|V(H)|)^3}|S|.$$

It follows from Lemma 5 that H contains a balanced separator (A, B) with $|A \cap B| \leq \mathrm{tw}(H)+1$. As $A\setminus B$ and $B\setminus A$ are disjoint, one of them contains at most half of the vertices. We may assume without loss of generality that $|A \setminus B| \leq |V(H)|/2$. As $N(A\setminus B) \subseteq A\cap B$, we get

$$|A \cap B| \geq \tfrac{1}{2^8(\log|V(H)|)^3}|A \setminus B|.$$

Since (A, B) is balanced, $|A \setminus B| + |A \cap B| \geq \frac{1}{3}|V(H)|$ and so

$$\tfrac{1}{3}|V(H)| \leq |A \cap B|(1 + 2^8(\log|V(H)|)^3).$$

Given that $|A \cap B| \leq \mathrm{tw}(H) + 1$, we deduce

$$\mathrm{tw}(H) \geq \tfrac{|V(H)|}{3\cdot 2^8(\log|V(H)|)^3+3} - 1 \geq \tfrac{|V(H)|}{2^{10}(\log|V(H)|)^3},$$

using that $|V(H)| \geq d(H) \geq \frac{255}{256} \cdot 10^8$.

By Theorem 8, H has a subgraph H' of maximum degree 3 such that

$$\mathrm{tw}(H') \geq \tfrac{\alpha\ \mathrm{tw}(H)}{(\log\mathrm{tw}(H))^{\beta}} \geq \tfrac{\alpha\,|V(H)|}{2^{10}(\log|V(H)|)^{\beta+3}},$$

since $\mathrm{tw}(H) \leq |V(H)|$. Note that H' is a subgraph of H (and F) and therefore also a depth-r minor of G. In G, H' corresponds to a subgraph G' (before contraction of the subgraphs of radius r) with $|V(G')| \leq (3r+1)|V(H')| \leq (3r+1)|V(H)|$. Indeed, since H' has maximum degree 3, each subgraph of radius at most r in G' whose contraction corresponds to a vertex of H' contains at most $3r+1$ vertices. Since H' is a minor of G', we have

$$\mathrm{tw}(G') \geq \mathrm{tw}(H') \geq \tfrac{\alpha\,|V(H)|}{2^{10}(\log|V(H)|)^{\beta+3}}.$$

Since $\mathcal{C}$ is monotone, every subgraph of G' is in $\mathcal{C}$ and thus has a balanced separator of order at most $C|V(G')|^{1-\delta}$. Hence, by Theorem 6,

$$\mathrm{tw}(G') \leq 105C|V(G')|^{1-\delta} \leq 2^7 C|V(G')|^{1-\delta}.$$

We just obtained lower and upper bounds on $\mathrm{tw}(G')$. Putting them together, we obtain:

$$\begin{aligned}
\tfrac{\alpha\,|V(H)|}{2^{10}(\log|V(H)|)^{\beta+3}} &\leq 2^7 C\,|V(G')|^{1-\delta}\\
&\leq 2^7 C\left((3r+1)|V(H)|\right)^{1-\delta}, \text{ and thus}\\
\tfrac{|V(H)|^{\delta}}{(\log|V(H)|)^{\beta+3}} &\leq \tfrac{2^{17}C}{\alpha}(3r+1)^{1-\delta}\\
&\leq \tfrac{2^{17}C}{\alpha}(3r+1), \text{ and}\\
|V(H)| &\leq \left(\tfrac{2^{17}C}{\alpha}(3r+1)(\log|V(H)|)^{\beta+3}\right)^{1/\delta}.
\end{aligned}$$

It follows that

$$\log|V(H)| \leq \tfrac{1}{\delta}\log\left(\tfrac{2^{17}C}{\alpha}(3r+1)\right) + \tfrac{\beta+3}{\delta}\log\log|V(H)|.$$

Since the function $n \mapsto \frac{\log n}{\log\log n}$ is increasing for $n \geq 16$, a direct consequence of our initial assumption that $|V(H)| \geq \exp\left(4\cdot\frac{\beta+3}{\delta}\log(2\cdot\frac{\beta+3}{\delta})\right)$ is that

$$\begin{aligned}
\tfrac{\log|V(H)|}{\log\log|V(H)|} &\geq 2\cdot\tfrac{\beta+3}{\delta}, \text{ and thus}\\
\log|V(H)| &\leq \tfrac{2}{\delta}\log\left(\tfrac{2^{17}C}{\alpha}(3r+1)\right).
\end{aligned}$$

We conclude that

$$|V(H)| \leq \left(\tfrac{2^{17}C}{\alpha}(3r+1)\left(\tfrac{2}{\delta}\log\left(\tfrac{2^{17}C}{\alpha}(3r+1)\right)\right)^{\beta+3}\right)^{1/\delta} \leq \tfrac{255}{256}c_1(r+1)^{1/\delta}(\tfrac{1}{\delta}\log(r+3))^{c_2/\delta},$$

for some constants c_1, c_2 depending only on C and the constants α, β of Theorem 8. Recall that $d(F) \leq \frac{256}{255}d(H)$. Since $d(H) \leq |V(H)|$, we obtain $d(F) \leq c_1\cdot(r+1)^{1/\delta}(\frac{1}{\delta}\log(r+3))^{c_2/\delta}$, as desired. This concludes the proof of Theorem 4. □

3. Open problems

A natural problem is to determine the infimum real $c > 0$, such that if a monotone class $\mathcal{C}$ has the property that every n-vertex graph in $\mathcal{C}$ has a balanced separator of order $O(n^{1-\delta})$, then $\mathcal{C}$ has expansion bounded by some function $r \mapsto O(r^{c/\delta})$. Theorem 4 implies that $c \leq 1$. On the other hand, it directly follows from Theorem 1 that $c \leq \frac{1}{4+\epsilon}$ would imply that if any n-vertex graph in $\mathcal{C}$ has a balanced separator of order $O(n^{1-\delta})$, then any n-vertex graph in $\mathcal{C}$ has a balanced separator of order $O(n^{1-(1+\epsilon/4)\delta})$. Therefore, Theorem 1 implies that $c \geq \frac{1}{4}$ (moreover, the proof of Theorem 1 in [5] can be slightly optimized to show that $c \geq \frac{1}{2}$). A good candidate to prove a better lower bound for c would be the family of all finite subgraphs of the infinite d-dimensional grid. The n-vertex graphs in this class have balanced separators of order $O(n^{1-1/d})$ (see [6]), and it might be the case that they have expansion $\Omega(r^{cd})$ for some $c > \frac{1}{2}$.

One way to measure the sparsity of a class of graphs is via its expansion (as defined in Section 1). Another way (which turns out to be equivalent) is via its *generalized coloring parameters*. Given a linear order L on the vertices of a graph G, and an integer r, we say that a vertex v of G is *strongly r-reachable* from a vertex u (with respect to L) if $v \leq_L u$, and there is a path P of length at most r between u and v, such that $u <_L w$ for any internal vertex w of P. If we only require that v is the minimum of the vertices of P (with respect to L), we say that v is *weakly r-reachable* from u. The *strong r-coloring number* $\mathrm{col}_r(G)$ of G is the minimum integer k such that there is a linear order L on the vertices of G such that for any vertex u of G, at most k vertices are strongly r-reachable from u (with respect to L). By replacing *strongly* by *weakly* in the previous definition, we obtain the *weak r-coloring number* $\mathrm{wcol}_r(G)$ of G. Note that for any graph G and any integer r, $\mathrm{col}_r(G) \leq \mathrm{wcol}_r(G)$. For more on these parameters and their connections with the expansion of graph classes, the reader is referred to [7].

As we have seen before, it follows from [5] that a monotone class of graphs has polynomial expansion if and only if, for some fixed $0 < \delta \leq 1$, each n-vertex graph in the class has a balanced separator of order $O(n^{1-\delta})$. Joret and Wood asked whether this is also equivalent to having weak and strong r-coloring numbers bounded by a polynomial function of r.

Problem 9 (Joret and Wood, 2017)**.** *Assume that $\mathcal{C}$ is a monotone class of graphs. Are the following statements equivalent?*

(1) $\mathcal{C}$ has polynomial expansion.
(2) There exists a constant c, such that for every r, every graph in $\mathcal{C}$ has strong r-coloring number at most $O(r^c)$.
(3) There exists a constant c, such that for every r, every graph in $\mathcal{C}$ has weak r-coloring number at most $O(r^c)$.

Note that clearly (3) implies (2). It was known that (3) implies (1) (this is a consequence of Lemma 7.11 in [7]), and Norin recently made the following observation, which shows that (2) implies (1).

Observation 10 (Norin, 2017)**.** *Every depth-r minor of a graph G has average degree at most $2\,\mathrm{col}_{4r}(G)$.*

Proof. Let L be a linear order on the vertices of G, such that for any vertex v of G, at most $\mathrm{col}_r(G)$ vertices are strongly r-reachable from v (with respect to L). Let H be a depth-r minor of a graph G. For any vertex u of H, let S_u be a subgraph of G of radius at most r, such that the S_u's are vertex-disjoint and for any edge uv of H, there is an edge in G between a vertex of S_u and a vertex of S_v. It is enough to prove that there is a linear order L' on the vertices of H such that any vertex u of H, at most $\mathrm{col}_{4r}(G)$ vertices of H are strongly 1-reachable from u.

We construct L' from L as follows: for u, v in H, we set $u <_{L'} v$ if and only if, with respect to L, the smallest vertex of S_u precedes the smallest vertex of S_v. This clearly defines a linear order on the vertices of H. Consider a vertex u of H and let x be the smallest vertex of S_u (with respect to L). Let v be a neighbor of u in H with $v <_{L'} u$ (i.e. v is strongly 1-reachable from u in H). Let $t \in S_u$ and $z \in S_v$ be such that tz is an edge of G. Observe that there is a path P_u from x to t in S_u (and x is the smallest vertex in this path with respect to L), and a path P_v from z to y in S_v. Let w be the first vertex of P_v such that $w <_L x$ (note that possibly $w = z$). The concatenation of P_u, zt, and the subpath of P_v between z and w has length at most $4r$ and thus shows that w is strongly $4r$-reachable from x in G. Hence, at most $\mathrm{col}_{4r}(G)$ vertices of H are strongly 1-reachable from u in H with respect to L', as desired. □

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