Graph decompositions and well-quasi-ordering

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Well-quasi-ordering

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- every decreasing sequence is finite;
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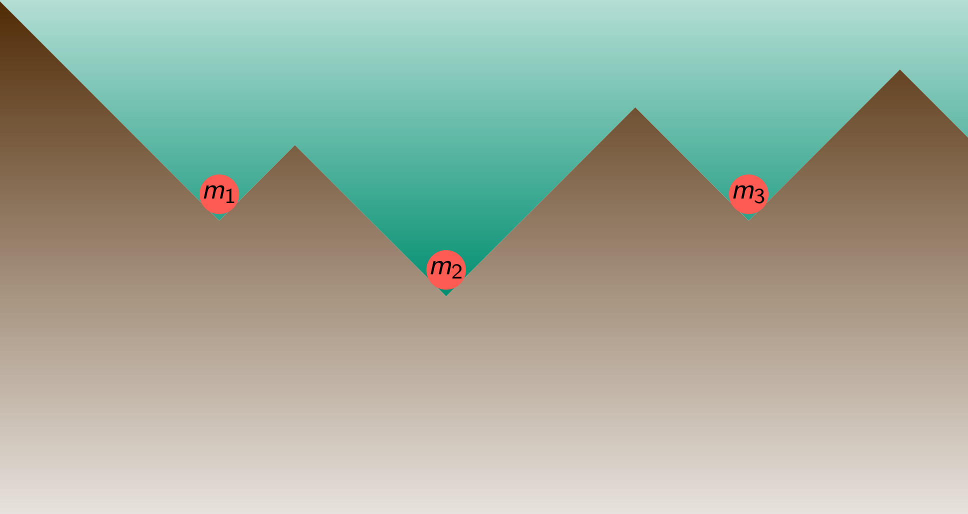
  $\to (\mathcal{P}(\mathbb{N}), \subseteq)$ is not a WQO;
- $(A^\star, \leqslant_{\mathrm{subseq}})$ with A finite: WQO;
- $(\text{graphs}, \leqslant_{\mathrm{minor}})$: WQO.

Why do we like well-quasi-orders?

Upwards closed classes have a finite number of minimal elements.



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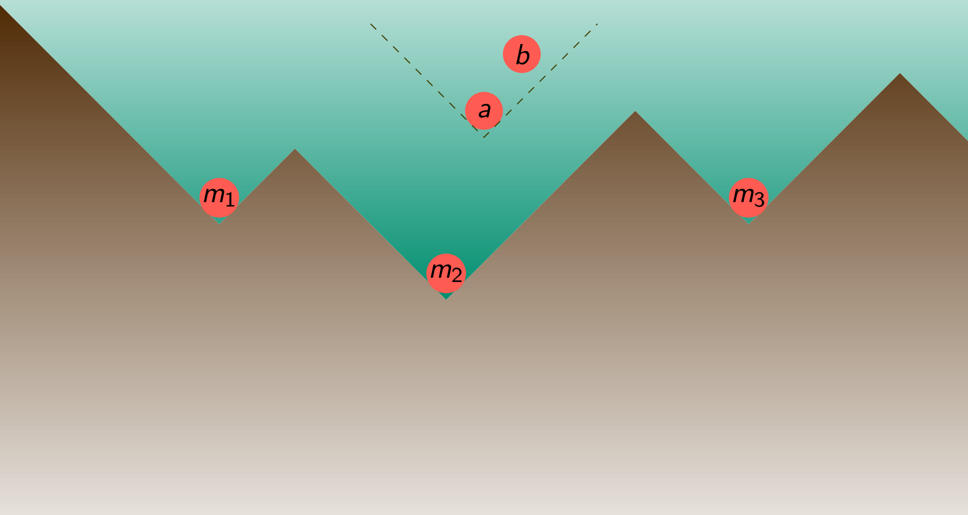

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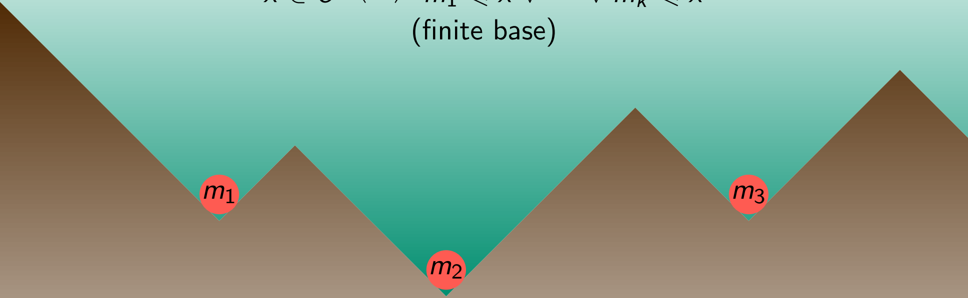

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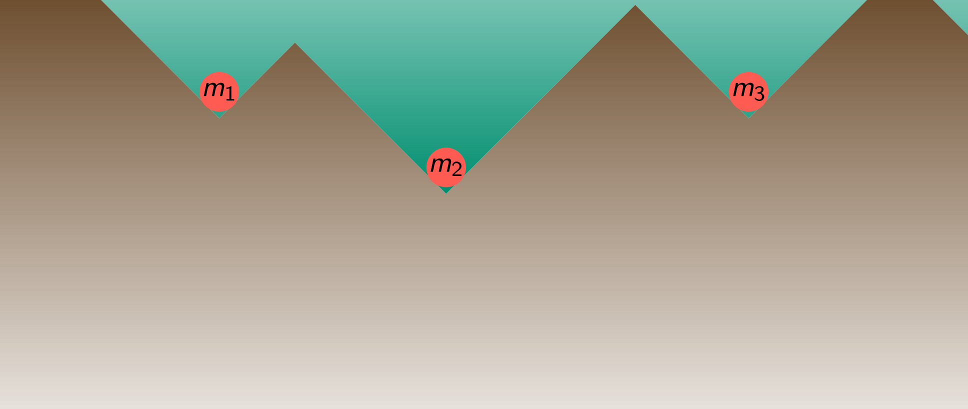

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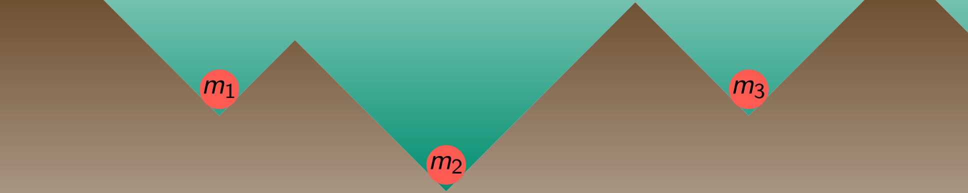

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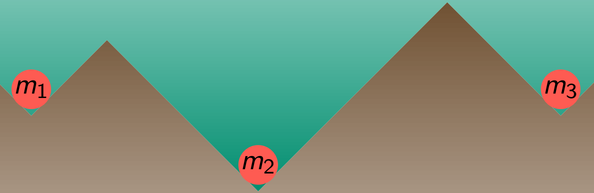

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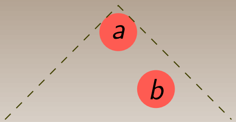

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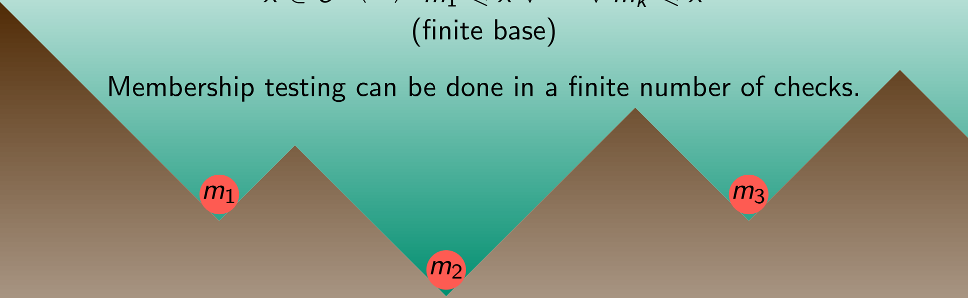

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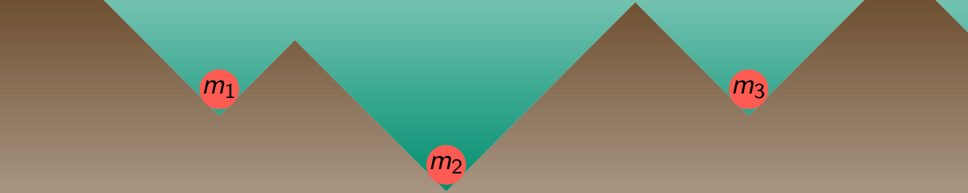

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graphs of genus $\geqslant g$ + minor relation

$(\geqslant k)$-colorable graphs + induced subgraph relation

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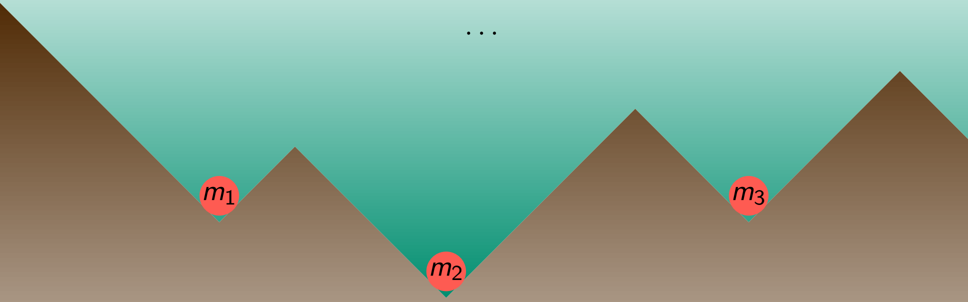

graphs of treewidth $\leqslant k$ + minor relation

trees + contraction relation

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- contraction relation $\leqslant_{\mathrm{ctr}}$: E contraction;
- induced minor relation $\leqslant_{\mathrm{im}}$: V deletion and E contraction;
- minor relation $\leqslant_{\mathrm{m}}$: V and E deletion, E contraction.

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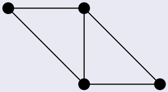
Main message: decomposition results (sometimes) imply WQO.

From structure to WQO

Toy example

Subdivisions of H are WQO by contraction.

$H =$ 

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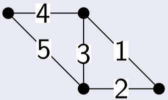
$H =$ 

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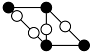
   e.g. # of subdivisions for each edge, in some chosen order;

$\text{enc}($  $) = (1, 0, 1, 0, 2)$

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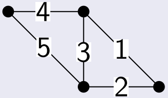
$H =$ 

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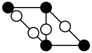
   enc() $= (1, 0, 1, 0, 2)$

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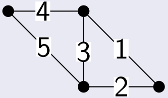
$H =$ 

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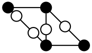
   enc() $= (1, 0, 1, 0, 2)$

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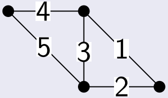
$H =$ 

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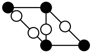
   enc() $= (1, 0, 1, 0, 2)$

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4. that's it!
   antichain $\{G_1, G_2, \dots\} \Rightarrow$ antichain $\{\mathrm{enc}(G_1), \mathrm{enc}(G_2), \dots\}$

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- $\mathcal{S} \subseteq \mathcal{G}^{\star}$ (Nash-Williams' minimum bad sequence).

First example

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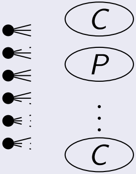
Lemma

*-induced minor-free graphs are WQO by induced minors.*

Lemma (Błasiok, Kamiński, R., Trunck '15)

Every 2-connected  *-induced minor-free graph is like this:*



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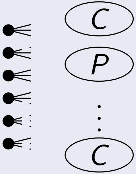

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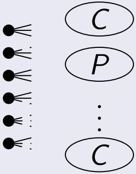

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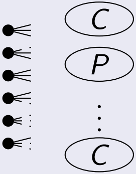

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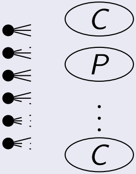

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(labeled) cographs and paths are easy to order.

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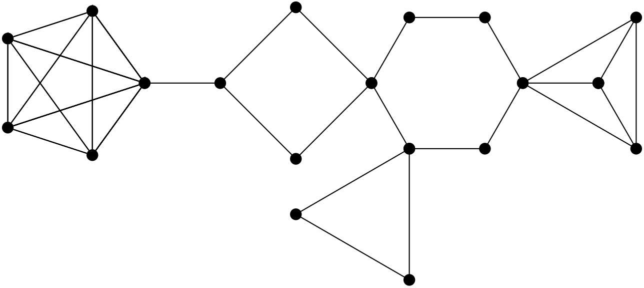
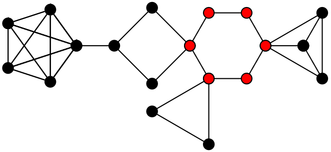
The decomposition

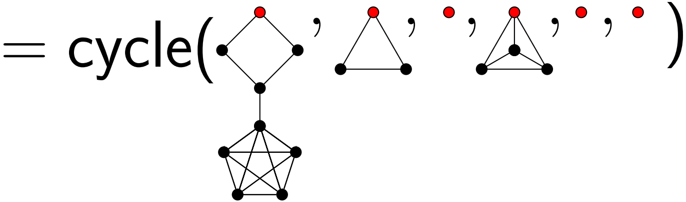
Lemma

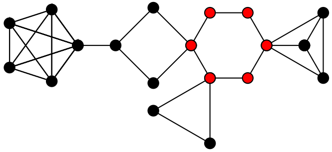
If G is  *-contraction-free then every block of G is either a clique or an induced cycle.*

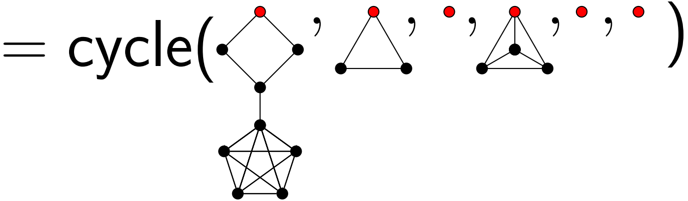


The encoding









If $(G_1, \ldots, G_p) \leqslant_{\mathrm{ctr}}{}^{\star} (H_1, \ldots, H_q)$

then $\mathrm{cycle}(G_1, \ldots, G_p) \leqslant_{\mathrm{ctr}} \mathrm{cycle}(H_1, \ldots, H_q)$

and $\mathrm{clique}(G_1, \ldots, G_p) \leqslant_{\mathrm{ctr}} \mathrm{clique}(H_1, \ldots, H_q)$

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6. $\{A_i\}_i$ is WQO, contradiction.

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1. Consider a minimal infinite antichain:

$$A_1, A_2, \ldots$$

2. $\mathcal{B} =$ "parts" of A_i's;
3. $(\mathcal{B}, \leqslant_{\mathrm{ctr}})$ is a wqo, by minimality of $\{A_i\}_i$;
4. $(\mathcal{B}^\star, \leqslant_{\mathrm{ctr}})$ is a wqo;
5. $(\mathrm{cycle}(\mathcal{B}^\star) \cup \mathrm{clique}(\mathcal{B}^\star), \leqslant_{\mathrm{ctr}})$ is a wqo;
6. $\{A_i\}_i$ is WQO, contradiction.

-contraction-free graphs are WQO by contractions.

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Thank you!