The Erdős-Pósa property

From combinatorics to approximation

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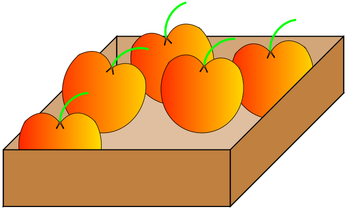
Joint work with D. Chatzidimitriou (Univ. Athens), I. Sau (LIRMM), and D. M. Thilikos (Univ. Athens and LIRMM).

Packing and Covering, in general

Packing problem: can I pack k apples in a box?



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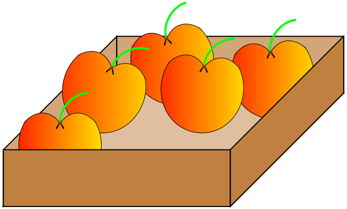
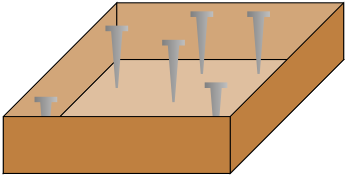


Covering problem: can I hammer k nails, s.t no apple fits in the box?



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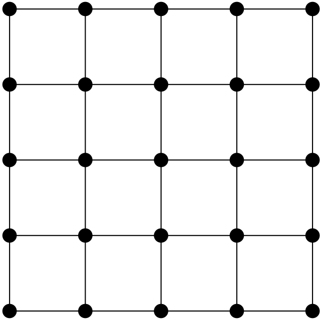

pack $=?$

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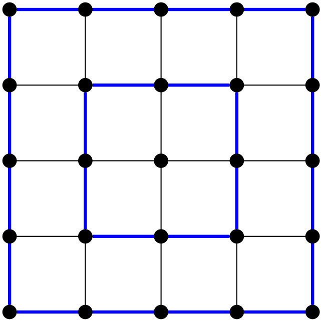

pack $\geqslant 2$

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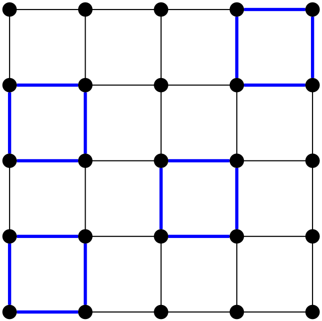

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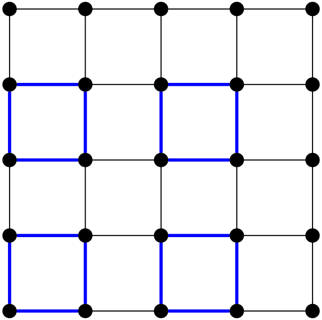

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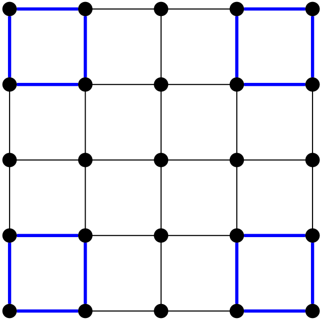

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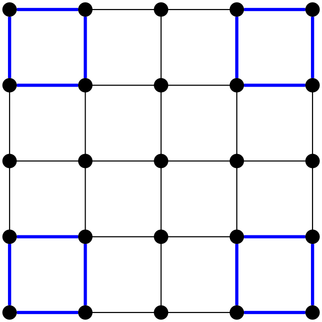

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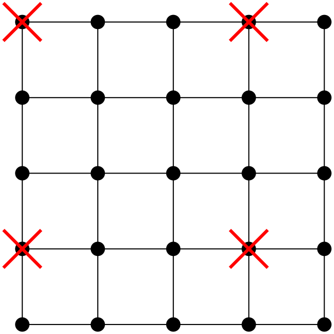

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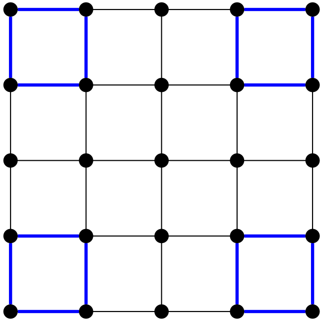

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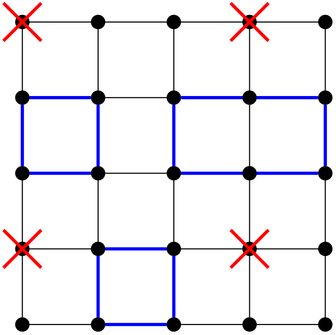

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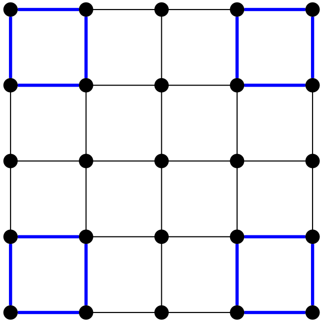

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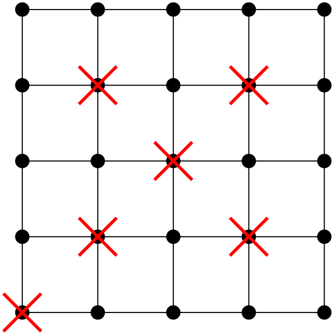
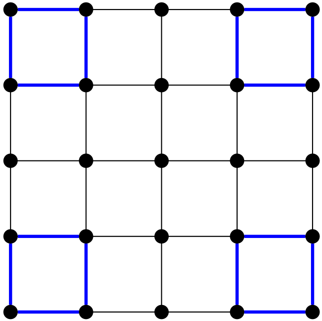


cover $\leqslant 6$

Packing and Covering numbers

Packing number **pack** maximum number of vertex-disjoint cycles in G;



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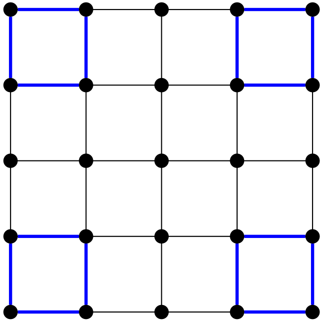
Packing and Covering numbers

Packing number **pack** maximum number of vertex-disjoint cycles in G;
Covering number **cover** minimum size of $X \subseteq V$ s.t. $G \setminus X$ forest.



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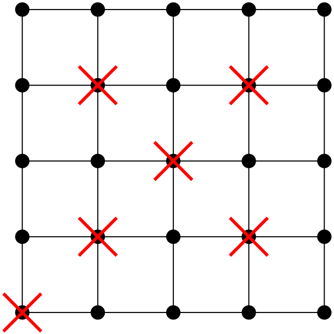

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Erdős-Pósa Theorem

Are **pack** and **cover** related?

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Theorem (Erdős & Pósa, 1962)

Every graph has k vertex-disjoint cycles or a set of $O(k \log k)$ vertices hitting every cycle.

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In other words: $f(k) = O(k \log k)$.

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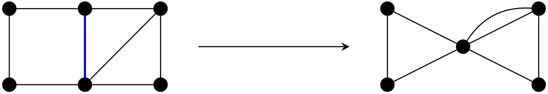
Generalizations?

Packing and covering models

Edge contraction:



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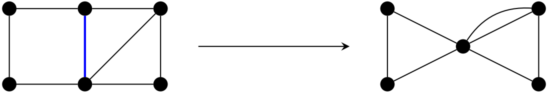

H-model: graph that can be contracted to *H*.

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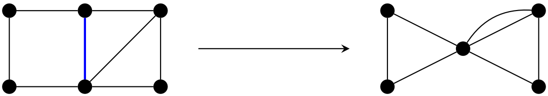

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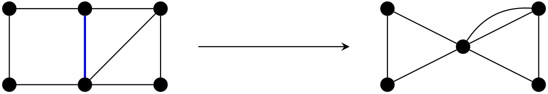

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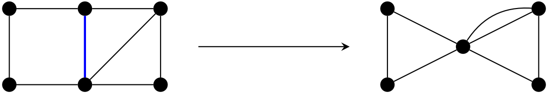

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Previously: $H =$ .

The Erdős-Pósa property

H has the Erdős-Pósa property if for some function f (gap),

$$\forall G,\ \mathbf{pack}_H(G) \leqslant \mathbf{cover}_H(G) \leqslant f(\mathbf{pack}_H(G)).$$

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Previously:  has the ĔṔ-property with gap $O(k \log k)$.

The Erdős-Pósa property of planar models

Theorem (Robertson & Seymour, 1986)

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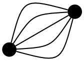
Theorem (Chekury & Chuzhoy, 2013)

$\forall H$ *planar,* $f_H(k) = O(k \operatorname{polylog} k)$.

Hitting and harvesting pumpkins

r-pumpkin θ_r: graph with 2 vertices and r edges. 

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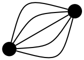
r-pumpkin θ_r: graph with 2 vertices and r edges. 

Theorem (Fiorini, Joret and Sau)

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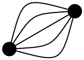
r-pumpkin θ_r: graph with 2 vertices and r edges. 

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Theorem (Joret, Paul, Sau, Saurabh and Thomassé, 2011)

There is a $O(\log(n))$-approximation for **pack**$_{\theta_r}$ *and* **cover**$_{\theta_r}$.

Main result

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Ingredients:

- a protrusion-based reduction;
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Also works for the edge variant!

The approximation

k	0	...	...	...	...	...	...	...	n
$\exists$ packing $\geqslant k$	Yes	...	...	...	...	Yes	No	...	No
$\exists$ cover $\leqslant k \log k$	No	...	No	Yes	...	...	...	...	Yes

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$\mathbf{pack}_{\theta_r}(G)$ ↓

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Possible $A(G,k)$	P	P	P	P/C	P/C	P/C	C	C	C

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Works only because $f_{\theta_r}(k) = O(k \log k)$.

The reduction

Protrusion: simple part of G separated by few edges.

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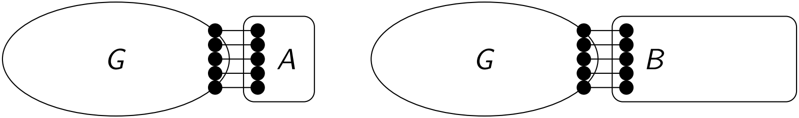
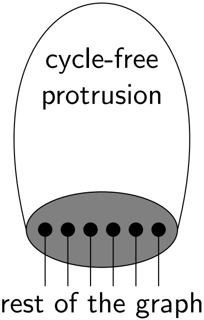
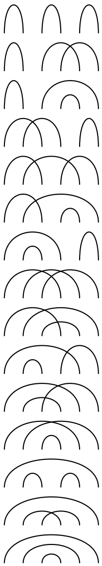
A and B are equivalent if $\mathbf{pack}_{\theta_r}$ and $\mathbf{cover}_{\theta_r}$ are the same in



for every graph G.

How can a cycle invade a given protrusion?





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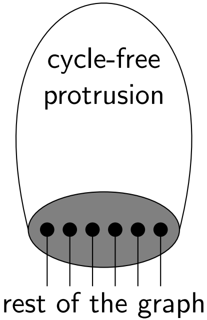

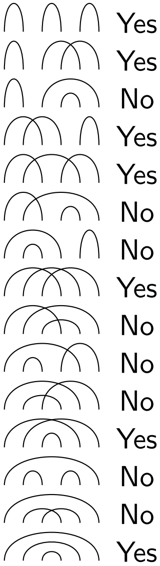

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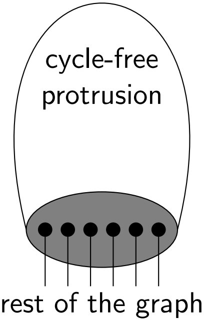
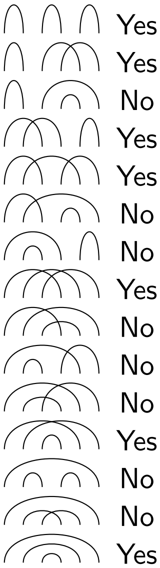
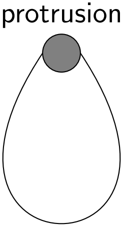




- equivalence relation on protrusions;
- safe move: reduce to an equivalent protrusion;
- how to compute the equivalence class?

Decomposing protrusions



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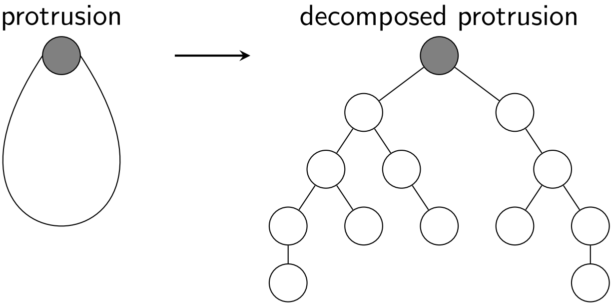

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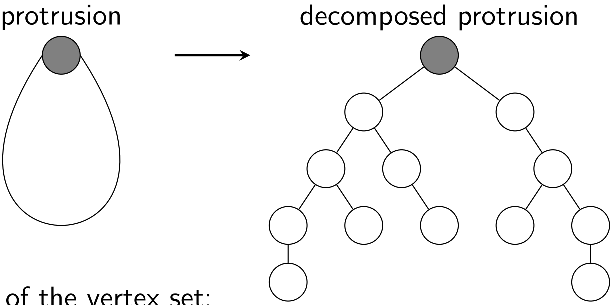

- partition of the vertex set;
- edges between adjacent nodes only;
- boundary of small size;
- node $\rightarrow$ protrusion $\rightarrow$ eq class.

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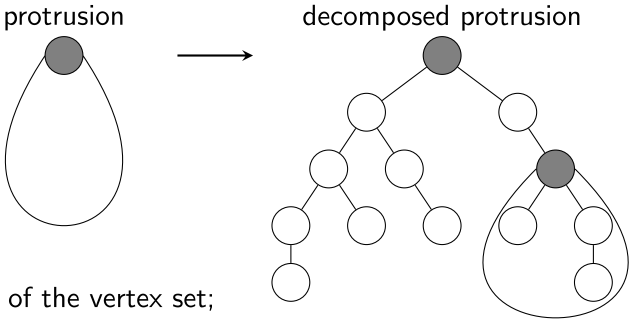

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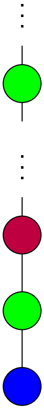
A long path in the decomposition tree



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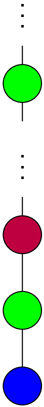

Long path → repetition.

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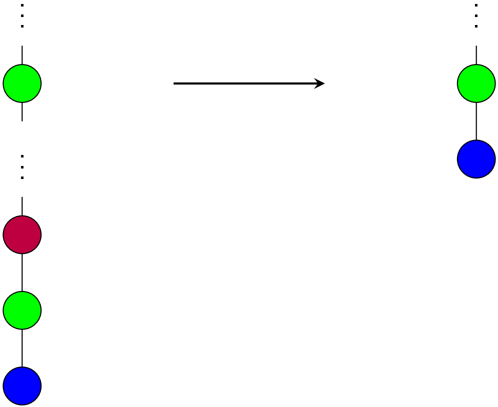

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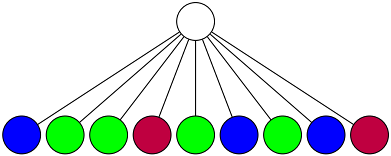

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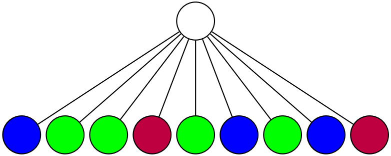

Large degree $\rightarrow$ repetitions.

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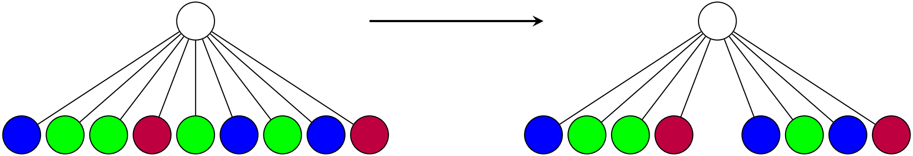
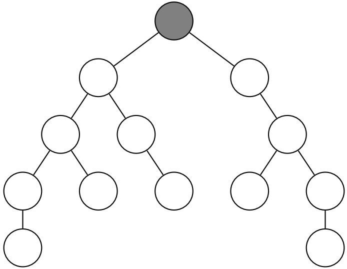


Large degree $\rightarrow$ repetitions.
Discard a redundant child.

When the reduction ends...



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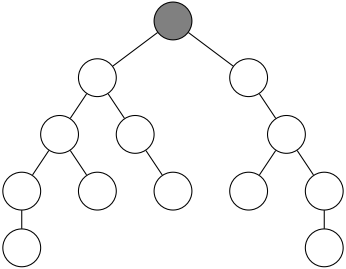

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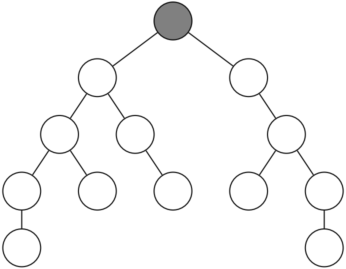

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- each reduction step takes $O(n)$ time (reduce from leaves to root).

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- the reduction takes $O(n)$ time.

Finding protrusions

How to find these protrusions?

Theorem (Chatzdimitriou, R., Sau, Thilikos)

Given G large enough, we can compute

- *a θ_r-model of G of small size, or;*
- *a large protrusion of small boundary, or;*
- *an subgraph contractible to some H with large δ,*

in $O(m)$ steps.

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$\rightarrow$ every tree has exponentionaly many leaves;

$\rightarrow$ many edges between trees;

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Finding a packing or a covering

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- an subgraph contractible to some H with large δ: $\exists$ a large packing.

Then:

- if $\mathcal{P} \geqslant k$ then we output $\mathcal{P}$;
- if G is θ_r-free, then $\mathcal{P}$ is a small cover.

A constructive algorithm?

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Can we find it in linear time?

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- the reduction algorithm;
- the algorithm to find protrusions,

we are able to

- answer $\textbf{pack}_{\theta_r} \geqslant k$ or $\textbf{cover}_{\theta_r} \leqslant k \log k$ in $O(n^2)$-time;
- extract the corresponding object in polynomial time.

The edge variant

Given G and H

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Not known whether all planar graphs have the edge-ĔṔ-property.

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Thank you!