Erdős-Pósa for pumpkins and approximation

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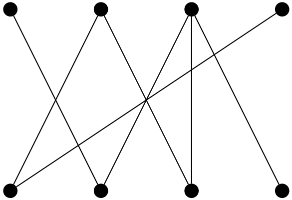
König's theorem

Theorem (König, 1931)

For every bipartite graph, $|$*maximum matching*$| = |$*minimum vertex cover*$|$.



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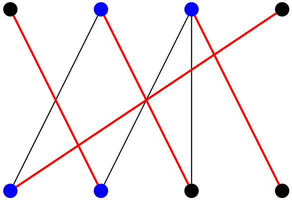

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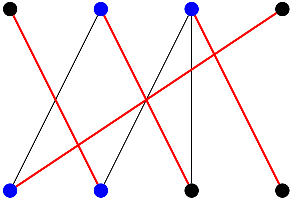

- min-max theorem for packing and covering edges;
- knowing one parameter gives the other one.

Packing and covering

Can we generalize it to

- packing/covering of larger classes; and
- in more general graphs?

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What about packing cycles in graphs?

The case of cycles

Packing number **pack** maximum number of vertex-disjoint cycles in G;
Covering number **cover** minimum size $X \subseteq V$ s.t. $G \setminus X$ forest (FVS).

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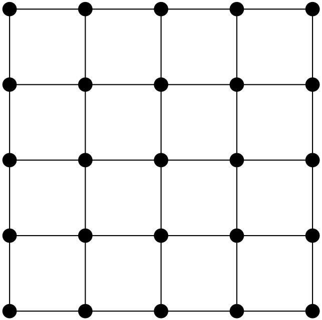

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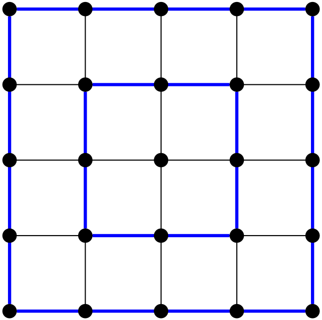

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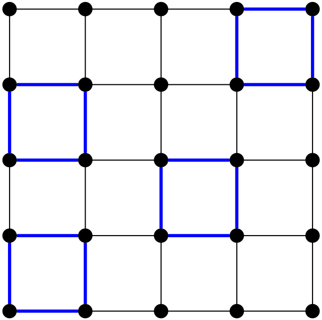

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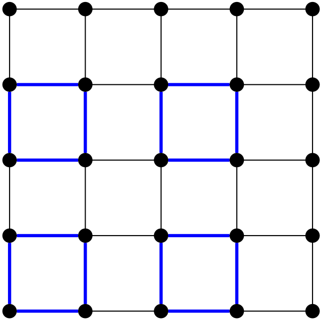

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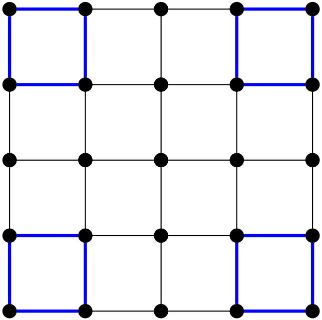

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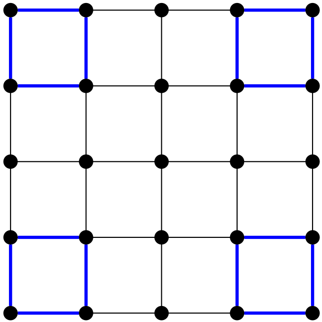

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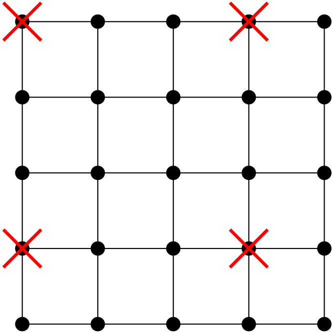

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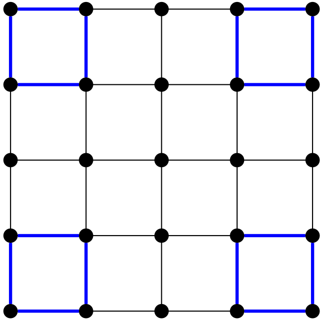

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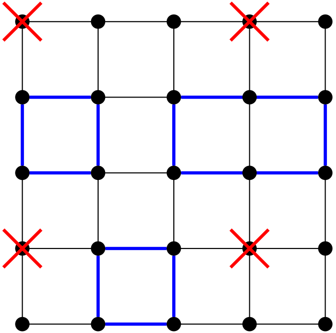

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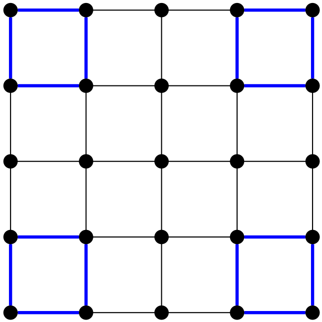

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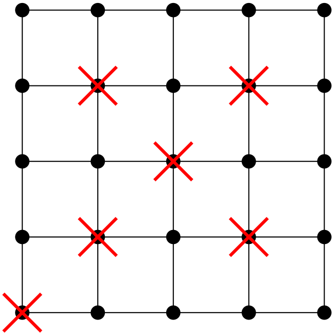

cover $\leqslant 6$

Erdős-Pósa Theorem

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Theorem (Erdős & Pósa, 1962)

Every graph has k vertex-disjoint cycles or a set of $O(k \log k)$ vertices hitting every cycle.

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Theorem (Erdős & Pósa, 1962)

Every graph has k vertex-disjoint cycles or a set of $O(k \log k)$ vertices hitting every cycle.

In other words: $f(k) = O(k \log k)$.

Packing and covering models

Given H and G,

Packing How many vertex-disjoint sgr. of G can be contracted to H?

Covering How many vertices to remove in G to get an H-minor-free graph?

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- Packing: How many vertex-disjoint sgr. of G can be contracted to H? (maximization problem)
  $\rightarrow$ packing number $\mathbf{pack}_H(G)$
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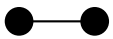
Previously: $H =$  and $H =$ .

The Erdős-Pósa property

H has the ΈΡ́-property if:

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f: gap.

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Two questions: What classes and with which gap?

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Two questions: What classes and with which gap?

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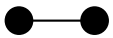
-  has the ĚṔ-property with gap k;
-  has the ĚṔ-property with gap $O(k \log k)$.

The Erdős-Pósa property of planar models

Theorem (Robertson & Seymour, 1986)

H has the $\acute{\text{E}}\acute{\text{P}}$-property $\Leftrightarrow$ H planar.

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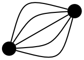
Theorem (Robertson & Seymour, 1986)

H has the ŐṔ-property $\Leftrightarrow$ *H planar.*

Theorem (Chekury & Chuzhoy, 2013)

$\forall H$ *planar,* $f_H(k) = O(k \operatorname{polylog} k)$.

Hitting and harvesting pumpkins

r-pumpkin θ_r: graph with 2 vertices and r edges. 

Theorem (Fiorini, Joret and Sau)

$f_{\theta_r} = O(k \log k)$.

Theorem (Joret, Paul, Sau, Saurabh and Thomassé, 2011)

There is a $O(\log(n))$-approximation for **pack**$_{\theta_r}$ *and* **cover**$_{\theta_r}$.

Main result

Theorem (Chatzdimitriou, R., Sau, Thilikos)

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Ingredients:

- a protrusion-based reduction;
- a algorithm to extract a big packing or a protrusion to reduce.

The approximation

k	0	...	...	...	...	...	...	...	n
$\exists$ packing $\geqslant k$	Yes	...	...	...	...	Yes	No	...	No
$\exists$ cover $\leqslant k \log k$	No	...	No	Yes	...	...	...	...	Yes

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k	0	…	…	…	…	$\mathbf{pack}_{\theta_r}(G)$ $\downarrow$	…	…	n
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Possible $A(G, k)$	P	P	P	P/C	P/C	P/C	C	C	C

Assume $A(G, k)$ outputs either a packing $\geqslant k$ or a cover $\leqslant k \log k$.

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1. run A on (G, k) for every $k \in \{0, \ldots, n\}$;

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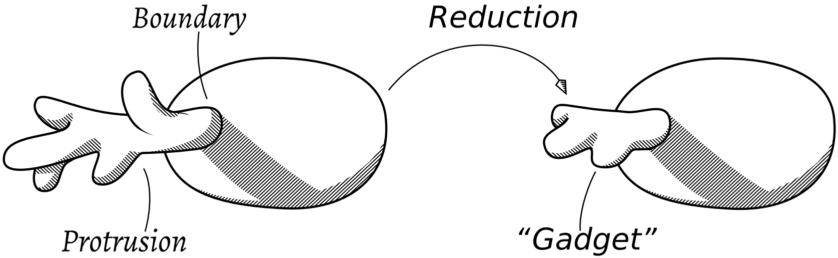
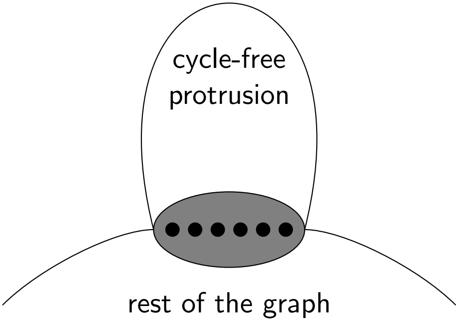
1. run A on (G, k) for every $k \in \{0, \ldots, n\}$;
2. returns the largest k s.t. $A(G, k)$ is a packing.

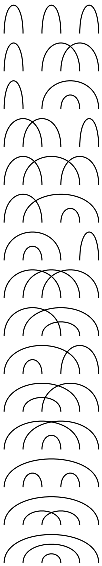
The reduction



Picture by Felix Reidl

How can a cycle invade a given protrusion?





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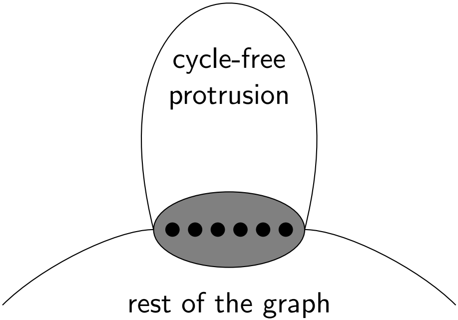

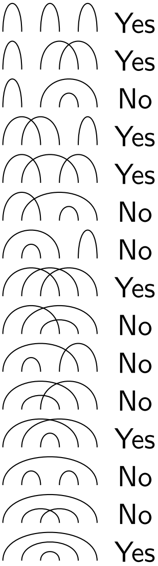

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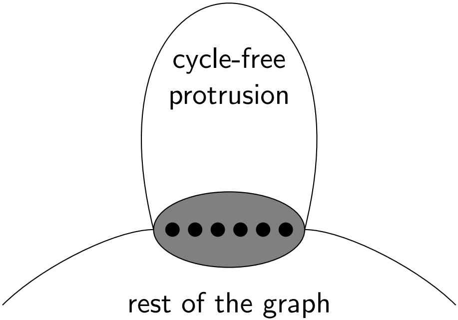
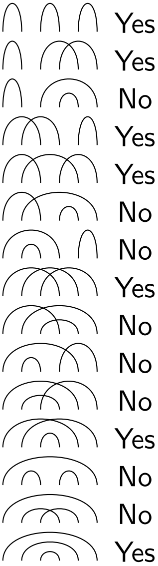
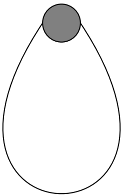




- defines an equivalence relation;
- safe move: reduce to an equivalent protrusion;
- how to compute the equivalence class?

Decomposing protrusions



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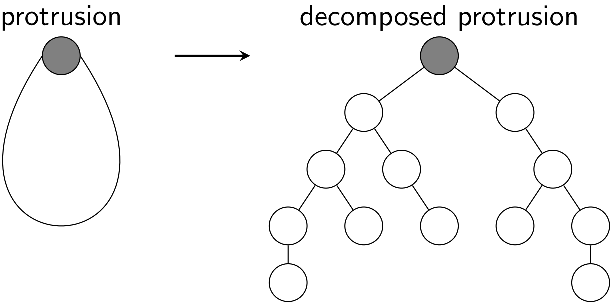

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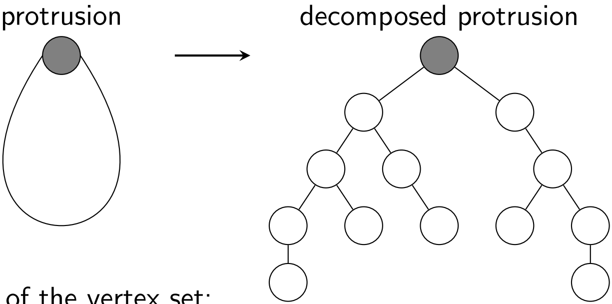

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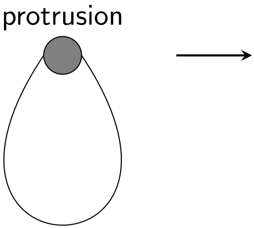
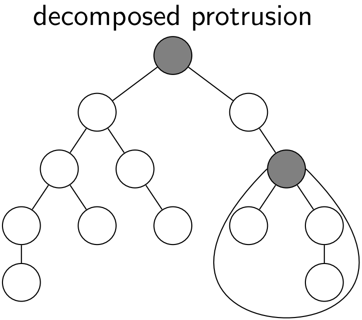
- edges between adjacent nodes only;
- boundary of small size;
- node → protrusion → eq class.





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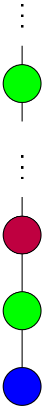
A long path in the decomposition tree



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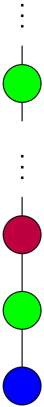

Long path $\rightarrow$ repetition.

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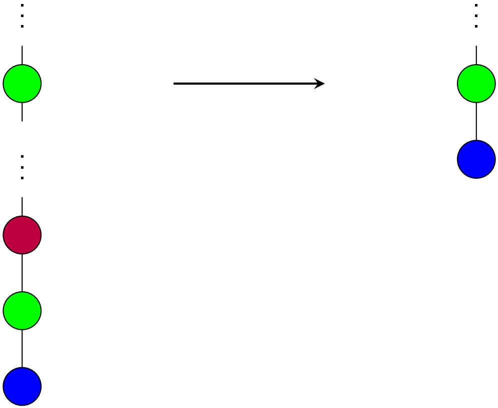

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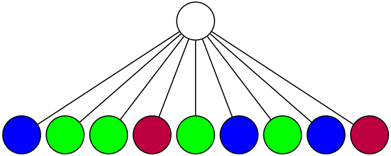

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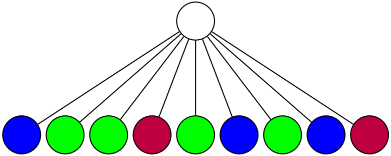

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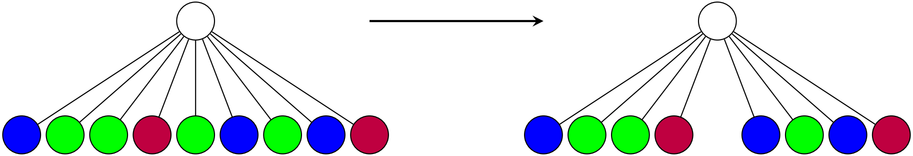
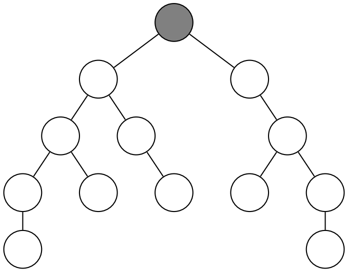


Large degree → repetitions.
Discard a redundant child.

When the reduction ends...



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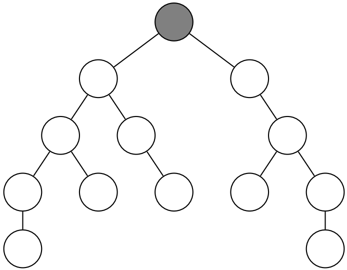

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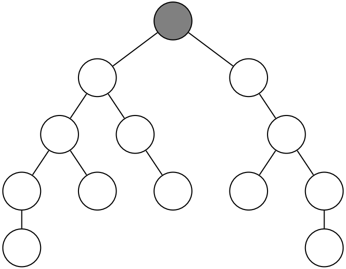

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- each reduction step takes $O(n)$ time (reduce from leaves to root).

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Given G and an a large protrusion of small boundary,

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- the reduction takes $O(n)$ time.

Finding protrusions

How to find these protrusions?

Theorem (Chatzdimitriou, R., Sau, Thilikos)

Given G large enough, we can compute

- *a θ_r-model of G of small size, or;*
- *a large protrusion of small boundary, or;*
- *an subgraph contractible to some H with large δ,*

in $O(m)$ steps.

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$\rightarrow$ every tree has exponentionaly many leaves;

$\rightarrow$ many edges between trees;

- an subgraph contractible to some H with large δ.

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Then:

- if $\mathcal{P} \geqslant k$ then we output $\mathcal{P}$;
- if G is θ_r-free, then $\mathcal{P}$ is a small cover.

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Can we find it in linear time?

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Given (G, k) and using both

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we are able to

- answer $\mathbf{pack}_{\theta_r} \geqslant k$ or $\mathbf{cover}_{\theta_r} \leqslant k \log k$ in $O(n^2)$-time;
- extract the corresponding object in polynomial time.

The edge variant

Given G and H

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Our algorithm can deal with this variant.

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$\rightarrow$ same complexity for both existential and constructive version.

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Thank you!