Linear Programming for Grownups

Inofficial lecture notes
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Contents

Literature

- Schrijver
- Günther Ziegler, “Lectures on Polytopes”

1 Coding length and the ellipsiod method

1.1 Definition. We first want to define the *size* of several input formats.

a) For $\alpha = \frac{p}{q} \in \mathbb{Q}$ with $p, q \in \mathbb{Z}$ coprime let

$$\text{size}(\alpha) = 1 + \lceil \log_2(|p| + 1) \rceil + \lceil \log_2(|q| + 1) \rceil.$$

The “1+” represents the sign.

b) For $c = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Q}^n$, let $\text{size}(c) = n + \sum \text{size}(\gamma_i)$. The summand n notes the delimiters, between the entries.

c) For $A = (a_{ij})_{i,j} \in \mathbb{Q}^{m \times n}$, let $\text{size}(A) = mn + \sum \text{size}(a_{ij})$.

d) To encode inequalities, either scalar or matrix, let $\text{size}(Ax \leq b) = 1 + \text{size}(A) + \text{size}(b)$. The same for equations.

1.2 Proposition. *Let* $A \in \mathbb{Q}^{m \times n}$ *with* $\text{size}(A) = \sigma$. *Then* $\text{size}(\det A) < 2\sigma$.

Proof. Write $A = \left(\frac{p_{ij}}{q_{ij}}\right)$ with $p_{ij}, q_{ij} \in \mathbb{Z}$ coprime and $q_{ij} > 0$. Further let $\det A = \frac{p}{q}$. We have the bound

$$q \leq \prod_{1 \leq i,j \leq n} q_{ij} = 2^{\sum \log q_{ij}} < 2^{\sum \log q_{ij} + \log p_{ij}} \leq 2^{\sigma - 1} \qquad \text{since } mn \geq 1$$

For p we get the upper bound

$$|p| = |\det A| \cdot q \leq \prod_{1 \leq i,j \leq n} (|p_{ij}| + 1) q_{ij} < 2^{\sigma - 1}$$

Together, this yields

$$\operatorname{size}(\det A) = 1 + \lceil \log_2(|p| + 1) \rceil + \lceil \log_2(|q| + 1) \rceil$$

□

1.3 Corollary. *For $A \in \mathrm{GL}_n(\mathbb{Q})$, we have* $\operatorname{size}(A^{-1}) \in \operatorname{poly}(\operatorname{size}(A))$.

Proof. Use Cramer's rule. □

1.4 Corollary. *Let $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$. If $Ax = b$ has a solution, then there is a solution x_0 with* $\operatorname{size}(x_0) \in \operatorname{poly}(\operatorname{size}(Ax = b))$.

Proof. Assume that A has linearly independent rows and $A = (A_1, A_2)$ where A_1 is non-singular. Then $x_0 := (A_1^{-1} b, 0) \in \mathbb{Q}^m \times \mathbb{Q}^{n-m}$ is a solution of the desired size. □

1.5 Corollary. *The decision problem*

> ***Linear Equation System***
> ***Input:*** *A, b rational*
> ***Question:*** *Does $Ax = b$ have a solution?*

has a good characterisation *(i.e. lies in* $\mathsf{NP} \cap \mathsf{coNP}$*).*

Proof. If the answer is positive, then Corollary 1.4 provides a certificate of polynomial size. Suppose $Ax = b$ does *not* have a solution. This happens if and only if there is some y with $y \cdot A = \mathbf{0}$ and $y \cdot b = 1$. Again by Corollary 1.4 there is such y of polynomial size. □

1.6 Corollary. *Let $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$ such that each row of the extended matrix (A, b) has size at most φ. If $Ax = b$ has a solution, then*

$$\{x : Ax = b\} = \{x_0 + \lambda_1 x_1 + \ldots + \lambda_t x_t : \lambda_i \in \mathbb{Q}\}$$

for certain vectors $x_i \in \mathbb{Q}^n$ such that $\operatorname{size}(x_i) \leq 4n^2\varphi$.

Proof. By Cramer's rule, the coefficients of x_i can be described as quotients of subdeterminants of the extended matrix of order at most n. By Proposition 1.2, these determinants have size $< 2n\varphi$. Taking quotients gives another factor 2, so each coefficient has size $< 4n\varphi$. Having n coefficients yields $\operatorname{size}(x_i) < 4n^2\varphi$. □

1.7 Remark. Gauß elimination transforms a given matrix A into a standard form

$$\begin{pmatrix} B & C \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

where B is non-singular, upper triangular; by row operations

$$a_{i,\cdot} \mapsto a_{i,\cdot} + \lambda a_{j,i} \text{for } \lambda \neq 0$$

and permutations of rows/columns.

Remark. *Sometimes, we can further reduce it to the case B invertible diagonal matrix.*

1.8 Theorem (Edmonds, 1967). *For A rational, Gauß elimination is a polynomial time algorithm within the bit-model.*

Proof. Wlog assume that no permutations are necessary. Polynomially many arithmetic operations suffice ($\mathcal{O}(n^3)$). But we still have to control the size of the intermediate results.
The procedure generates matrices

$$A = A_0, A_1, A_2, \dots \qquad\qquad A_k = \begin{pmatrix} B_k & C_k \\ \mathbf{0} & D_k \end{pmatrix}$$

where B_k is non-singular, upper triangular of order k. Then A_{k+1} is obtained form A_k by row operations with pivot $\delta_{11} \neq 0$, where $D_k = (\delta_{ij})$. If we can show $\text{size}(A_k) \in \text{poly}(\text{size}(A))$, this also holds for the result and we are done.
We have

$$\delta_{ij} = \frac{\det(A_k[1,\dots,k,k+i;1,\dots,k,k+j])}{\det(\underbrace{A_k[1,\dots,k;1,\dots,k]}_{=B_k})} \qquad \text{pick given rows and columns}$$

By Proposition 1.2, $\text{size}(\delta_{ij}) < 4\,\text{size}(A)$, since each entry of B_k and C_k have been coefficients of D_j for some $j < k$. □

1.9 Corollary. *The following problems are solvable in polynomial time.*

(i) determining the rank of a rational matrix

(ii) computing the determinant of a rational matrix

(iii) computing the inverse of a rational matrix

(iv) testing vectors for linear (in)dependence

(v) solving a system of linear equations

1.10 Theorem. *If the system $Ax \leq b$ of rational linear inequalities has a solution, then it has a solution of size* $\text{poly}(\text{size}(A, b))$.

Proof. Let $\{x : A'x = b'\}$ describe a minimal face of the polyhedron $\{x : Ax \leq b\}$, where $[A', b']$ is a submatrix of $[A, b]$. That minimal face contains a point of size $\text{poly}(\text{size}(A, b))$. □

1.11 Lemma (Farkas' Lemma). *Let A be a matrix and b some vector. Then there exists a vector $x \geq 0$ with $Ax = b$ iff $yb \geq 0$ for each row vector y with $yA \geq 0$.*

The important part is the negation of the statements, since it yields a certificate, that some LP has no solution. If the system $yA \geq 0$, $yb < 0$ has a solution, then $Ax = b, x \geq 0$ has no solution.

1.12 Corollary. *The following problems have good characterisations (i.e. $\in \mathsf{NP} \cap \mathsf{coNP}$).*

- *Given A and b rational, does $Ax \leq b$ have a solution?*
- *Given A and b, does $Ax = b$ have a nonnegative solution?*
- *Given A, b, c and δ, does $Ax \leq b$, $cx > \delta$ have a solution?*

1.13 Definition. Let us denote a nonempty polyhedron as

$$P = P(A,b) = \{x \in \mathbb{R}^n : Ax \le b\}$$

with minimal faces $F_1, \dots, F_r$. Pick a point x_i from each minimal face. Then

$$P = \operatorname{conv}\{x_1, \dots, x_r\} + \operatorname{rec} P$$

where

$$\operatorname{rec} P := \{y \in \mathbb{R}^n : \forall x \in P. \forall \lambda \ge 0. x + \lambda y \in P\}$$

is the *recession cone*, i.e. the cone of all unbounded directions. Furthermore set

$$\operatorname{lin} P := \{y \in \operatorname{rec} P : -y \in \operatorname{rec} P\} = \{x : Ax = 0\}$$

the *lineality space.* kernel

1.14 Definition. Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron. The *facet complexity* of P is the smallest number $\varphi \ge n$ such that there exist rational A, b with $P = P(A,b)$ and each inequality in $Ax \le b$ has size $\le \varphi$. The *vertex complexity* is the smallest number $\nu \ge n$ that that there exist $x_1, \dots, x_k$ and $y_1, \dots, y_t \in \mathbb{Q}^n$ with

$$P = \operatorname{conv}\{x_1, \dots, x_k\} + \operatorname{pos}\{y_1, \dots, y_t\} \qquad \text{pos} = \text{nonnegative cone}$$

where each x_i, y_j have size $\le \nu$.

1.15 Remark. Both notions are defined for any polyhedron, even if there are no vertices.

1.16 Theorem. *Let $P \subseteq \mathbb{R}^n$ be a rational polyhedron with facet complexity φ and vertex complexity ν. Then $\nu \le 4n^2\varphi$ and $\varphi \le 4n^2\nu$.*

Proof. Let P be such that each inequality in $Ax \le b$ has size $\le \varphi$.

- Let $F_1, \dots, F_k$ be the minimal faces of P. Then $F_i = \{x : A'_i x \le b'_i\}$ for some submatrix $[A_i, b_i]$ of $[A, b]$. Each equation in $A'_i x = b'_i$ has size at most φ. By Corollary 1.6, F_i contains a solution x_i of size $4n^2\varphi$.
- Similarly, $\operatorname{lin} P = \{x : Ax = 0\}$ has a basis where vector has size at most $4n^2\varphi$.
- Each minimal proper face F of $\operatorname{rec} P$ contains a vector $y \notin \operatorname{lin} P$ of size $\le 4n^2\varphi$, since

  $$F = \{x : A'x = 0, ax \le 0\}$$

  for some submatrix A' of A and some row a of A. $\square$

The important part is that we relate the sizes of the points and constraints. We do not relate their numbers to each other.

1.17 Corollary. *Let $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^m$ and $c \in \mathbb{Q}^n$ such that the optima*

$$\max\{cx : Ax \le b\} = \min\{yb : y \ge 0, yA = c\} \qquad (*)$$

are finite. Let σ be the maximal size of the coefficients in the input.

(i) The maximum in (*) has optimal solution of size $\text{poly}(n, \sigma)$*.*

(ii) The minimum in (*) has optimal solution of size $\text{poly}(n, \sigma)$*.*

(iii) The optimal values in (*) are in $\text{poly}(n, \sigma)$, $\text{poly}(m, \sigma)$*.*

1.18 Definition (LP-optimisation problem). Given A, b, c rational, test if $\max\{cx : Ax \leq b\}$ is feasible, finite or unbounded. If feasible and finite, find an optimal solution. If unbounded, find feasible solution x_0 and vector z with $Az \leq 0$ and $cz > 0$.

1.19 Remark. LP-feasibility $\implies$ LP-optimisation. Assume, we have an oracle for LP-feasibility an want to us it for optimisation. Given A, b, c

(i) check $Ax \leq b$ and find solution x_0.

(ii) check if $y \geq 0$ and $yA = c$ feasible.

(iii) Then $Ax \leq b$, $y \geq 0\check{}$, $yA = c$ and $cx \geq yb$ has a solution (x^*, y^*), which is an optimal solution for the dual pair (*).

1.20 Remark. LP-optimisation $\implies$ LP-optimisation: Naive: take $c = \mathbf{0}$ as objective function.

1.21 Remark. Again, let $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$. A point $x \in P$ is *interior*, if $Ax < b$. This exists iff $\dim P = m$. Consider the LP

$$\max\{\varepsilon : Ax + \mathbf{1}_m \cdot \varepsilon \leq b, 0 \leq \varepsilon \leq 1\} \tag{***}$$

(i) The LP (***) is feasible iff $Ax \leq b$, i.e. $\dim P \neq \emptyset$.

(ii) The LP (***) has an optimal solution with $\varepsilon > 0$ iff $\operatorname{int} P \neq \emptyset$ iff $\dim P = n$.

Ellipsoid Method (Khachyan, 1979)

1.22 Definition. A symmetric matrix $D \in \mathbb{R}^{n \times n}$ is *positive definite* if all its eigenvalues are positive.

1.23 Theorem (Principal axis theorem). *The following are equivalent:*

(i) D *is positive definite*

(ii) $D = B^T B$ *for some* $B \in \mathrm{GL}_n(\mathbb{R})$

(iii) $x^T D x > 0$ *for all* $X \in \mathbb{R}^n \setminus \{\mathbf{0}\}$

1.24 Definition. Let $z \in \mathbb{R}^n$ and D positive definite. Then the set

$$\operatorname{ell}(z, D) := \left\{x : (x - z)^T D^{-1} (x - z) \leq 1\right\}$$

is called the *ellipsoid* with centre z.

Remark.

- *For a given ellipsoid,* z *and* D *are unique.*
- *An ellipsoid is the same as an affine image of the unit ball.*

1.25 Theorem. *Let $E = \text{ell}(z, D) \subseteq \mathbb{R}^n$ be an ellipsoid, and let $a \in \mathbb{R}^n$ be a row vector. Let further E' be an ellipsoid, that*

- *contains $E \cap \{x : ax \leq az\}$ (intersection of E with a half-space) and*
- *E' has smallest volume with this property.*

Then $E' = \text{ell}(z', D')$ is unique and is given by

$$z' = z - \frac{1}{n+1} \cdot \frac{Da^T}{\sqrt{aDa^T}} \qquad D' = \frac{n^2}{n^2-1}\left(D - \frac{2}{n+1}\frac{Da^TaD}{aDa^T}\right)$$

Moreover, the volume is bounded by

$$\frac{\text{vol}\, E'}{\text{vol}\, E} < e^{-\frac{1}{2n+2}}$$

1.26 Algorithm (Outline for LP-feasibility). Given $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$, does $Ax \leq b$ have a (rational) solution? We make the following assumption:

- $P := P(A, b)$ is bounded and full dimensional.
- We have exact arithmetic, i.e. infinite precision.

(i) Let φ be the maximal size of a row of $[A, b]$ and put $\nu := 4n^2\varphi$. (In particular, the facet complexity is $\leq \varphi$.) Then by Theorem 1.16, each vertex of P has size $\leq \nu$ (since ν is an upper bound for the actual vertex complexity). Let $R := 2^\nu$. Then $P \subseteq \{x : \|x\| \leq R\}$, because R is the largest number we can represent with ν bits.

(ii) We construct a sequence $z^0, z^1, z^2, \ldots$ and $D_0, D_1, \ldots$ such that $E_i := \text{ell}(z^i, D_i)$ is a sequence of ellipsoids with $\text{vol}(E_{i+1}) < \text{vol}(E_i)$. Set $z^0 = 0$ and $D_0 = R^2 \cdot I$, so $E_0 = \{x : \|x\| \leq R\} \supseteq P$.

(iii) Suppose we have z^i and D_i. If $z^i \in P$, we have a feasible solution, so we stop. Otherwise, there is some row index k such that $a_k z^i > b_k$ (where a_k is the k-th row of A, i.e. we pick a row violating the inequality). Then define E_{i+1} as the ellipsoid obtained by Remark 1.39, with row vector a_k. Then

$$E_{i+1} \supseteq E_i \cap \left\{x : a_k x \leq a_k z^i\right\} \supseteq E_i \cap \{x : a_k x \leq b_k\} \supseteq P$$

Our volume decreases exponentially, via

$$\text{vol}\, P \leq \text{vol}\, E_i \leq e^{-\frac{i}{2n+2}} \cdot (2R)^n$$

(iv) Since P has full dimension, there are $x_0, \ldots, x_n \in P$ affinely independent. So we can estimate the volume

$$\text{vol}\, P \geq \text{vol}\,\text{conv}\{x_0, \ldots, x_n\} \geq \frac{1}{n!}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_n \end{pmatrix} \geq n^{-n} n^{-n\nu} \geq 2^{-2n\nu}$$

by Hadamard's bound for the determinant.

(v) Put $N := 16n^2\nu \in \text{poly}(\text{size}(A, b))$. Then assuming, we arrive at E_N, we reach a contradiction

$$2^{-2n\nu} \leq \text{vol}\, P \leq \text{vol}\, E_N < e^{-\frac{N}{2n+2}} (2R)^n \leq 2^{-2n\nu} \quad \lightning$$

So we must have found an interior point in less than N steps.

missing lecture

1.27 Theorem (Koebe 1936, Adreev, Thurston). *Every planar graph is a disk graph.*

Proof. Let $\Gamma = ([n], E)$ be our graph.

i) Wlog *all* regions are triangles: We can add new edges. For the outer region, we put a large triangle that includes everything in its interior and then triangulate again, what is missing. Also assume the outer region is given by $\{1, 2, 3\}$. By linear transformations, this outer triangle can be made regular.

ii) By Euler's formula, we have $2n - 4$ regions.

iii) Let $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_{>0}$ with $\sum r_i = 1$, which will be our radii. For i, j, k vertices of some region. Then we have a unique triangle $\Delta_r(i, j, k)$ given by the sides $r_i + r_j$, $r_i + r_k$ and $r_j + r_k$. Let $\alpha_r(i; j, k)$ be the angle at i in this triangle $\Delta_r(i, j, k)$. For the outer region, we still take the small angle in the interior. (Let $\alpha_r(i; j, k) = 0$ if they do not form a region.)

iv) Put $\sigma_r(i) := \sum_{j,k} \alpha_r(i; j, k)$. We need to show that for r with

$$\sigma_r(i) = \begin{cases} \frac{2\pi}{3} & : i \in \{1, 2, 3\} \\ 2\pi & : \text{otherwise} \end{cases}$$

there exists a realisation of Γ as a disk graph with radius vector r.

Proof. omitted □

v) Now our set of candidates $\Delta := \{r \in \mathbb{R}^n : r_i > 0, \sum r_i = 1\}$ is the open $(n-1)$-simplex. Let

$$H := \left\{ x \in \mathbb{R}^n : \sum x_i = (2n - 4)\pi \right\}$$

be an affine hyperplane. Consider the continuous map $f : \Delta \to H$ with $f(r) = (\sigma_r(i))_{i=1,\ldots,n}$. We need to show $x^* = \left(\frac{2\pi}{3}, \ldots, \frac{2\pi}{3}, 2\pi, \ldots\right) \in f(\Delta)$. Then the claim follows from item iv.

vi) This function has the following properties

- f is injective
- f has a continuous extension to the boundary $\partial\Delta$ (meaning some $r_i = 0$), call it $\overline{f}$.
- Domain and image are both homeomorphic to unit balls $\overline{B}^{n-1}$ with boundary S^{n-2}. So we get an extended version

$$h_1 \circ \overline{f} \circ h_2 : \left(\overline{B}^{n-1}, S^{n-2}\right) \to \left(\overline{B}^{n-1}, S^{n-2}\right)$$

By a Theorem of , this function in surjective on its interior. Therefore, by transforming back, f is surjective. names □

Non-uniqueness

1.28 Notation. Let $C_{r,n}$ be the cyclic polytope of dimension n with r vertices.

1.29 Proposition. *Each l-subset of the vertices of $C_{r,n}$ forms a face if $l \leq \lfloor \frac{n}{2} \rfloor$.*

1.30 Theorem (McMullen, 1970). *Let P be an n-dimensional polytope with r vertices. Then $f_k(P) \leq f_k(C_{r,n})$ for all $0 \leq k \leq n$.*

1.31 Theorem (Van-Kampen, Flores, Grünbaum). *Let P, Q be polytopes of dimensions $n = \dim P < \dim Q = n'$. If $P_{\leq k} \cong Q_{\leq k}$ then $k \leq \lfloor \frac{n}{2} \rfloor - 1$.*
Here $P_{\leq k}$ is the k-skeleton, i.e. the poset of faces of dimension at most k.

1.32 Example. Let Δ_n be the n-dimensional simplex. Then $\Gamma(\Delta_n) \cong K_{n+1}$. If $n \geq 4$, then $\Gamma(C_{r,n}) \cong K_r$.

1.33 Example (Joswig, Ziegler). Let $S := [-1,1]^2$, the 2-dimensional square and

$$Q := \mathrm{conv}(S \times 2S \times \{-1\} \cup 2S \times S \times \{1\})$$

In terms of inequalities it is given by

$$\begin{aligned} -1 &\leq x_5 \leq 1 \\ \pm 2x_1 &\leq 3 - x_5 \\ \pm 2x_2 &\leq 3 - x_5 \\ \pm 2x_3 &\leq 3 - x_5 \\ \pm 2x_4 &\leq 3 - x_5 \end{aligned}$$

The graph is the same as the cube's.
Let $\pi : \mathbb{R}^5 \to \mathbb{R}^4$ be the linear projection onto the first 4 coordinates. Then $P := \pi(Q) = \mathrm{conv}(S \times 2S \cup 2S \times S)$ again is a polytope. If terms of face description, we do the Fourier-Motzkin elimination.

$$\begin{aligned} \pm x_i &\leq 2 & 1 &\leq i \leq 4 \\ \pm x_i \pm x_k &\leq 3 & j &= 1,2 \quad k = 3,4 \end{aligned}$$

Check $\Gamma(P) \cong \Gamma(Q) \cong \Gamma([0,1]^5)$.

Simplex Polytopes

1.34 Proposition. *Let P be an n-dimensional polytope. TFAE*

i) Each vertex is contained in precisely n facets.

ii) The graph $\Gamma(P)$ is n-regular.

iii) The dual polytope P^v is simplicial, i.e. each face is a simplex.

1.35 Definition. Let P be an n-dimensional polytope. We say P is *simple* iff

- $\Gamma(P)$ is n-regular
- exactly n facets through each vertex

- P^v is simplicial, i.e. every proper face is a simplex.

In the literature, this sometimes is called "primally non-degenerate".

1.36 Example.
- Every polygon is simple and simplicial
- Every simplex is simple and simplicial. These two are the only types, that have both properties.
- Cubes $[-1,1]^n$ are simple. Their cross polytopes $\operatorname{conv}\{\pm e_i : i = 1,\dots,n\}$ are simplicial.
- Cyclic polytopes are simplicial.

1.37 Theorem (Bluid, Mani, 1987; Kalai 1988). *Let P be a simple polytope. Then the graph $\Gamma(P)$ determines the combinatorial type of P.*

Proof. Let $\Gamma(P) = (V,E)$ and $\dim P = n$. Consider all acyclic orientations of Γ (always exist). If F is a nonempty face of P, then each acyclic orientation has at least one sink in $\Gamma(F)$ (e.g. the last node of a topological ordering). An acyclic permutation is called *good* if it has a unique sink on each nonempty face. Each generic linear objective function on the polytope induces a good orientation on $\Gamma(P)$. (In the literature, this is referred to as "dually non-degenerate.) Hence good orientations exist.
Let O be some (not necessarily good) acyclic permutation. Define

$$h_k^O := \#\{v \in V : \operatorname{indeg}(v) = k\} \qquad \text{for } k = 0,\dots,n$$

Then $h := (h_0,\dots,h_n)$ is called the h-vector of P w.r.t. O.
Let $v \in V$ with $\operatorname{indeg}(v) = k$. So v is contained in k edges as target. Every subset of them spans a face and all of its edges containing v, point to v. Thus, v is a sink in 2^k nonempty faces of P.

$$f^O := \sum_{i=0}^{n} h_i^O = \#\{(v,F) : v \in V, v \text{ is sink in face } F \neq \emptyset\}$$
$$f := \sum_{i=0}^{n} f_i = \#\{F : \emptyset \neq F \text{ face}\}$$

Then we observe

i) $f^O \geq f$, since f^O counts each nonempty face at least once

ii) O is good iff $f^O = f$: only in this case, we count each nonempty face only once

Now the claim follows from the subsequent Lemma 1.38. □

1.38 Lemma. *The facets of P bijectively correspond to the connected $(n-1)$-regular induced subgraphs of Γ, which are initial w.r.t. some good orientation.*

Proof. $\Rightarrow$ "clear"$^{\text{TM}}$, once you understand the definition, and what "initial" means.

$\Leftarrow$ Let Φ be a connected $(n-1)$-regular induced subgraph, which is initial w.r.t. some good orientation O. Let $s \in V$ be a sink in Φ. Then there exists a unique facet F of P through s spanned by $n-1$ edges through s in Φ. Since O is good, s is the unique sink in $\Gamma(F)$, i.e. all vertices in F come before s in O. Since Φ is initial, Φ contains all vertices before s. Hence $\Phi \supseteq \Gamma(F)$. Both are connected and $(n-1)$-regular. So $\Phi = \Gamma(F)$. □

1.39 Remark.

- J. Karbel-Köiur 2002: primal-dual
- Friedman 2009: polynomial time algorithm
- Stanley/Billera-Lee 1980: g-theorem (and the g-conjecture), characterises the f-vectors of all simplicial/simple polytopes
- dual good acyclic orientations = shelling; connection to classical/discrete Morse theory

Perturbation

1.40 Proposition. *For each n-polytope P there exists a simplicial n-polytope Q with the same number of vertices, such that $f_k(Q) \geq f_k(P)$ for all k and the (Hausdorff) distance between P and Q is arbitrarily small.*

Proof. See book by Joswig and Theobald; Lemma 3.48 (replace “simplex” by “pyramid”). □

1.41 Corollary. *For each n-polytope P there exists a simple n-polytope Q with the same number of facets such that $f_k(Q) \geq f_k(P)$ and P and Q are arbitrarily close.*

Proof. Dualise, then perturb it slightly according to Proposition 1.40, and then dualise back. □

1.42 Remark. It suffices to study simple polytopes for the (polynomial) Hirsch conjecture.

2 lectures missing

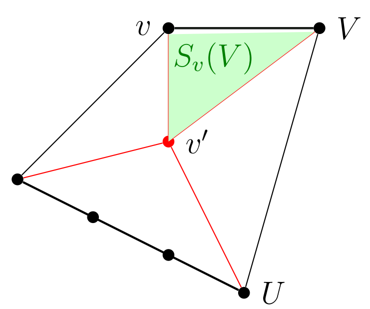
1.43 Definition. A *prismatoid* is a polytope with two distinguished disjoint facets U, V, that contain all the vertices. The *width* of a prismatoid is the dual graph distance between the two facets .

1.44 Theorem (Strong n-step theorem for prismatoids). *If Q is a prismatoid of dimension n with m vertices and width l, then there is another prismatoid Q' of dimension $m-n$ with $2m-2n$ vertices and width $l \geq l + m - 2n$. In particular for $l > n$ this exceeds the Hirsch-conjecture.*

Proof. Let $s := m - 2n$ be the *asymplicciality* of Q. Note $a \geq 0$, since U and V are disjoint. We do induction on s.

Base $s = 0$**:** Since $l \leq m$, our condition says Q' must have width ≥ 0, so we just pick $Q' := Q$.

Step: We want to construct a prismatoid $\widetilde{Q}$ of dimension $n+1$, $m+1$ vertices and width $> l$. Since $s > 0$, at least one facet U or V is not a simplex, wlog say U. Pick $v \in V$. Let $S_v(Q)$ be the 1-point-suspension of Q at v. This has $S_v(V)$ as a facet. Side U represents several



points (just hard to draw in the plane). We embed the original $Q \subseteq \mathbb{R}^n$ into $\mathbb{R}^{n+1}$, by setting height 0.

Now $S_v(Q)$ is almost a prismatoid, with facets $S_v(V)$ and U, but U is not a facet of $S_v(Q)$. Pick some vertex $u \in U$ and perturb it slightly (yields a facet, since U was not a simplex). This yields our facet, and by the perturbation the width increases by 1.

Each step introduces 1 new points, increases the dimension and width by 1. So after $m - 2n$ steps, we have the claim. □

Projective Geometry

Let K be an arbitrary field (although for later use in LP, we need K to be ordered). Then we define $\mathrm{PG}_d\, K$ to be the projective geometry of dimension d over K.

$$u \sim v :\Leftrightarrow \exists \lambda \in K . u = \lambda v$$
$$\mathrm{PG}_d\, K := K^{d+1} - \{\mathbf{0}\} / \sim$$

The projective geometry consists of all the linear subspaces of K^{d+1}. We have the embedding $K^d \hookrightarrow \mathrm{PG}_d\, K$ via $x \mapsto (1 : x)$. This just means we use the representative whose first entry is 1. But we additionally have points $(0 : x)$, which form our *points at infinity*.

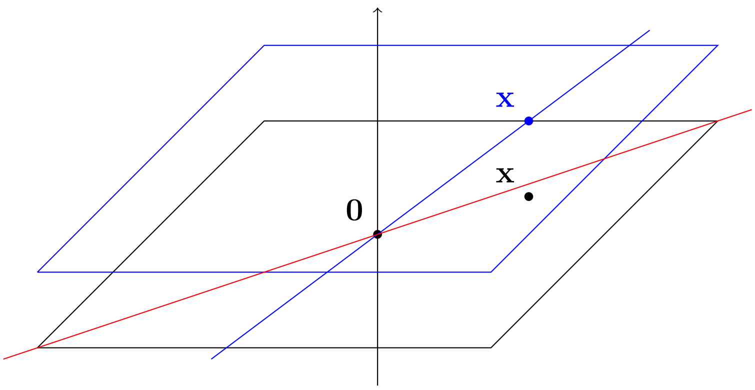


Figure 1: Any line with upwards slope hits $\{1\} \times K^2$ exactly one, so x (black) just gets additional entry 1 (blue). But the red line does not meet the blue plane, so it is considered as infinity.

As a special case, we have $\mathrm{PG}_1\, K = K \cup \{\infty\}$. The generalisation is

$$\mathrm{PG}_d\, K = K^d \cup \mathrm{PG}_{d-1}\, K = \bigcup_{k=1}^{d} K^k \cup \{\infty\}$$

A projective transformation is a bijection on $\mathrm{PG}_d\, K$, which maps subspaces to subspaces and preserves the cross ratio.

Example. *Each element in $\mathrm{GL}_{d+1}(K)$ induces a projective transformation. Any non-zero multiple of $\mathbf{1}$ induces the identity.*

This gives rise to the projective general linear group $\mathrm{PGL}_{d+1}(K) = \mathrm{GL}_{d+1}(K)/(K^* \cdot \mathbf{1})$.
Now consider the case $K = \mathbb{R}$. A set of linear inequalities may describe one of the following

1. polytope

2. unbounded polyhedron
3. something else

We say, transformation π is admissible for P, if we have one of the first two cases.

- All affine transformations are admissible for all polytopes.
- Consider polytopes in $\mathbb{R}^d_{\geq 0} \subseteq \mathrm{PG}_d\,\mathbb{R}$.. Take any projective transformation $\pi \in \mathrm{GL}_{d+1}\,\mathbb{R}$ such that all coefficients are nonnegative. Then π is admissible for hyperplanes.

2 Pivoting, Klee-Minty-Cubes, etc.

Simplex Method

Input : A, b, c, vertex v

Output optimal vertex, or "unbounded"

- $I \leftarrow$index set of a basis of v
- (*) determine $y \in (\mathbb{R}^m)^*$ with $yA = c$ and $y_i = 0$ for all $i \notin I$
- if $y \geq 0$, return v
- (**) $i =$ an index with $y_i < 0$.
- $s =$ column of $-(a_I)^{-1}$ with index i such that $A_{I-i}s = 0$ and $a_i s = -1$
- if $As \leq 0$: return "unbounded"
- $\lambda_s = \min\left\{\frac{b_j a_{-j} v}{a_j s} : j.a_j s > 0\right\}$, $j =$ row where minimum is attained
- $I = (I - i) \cup \{j\}$, $v = v + \lambda_s s$
- goto (*)

Bland's pivot rule, to avoid cycling

(**) choose i minimal

(***) choose j minimal

Klee-Minty

$$\mathrm{KM}(n, \varepsilon) = \min\{x_n : 0 \leq x_1 \leq 1, \forall i \geq 2. \varepsilon x_{i-1} \leq x_i \leq 1 - \varepsilon x_{i-1}\}$$

2.1 Lemma. *For $0 \leq \varepsilon < \frac{1}{2}$ the polytope is combinatorially equivalent to $[0,1]^n$.*

Proof. Consider disjoint pairs of facets. This yields our binary enumeration of the vertices of the cube. $\square$

Let x be a vertex of $\mathrm{KM}(n,\varepsilon)$, which corresponds to a bitstring with exactly k ones at $s_1 < \ldots < s_k$. Then the last coordinate is

$$\varepsilon^{n-s_1} - \varepsilon^{n-s_2} + \ldots + (-1)^{k-1}\varepsilon^{n-s_1} = \sum_{i=1}^{k}(-1)^{k-i}\varepsilon^{n-s_i}$$

Proof by induction on n. We impose direction on the edges of our feasible polytope, by $v \to v'$, if v' has a better objective value.
This we can characterise as follows: Now let x, x' be two bit-encodings of vertices which differ in exactly their i-th bit. Then the edge is directed as $x \to x'$ iff $\sum_{j=i}^{n} x_j$ is odd. This gives a recursive model of the vertex-edge graph $\Gamma(\mathrm{KM}(n,\varepsilon))$.
The $n-1$ dimenisonal KM-cube appears as faces of the n-dimensional KM-cube.

2.2 Theorem. *Bland's simplex algorithm applied to* $\mathrm{KM}(n,\varepsilon)$ *starting at* $0\ldots01$ *visits all* 2^n *vertices.*

2.3 Corollary. *Bland's Simplex algorithm has exponential worst-case running time.*

Proof. Fix constant ε, e.g. $\varepsilon = \frac{1}{4}$. Then we have $\Theta(n)$ inequalities of constant size, so input size $\Theta(n)$, but running time $\Omega(2^n)$. □

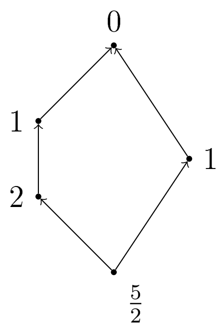
2.1 Random Pivoting

Fix the LP (A, b, c) given by

$$\max\{cx : Ax \leq b\}$$

and set $P := P(A, b)$. Assume P is bounded and c generic (distinct vertices have different values). The analysis can be generalised to DAGs (directed, acyclic graphs).
Our measure is *the expected path length* (the expected number of steps) from any given vertex of P to the (unique) optimum $\mathrm{top}(P)$.

Random-Edge At any non-optimal vertex x of P, follow one of the improving edges leaving x with equal probability.



2.4 Theorem (Gärtner, Henk & Ziegler, 1998). *The expected number* E_n *of steps that Random-Edge will take, starting at a random vertex of of* $\mathrm{KM}(n,\varepsilon)$ *is bounded by*

$$\frac{n^2}{4(H_{n+1}-1)} \leq E_n \leq \binom{n+1}{2}$$

Proof of $\le$, following Kelley. We have the recursion

$$E_n(x) = 1 + \frac{1}{\#\{x' : x \to x'\}} \sum_{x' : x \to x'} E_n(x')$$

Denote by $i(x)$ the highest index i such that the corresponding bit is $x_i = 1$. Then

$$i(x) \le E_n(x) \le \binom{i(x)+1}{2} \le \binom{n+1}{2}$$

which gives the upper bound. The lower bound is more involved, so we skip it here. □

Random-Facet (Kalai, 1992) If x has only one improving edge, take it. Otherwise randomly choose some facet F containing x, recursively call Random-Facet(F, x), arriving at top(F) and continue recursively.

Earlier, we selected oone edge, that we took. Now, for simplex polytopes, choosing a facet is the same as discarding an edge. So this strategy is "opposite" to Random-Edge.

2.5 Theorem (Gärtner, Henk & Ziegler, 1998). *Random-Facet on* KM(n, ε) *started at* $\mathbf{1}$ *to* top $= 0$ *takes expected number of steps*

$$F_n(\mathbf{1}) = n + 2\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k+2}\binom{n-k}{2} \approx \left(\frac{\pi}{4} - \frac{1}{2}\right) n^2$$

For random starting vertex we have

$$F_n = \frac{1}{2^n} \sum_x F_n(x) = \frac{n^2+3n}{8}$$

Proof. Observation: For unit vectors, we have $F_n(e_i) = F_n(e_1 + e_{i-1}) = i$, so $F_n(e_n) = n$.
Let $x \in \{0,1\}^n$. Restricting to facet with $x_i = 0$ yields top $= \mathbf{0}$. Restricting to facet with $x_i = 1$ yields $e_1 + e_{i-1}$, where $e_0 = \mathbf{0}$. Denoting

$$x^{(i)} = (x_1, \ldots, x_{i-2}, x_{i-1} - x_i, x_{i+1}, \ldots, x_n) \in \{0,1\}^{n-1}$$

we get

$$F_n(x) = \frac{1}{n}\left(\sum_{i=1}^{n} i x_i + \sum_{i=1}^{n} F_{n-1}\big(x^{(i)}\big)\right)$$

What happens with $x = \mathbf{1}$?

□

several lectures missing

3 Interior Point Methods

We can solve linear equation systems by Gauss and if we generalise this to LP, we arrive at the Simplex Method. If we generalise Newton's Method, we end up with interior point methods.