Public Key Cryptography

Henning Seidler

April 25, 2024

Organisation

- one lecture a week
- irregular example sheets, including programming tasks
  - Install Python, including IPython
  - at least one task needs SageMath
  - you are advised to create your own tools collection
- notes/slides, example sheets on ISIS

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Exam

- part of "Secure Cryptography"
  cannot be examined alone
- oral exam at the end, about PKC and CryptoSec
- details will be announced towards the end of the semester
- prior to the exam no registration necessary

Organisation – Further Information

Questions?

- check course description(!)
- read announcements
- ask in the forum
- only if question contains private information, mail henning.seidler@tu-berlin.de

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Literature

- Galbraith – "Mathematics of Public Key Cryptography"
- "Handbook of Applied Cryptography" (?)
- Wikipedia/Write-Ups/papers/...

Tell me, if you find a matching text book.

Goals

There are two kinds of cryptography in this world: cryptography that will stop your kid sister from reading your files, and cryptography that will stop major governments from reading your files. This [lecture] is about the latter.

(Bruce Schneier)

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>
> *(Bruce Schneier)*

- give you an overview about Public Key Crypto
- typical encryption schemes
- also tell you, what can go wrong
- including practical tasks
- prepare you for CTFs

$> whoami && jobs

AG Rechnersicherheit

- We're a registered student organization. Basically a group of students interested in (IT) security topics.
- Weekly Meetups:
  - Tue, 6pm - 8pm
  - TEL 20, Auditorium 3 and via Jitsi
  - Techtalks and discussions about recent events or techniques
  - No knowledge needed - just be interested and eager to learn new things! :-)
- We participate in hacking contests (CTFs) as ENOFLAG/LEGOFAN/Last-Email!

$> cat ~/.todo

Upcoming events

this Saturday: Bambi-CTF (beginner)

- Attack-Defense CTF
- Exploit other teams while fixing our own vulnerabilities

several weekends:

play Jeopardy CTFs

- tasks with security flaws
- find secret code (Flag)

?? 2024: FaustCTF

8.6.2024: CryptoCTF

22.6.2024 ~~LNDW~~

$> exit

See/Hear you on Tuesday :-)

- Auditorium 3 @ TEL 20. floor on Tuesday 6 - 8 pm
- `https://meet.enoflag.de/erstis` on Tuesday 6 - 8 pm
- E-Mail: hi@enoflag.de / mailing list
- Links: `https://enoflag.de` and `https://www.agrs.tu-berlin.de`

Story

How this lecture was created:

- played RuCTFe, had an afterparty
- after drinks and pizza (ca. 1 AM),
  Júlia: "We should teach each other, what we know."
  me: "I could teach crypto."
  next morning, it still seemed a good idea
- Winter 18/19: course of 8 lectures and exercises in AGRS
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Questions so far?

What is Cryptography?

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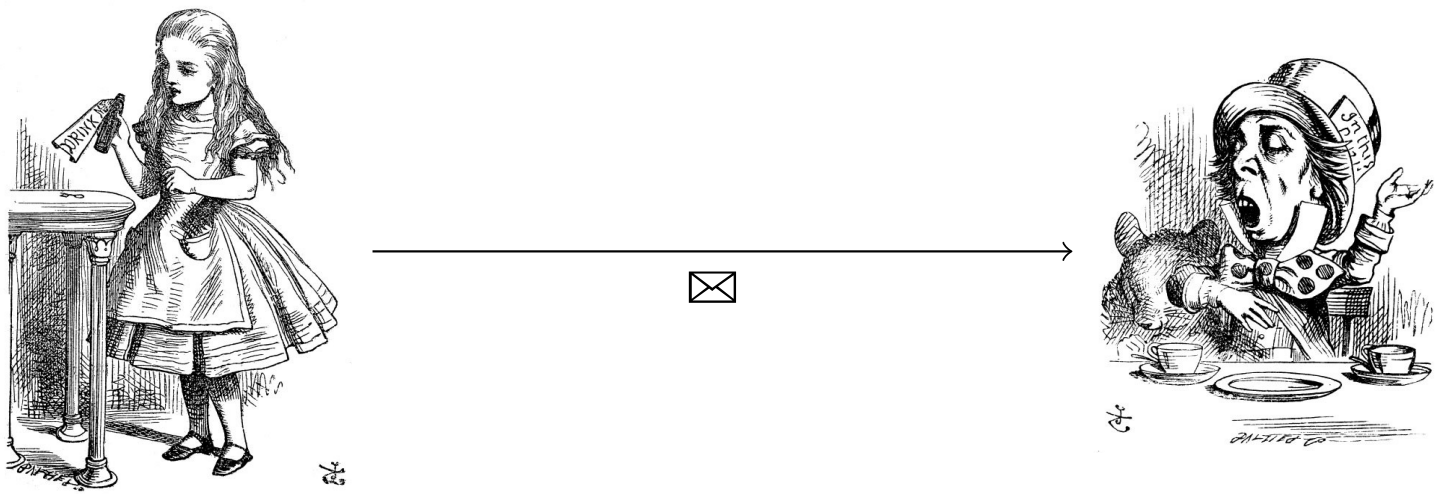

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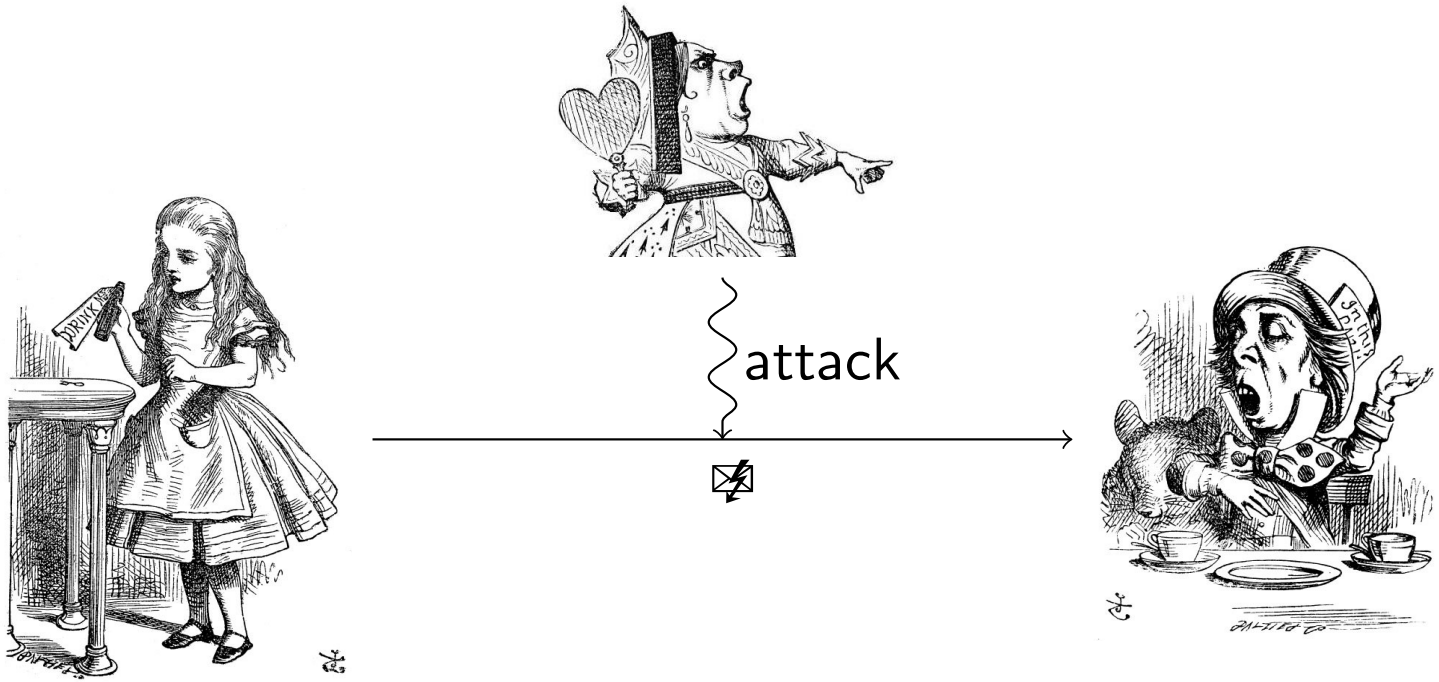

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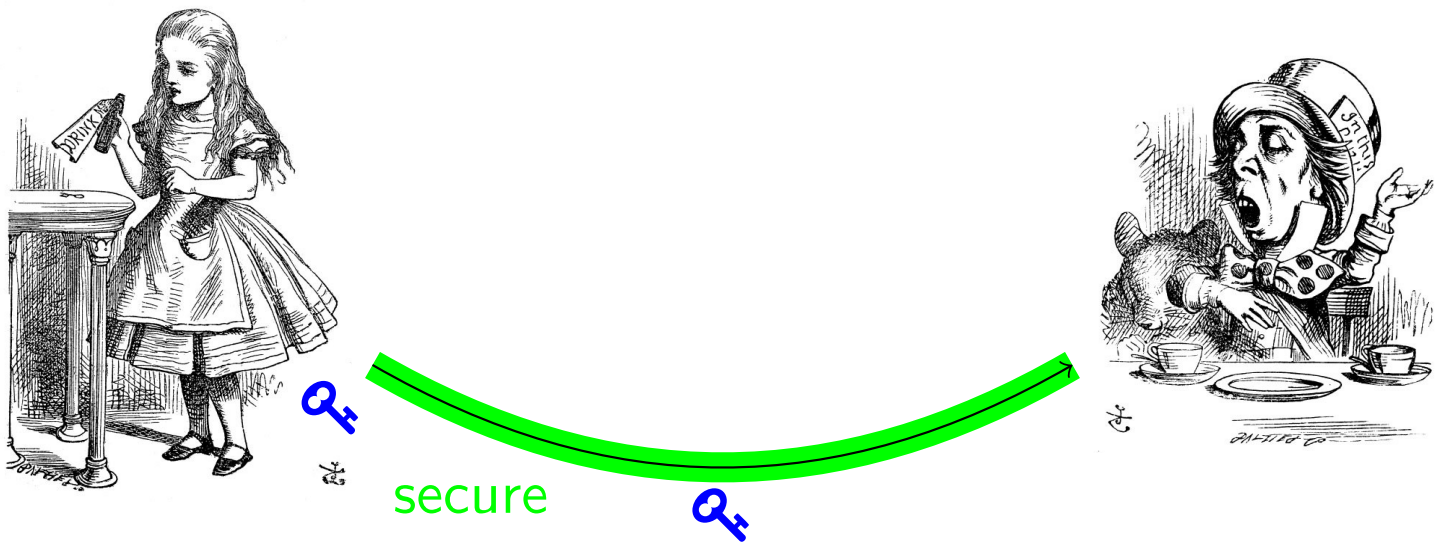

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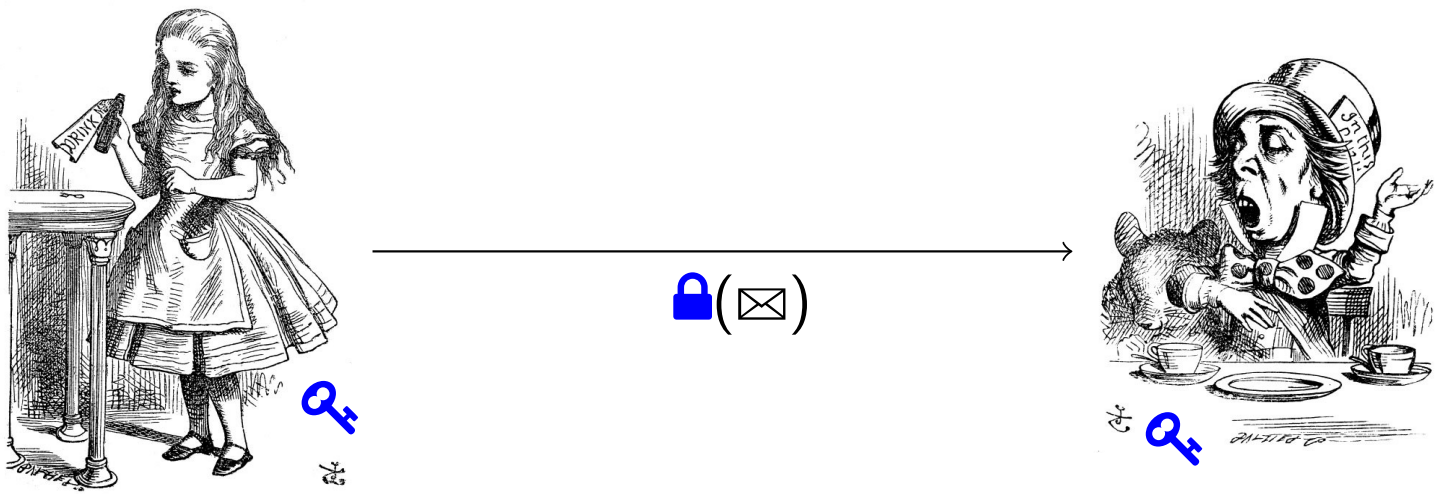

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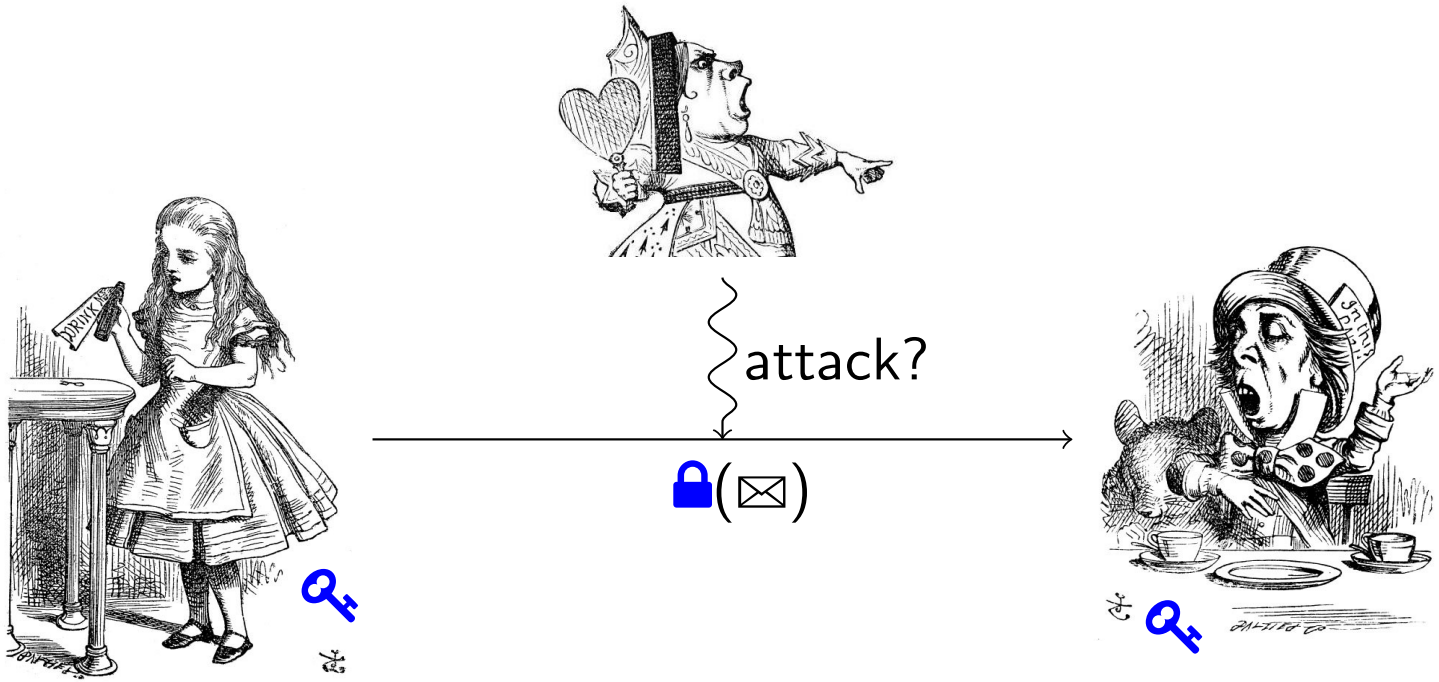

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Asymmetric:

What if we do not have a secure connection?

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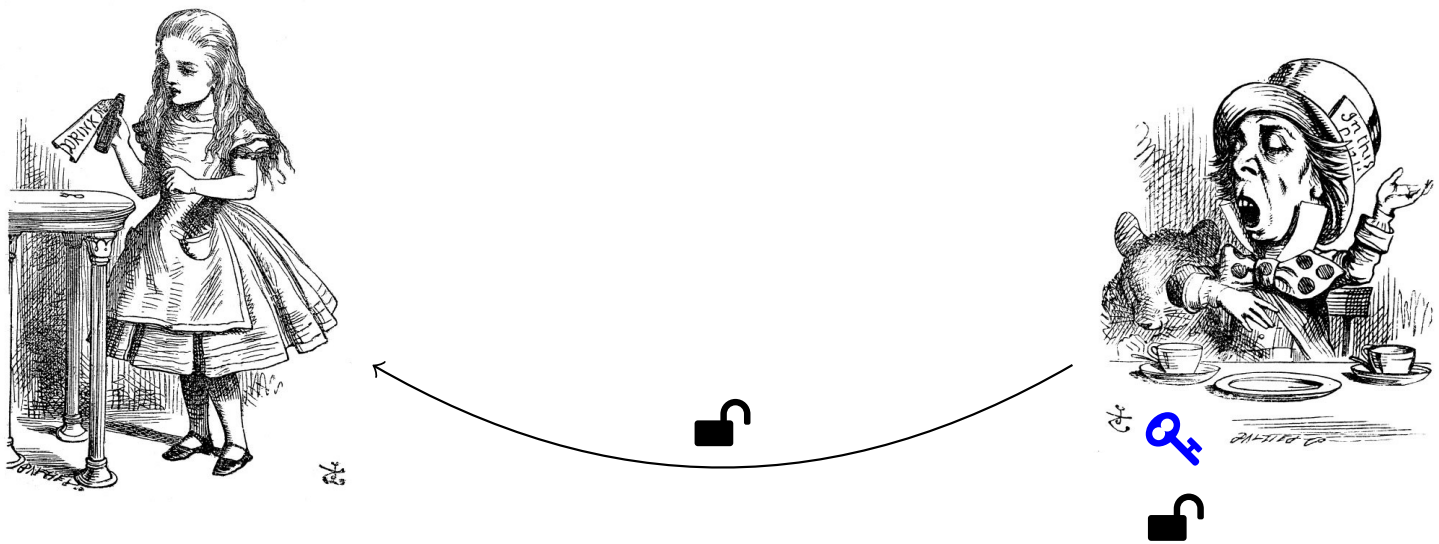

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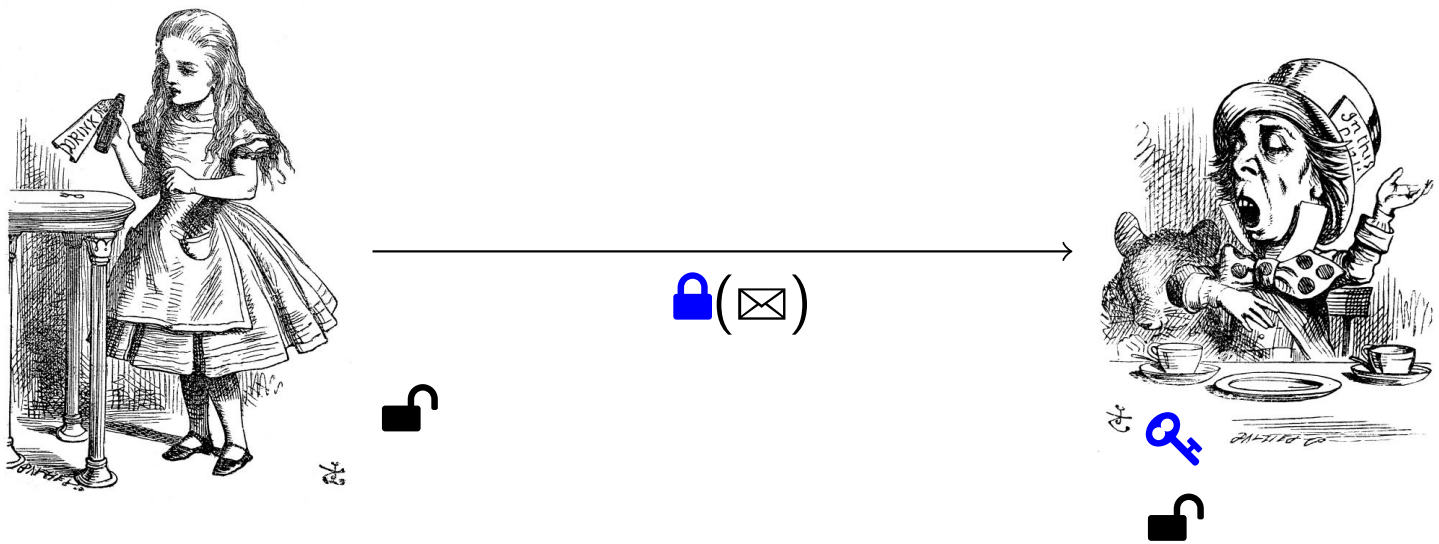

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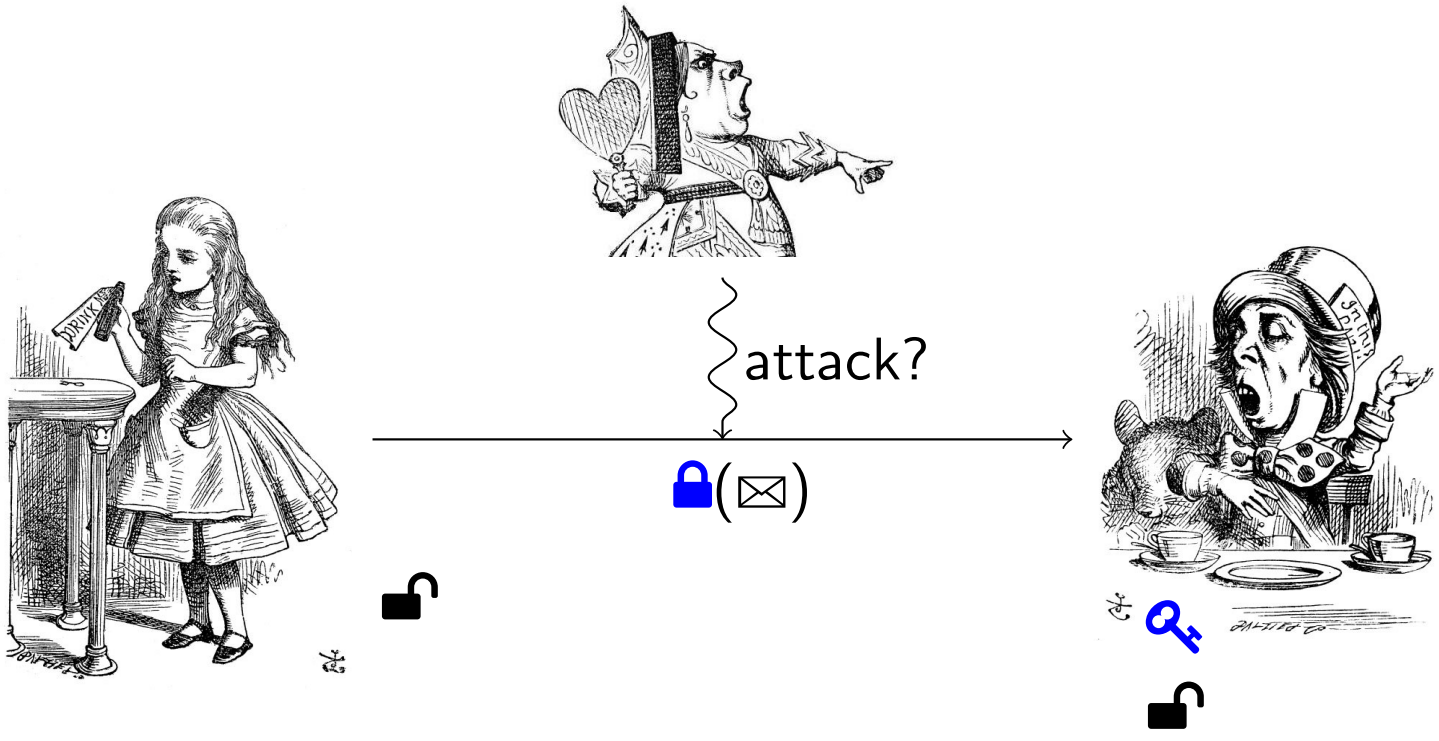

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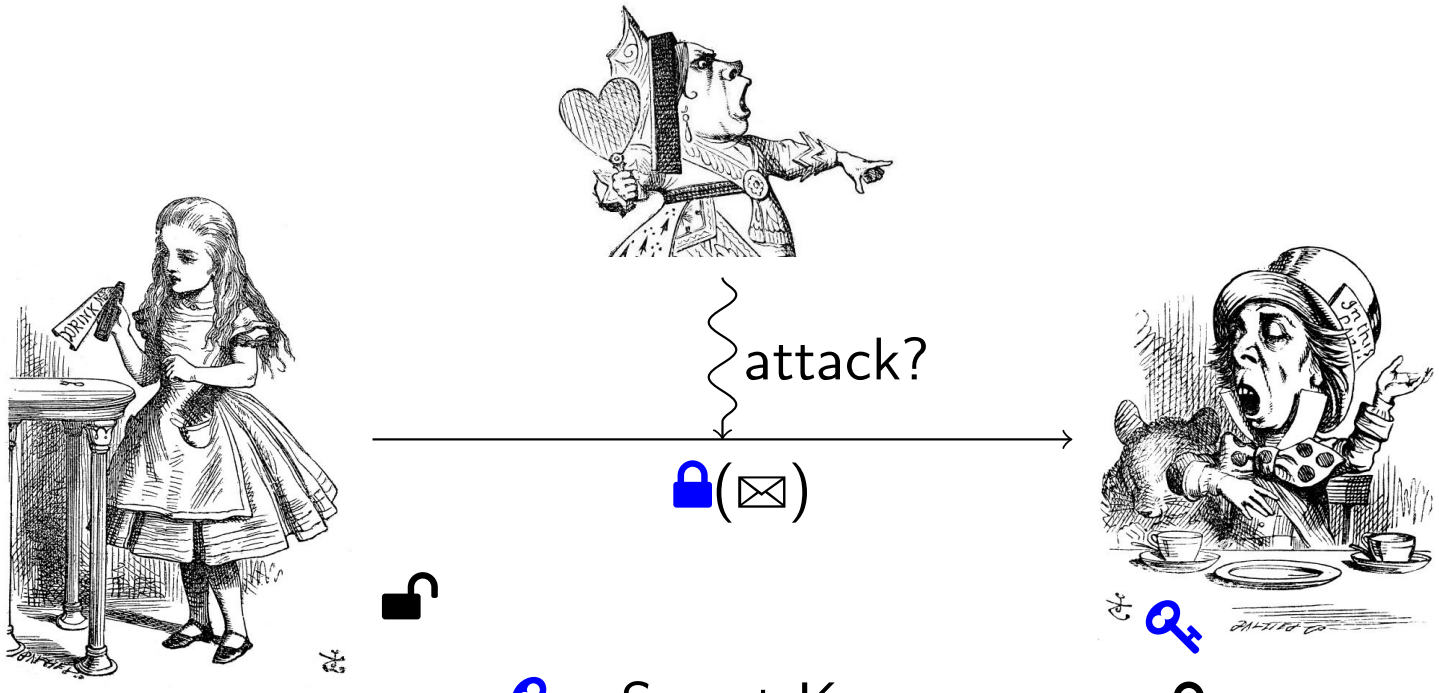

🔑: Secret Key

🔓: Public Key

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cryptography: (in strict sense) design of cryptosystems

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Kerckhoff's Principle (Open Design)

enc and dec are known,
only key k_{dec} secret (and k_{enc} if both same)

Mathematical Model

Definition (Cryptosystem)

A cryptosystem is a quintuple $(P, C, K, \mathrm{enc}, \mathrm{dec})$ where

- P is the set of all plaintexts
- C is the set of all ciphers
- K is the set of all keys/key pairs
- $\mathrm{enc} : P \times K \rightsquigarrow C$ is the encryption relation (not necessarily a map)
- $\mathrm{dec} : C \times K \to P$ is the decryption function
- $\forall m \in P, k \in K \,.\, \mathrm{dec}\left(\mathrm{enc}\left(m, k\right), k\right) = m$, or $\forall m \in P, (k_{\mathrm{dec}}, k_{\mathrm{enc}}) \in K \,.\, \mathrm{dec}\left(\mathrm{enc}\left(m, k_{\mathrm{enc}}\right), k_{\mathrm{dec}}\right) = m$
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Kerckhoff: The whole cryptosystem in known.

Attack Scenarios

What does Eve know?

CO: ciphertext only

KP: known plaintext, i.e. pairs of cipher and message

CPA: chosen plaintext attack

CCA1: chosen cipher attack, at the beginning, Eve can request decryption for chosen ciphers

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Example

- CPA: minimum for public key crypto
- CCA2: impersonate authentication server (ssh login)

Attack Scenarios

What is a success?

OW: one-way, decrypting cipher

NM: non-malleability, change cipher that decryption still yields a meaningful message

PA: plaintext awareness, generate a cipher, whose decryption yields a meaningful message

IND: indistinguishability, which given cipher matches given message answer must be significantly better than guessing

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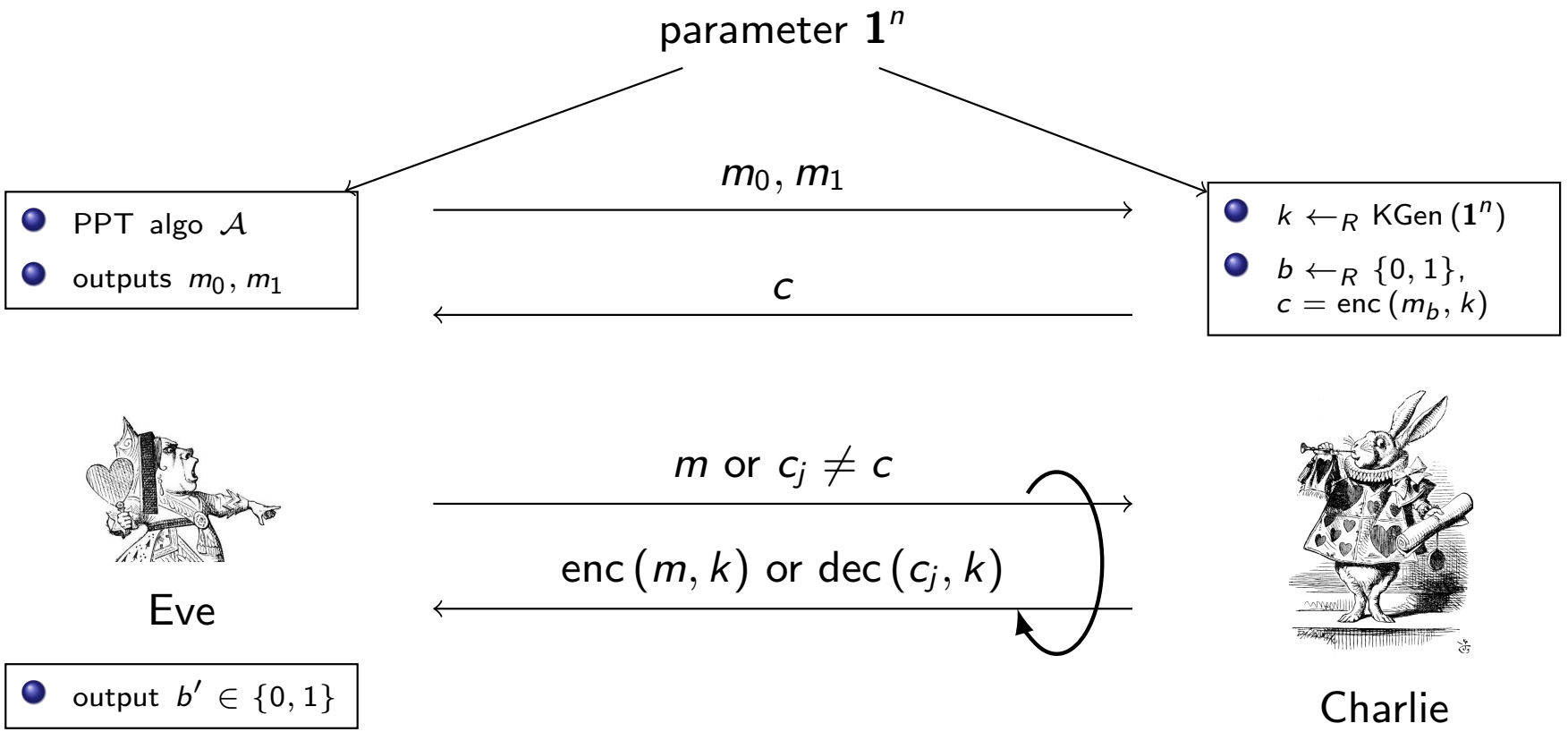
- combine attacker's power and notion of success
- Strongest goal: IND-CCA2

The IND-CCA2 Game



- Eve wins if $b' = b$
- PPT = probabilistic polynomial time
- secure if for every $\mathcal{A}$ and every polynomial p, Eve's winning chance is less than $\frac{1}{2} + \frac{1}{p(n)}$

Caesar-Cipher

by Albert Uderzo,

taken from

https://asterix.fandom.com/de/wiki/Julius_C%C3%A4sar

100 BC - 44 BC

Simple substitution

- A $\rightarrow$ D
- B $\rightarrow$ E
- C $\rightarrow$ F
- ...

(much) later with arbitrary shift

Polyalphabetic Ciphers

Renaissance

- Johannes Trithemius
- Giovan Battista Bellaso
- Leon Battista Alberti
- **Blaise de Vigenére**

Different Ceaser-ciphers for different letters, depending on keyword

Broken by

- Charles Babbage (1854)
- **Friedrich Wilhelm Kasiski** (1863)

find length of keyword

- search for blocks that occur multiple times
- greatest common divisor of differences of their occurrences $\leadsto$ keylength
- then break separate indices by frequency

Rotor Machines

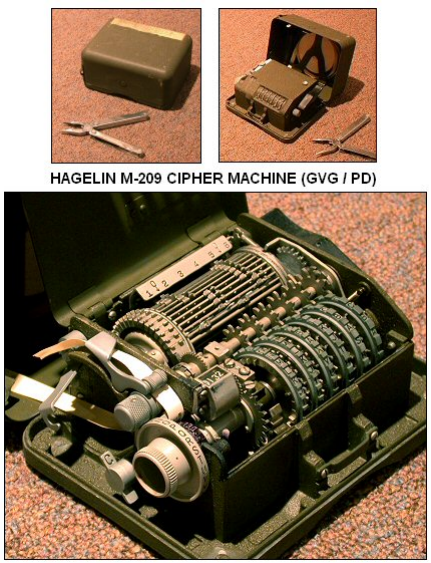
Starting during and after World War I

Enigma by Arthur Scherbius, broken by project "Ultra" (Alan Turing)

M-209 by Boris Hagelin, used by USA,
broken by German cryptoanalysts, from 1943 on

several others

Enigma



M-209

Public Key Cryptography

DH key exchange: Whitfield Diffie, Martin Hellman, 1976

- works in a group
- nowadays mostly elliptic curves over a finite field

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Enter Mathematics

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multiplicative inverse: a^{-1} is the number with $a \cdot a^{-1} = 1$

- not always possible
- works iff $\gcd(a, n) = 1$
- if n prime, works for all $0 < a < n$

Example Ring – CPU/ALU

- modern CPU uses 64 Bit $\rightsquigarrow$ can save 2^{64} numbers
- all computations run modulo 2^{64}
- $1 \ldots 1_2 = 2^{64} - 1 = -1$
- for arithmetic, ALU does not care about signed/unsigned

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"Negative" Numbers

- $-a$ is the number that satisfies $a + (-a) = 0$
- say $\overline{a}$ is a with all bits flipped,
- $a + \overline{a} = 1 \ldots 1$ (in every bit add 0 and 1)
- $a + \overline{a} + 1 = (1)0 \ldots 0 = 0$ (overflow)
- hence $-a = \overline{a} + 1$

Algorithms

Need algorithms for the following: (b bits input)

- addition, subtraction,
- efficient multiplication
- division with remainder in $\mathbb{Z}$, in particular modulo-operator
- division/multiplicative inverse in $\mathbb{Z}_p$ (or $\mathbb{Z}_n$, if possible)

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- naive/school-method: $\mathcal{O}(b^2)$
- Karatsuba: divide-and-conquer, $\mathcal{O}\left(b^{\log_2 3}\right)$
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Division

reduce to multiplication, same complexity

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Theorem (Extended Euclidian Algorithm)

For all $a, b \in \mathbb{Z}$ there are $s, t \in \mathbb{Z}$ such that

$$s \cdot a + t \cdot b = \gcd(a, b)$$

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  if b == 0: return (a,1,0)
  d,s,t = EEA(b, a % b)
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- compute $d, s, t = \mathrm{EEA}(a, n)$, clearly $d = 1$
- $1 = s \cdot a + t \cdot n$ means $s \cdot a \equiv 1 \mod n$
- so $s = a^{-1}$ in $\mathbb{Z}_n$

Chinese Remainder Theorem

Theorem (Chinese Remainder Theorem (CRT))

Let $n_i \in \mathbb{Z}$ (pairwise) coprime, $a_i \in \mathbb{Z}$ arbitrary for $i = 1, \ldots, k$. Then the system

$$a_i \equiv x \mod n_i \qquad i = 1, \ldots, k$$

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Algorithm for 2 congruences:

$$1, s, t = \text{EEA}(n_1, n_2) \rightsquigarrow \qquad 1 = s \cdot n_1 + t \cdot n_2$$
$$\text{solution} \qquad x := a_2 \cdot s \cdot n_1 + a_1 \cdot t \cdot n_2$$

continue recursively with: $a' = x$ and $n' = n_1 \cdot n_2$

Example (CRT)

- Consider the system

$$x \equiv 1 \mod 3$$
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$$\begin{aligned} x &\equiv 1 \mod 3 \\ x &\equiv 2 \mod 5 \\ x &\equiv 3 \mod 7 \end{aligned}$$

- combine first two:

$$\mathrm{EEA}(3,5) = (1,2,-1) \qquad \text{as } 1 = 2 \cdot 3 + (-1) \cdot 5$$
$$\rightsquigarrow a' = 2 \cdot 2 \cdot 3 + 1 \cdot (-1) \cdot 5 = 7$$

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- reduced system, continue recursively

$$\begin{aligned} x &\equiv 7 \mod 15 \\ x &\equiv 3 \mod 7 \end{aligned}$$

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Theorem

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$$(a+1)^p = \sum_{k=0}^{p} \binom{p}{k} a^k = a^p + 1 + \sum_{k=1}^{p-1} \underbrace{\frac{p(p-1)\dots(p-k+1)}{1 \cdot 2 \cdot \dots \cdot k}}_{p \text{ divides this}} a^k$$

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□

Corollary (Alternative formulation)

p prime, a coprime to p (i.e. no multiple), then $a^{p-1} \equiv 1 \mod p$.

Euler's Theorem

Generalise Fermat's Little Theorem:

Definition (Euler's Phi-Function)

$$\varphi(n) := |\mathbb{Z}_n^*| = |\{a \in \mathbb{Z}_n : \gcd(a, n) = 1\}|$$

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Theorem (Euler)

Let $n \geq 2$ and $a \in \mathbb{Z}_n^*$. Then $a^{\varphi(n)} \equiv 1 \mod n$.

Euler's Theorem

Generalise Fermat's Little Theorem:

Definition (Euler's Phi-Function)

$$\varphi(n) := |\mathbb{Z}_n^*| = |\{a \in \mathbb{Z}_n : \gcd(a, n) = 1\}|$$

Lemma (How to compute $\varphi(n)$?)

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Theorem (Euler)

Let $n \geq 2$ and $a \in \mathbb{Z}_n^*$. Then $a^{\varphi(n)} \equiv 1 \mod n$.

Special case: $n = p$ prime, $\varphi(p) = p - 1$, exactly Fermat
Proof e.g. via group theory (Lagrange's Theorem).

Prime Number Theorem

Theorem (Prime-Number-Theorem)

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$$\pi(100) = 25 \qquad \sim 1/4 \text{ of numbers}$$
$$\pi(10\ 000) = 1229 \qquad \sim 1/8 \text{ of numbers}$$

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RSA

Setup:

- p, q primes
- $n := p \cdot q \implies \varphi(n) = (p-1)(q-1)$
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Usage:

- encrypt $c = m^e \bmod n$
- decrypt $m = c^d \bmod n$

Example (Key Generation)

- choose primes $p = 47$ and $q = 97$, yields $n = 4559$
- choose $e = 17$
- $\varphi(n) = (p-1)(q-1) = 4416$
- `_,d,_ = EEA(e, phi)`, yields $d = 3377$

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Example (En-/Decryption)

- message $m = 102$ (first letter of `flag...`)
- encrypt: cipher

$$c = m^e \bmod n = 102^{17} \bmod 4559 = 2993$$

- decrypt: get back message

$$m = c^d \bmod n = 2993^{3377} \bmod 4559 = 102$$

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Theorem

For every message $0 \leq m < n$ *we have* $m = (m^e)^d \bmod n$.

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If $p \mid m$, then $m^{ed} \equiv 0 \equiv m \mod p$. Else

$$m^{ed} \equiv m^{1+k(q-1)(p-1)} \equiv m \cdot \left(m^{p-1}\right)^{k(q-1)} \equiv m \cdot 1^{k(q-1)} \equiv m \bmod p$$

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So $m^{ed} \equiv m \mod p$. Analogue for q.
Hence $m^{ed} \equiv m \mod n$ by CRT. □

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Generating primes of B bit

- generate random bit sequence $p = p_{B-1} \dots p_1 1$ (last bit 1)
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  $e \in \{3, 5, 17, 257, 65537\} = \{2^{2^k} + 1 : k = 0, \dots, 4\}$
  Fermat primes: coprime iff $e \nmid \varphi(n)$,
  $e = 10 \dots 01_2$, only $2^k + 1 \leq 17$ multiplications $\rightsquigarrow$ fast

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Need to compute $a^b \mod n$

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1: **function** $\textsc{Pow}(a, b, n)$
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7: $\quad$ **return** c

total: $\leq 2 \log b$ mult. and mod of size $\log n$

Example

- modulus $n = 4559$
- public exponent $e = 17 = 2^4 + 1$
- message $m = 102$

Further computations in $\mathbb{Z}_n$:

$$
\begin{aligned}
c &:= 102 && && && m^1 \bmod n \\
c &:= 102^2 && = 1286 && && m^2 \bmod n \\
c &:= 1286^2 && = 3438 && && m^4 \bmod n \\
c &:= 3438^2 && = 2916 && && m^8 \bmod n \\
c &:= 2916^2 && = 521 && && m^{16} \bmod n \\
c &:= 521 \cdot 102 && = 2993 && && m^{17} \bmod n
\end{aligned}
$$

In total: 5 multiplications

Further Optimisation

During setup also compute (once)

$$d_p = d \bmod p - 1 \qquad d_q = d \bmod q - 1 \qquad q_{\text{inv}} = q^{-1} \bmod p$$

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(Special cases $p \mid c$ and $q \mid c$.) Hence, $m = c^d \bmod n$ by CRT. □

Complexity Analysis

$$c_p = c \bmod p \qquad c_q = c \bmod q$$

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via CRT $3 + 2 \cdot \frac{3}{2} \cdot \frac{B}{2}$ op.s of size $\frac{B}{2}$, 1 mod of size B
$\sim \frac{3}{2}B$ op.s of size $\frac{B}{2}$
factor 2-4, depending on multiplication method
can be run in parallel $\rightsquigarrow$ another factor 2

Example

- private key: $n = 4559$, $d = 3377$, $p = 47$, $q = 97$
- compute once: $d_p = 19$, $d_q = 17$, $q_{inv} = 16$

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- decrypt $c = 2993$

$$c_p = 32 \qquad c_q = 83$$

$$m_p = c_p^{d_p} \bmod p = 8 \qquad m_q = c_q^{d_q} \bmod q = 5$$

$$h = q_{\text{inv}}(m_p - m_q) \bmod p = 1$$

$$m = m_q + hq \bmod n = 102$$

Note: These computations are nearly possible by hand.

Definition (Addition Chain, D. Knuth, TAOCP Vol. 2)

An addition chain for integer n of length l is a sequence

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$$x^2 = x \cdot x \mod n \qquad x^{12} = x^6 \cdot x^6 \mod n$$
$$x^3 = x^2 \cdot x \mod n \qquad x^{15} = x^{12} \cdot x^3 \mod n$$
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Problem with Addition Chains

Let $l(n)$ denote length of smallest addition chain for n.

Finding $l(n)$ is hard

Let $|n|_1$ denote number of 1s. Known bounds are:

$$\log n + \log |n|_1 - 2.13 \leq l(n) \leq \log n + |n|_1 - 1$$

$$l(n) \in \log n + (1 + o(1)) \cdot \frac{\log n}{\log \log n}$$

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Given $a_1, \ldots, a_k$, find smallest chain containing them all is NP-complete.

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Exercise (Challenge)

Find a small addition chain for $2^{127} - 3$.

Parameter Size

- strength of key given in bit size of n
- `ssh-keygen` currently has default 3072
- secure key should have 4096; more threatened by Quantum Computers, than classical factoring
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Assume $n \sim 4096$ Bit, $e = 65537$.

- Encryption: 17 op.s of size 4096
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Can we swap the effort? NO! (see later)

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Corollary

If d is known, it is not sufficient to just replace e and d.
need new primes (computational effort)

Exercise

We can also break the key, if d_p or d_q is given beside the public key.

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p, q **known** see key generation

$\varphi(n)$ **known** solve quadratic equation:

$$a := n - \varphi(n) = p + q - 1 \qquad \text{known}$$
$$n = p \cdot q = p \cdot (a + 1 - p)$$

Equation $x^2 - (a+1)x + n = 0$ has two solution: p, q

Theorem of Secret Parameters – d known

Assume d is known.

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2: $s \leftarrow \mathsf{M}_2(ed - 1)$ ▷ multiplicity of 2
3: $k \leftarrow \frac{ed-1}{2^s}$ ▷ "odd part"

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    pick random $0 < a < n$
    if $\gcd(a, n) > 1$ **then** ▷ very low chance
      return gcd
    for $i = 0, \ldots, s - 1$ **do**
      if $\gcd\left((a^k)^{2^i} - 1, n\right) \notin \{1, n\}$ **then**
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9: **if** $\gcd\left((a^k)^{2^i} - 1, n\right) \notin \{1, n\}$ **then**
10: **return** gcd

Chance of success $\geq \frac{1}{2}$ per loop,
but the "why" is more complicatded

Notation: $ed - 1 = * \cdot \varphi(n) = k \cdot 2^s$, k odd

Interesting Code part:

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Look for first step i with $\gcd\left((a^k)^{2^i} - 1, n\right) > 1$, (could be n)
but if not, the gcd is p or $q \rightsquigarrow$ know everything

Proof Idea.

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$$(a^k)^{2^s} - 1 \equiv 0 \mod n$$

- congruence also holds modulo p, q
- also possibly for smaller exponents x, y (pick smallest)

$$(a^k)^{2^x} - 1 \equiv 0 \mod p \qquad (a^k)^{2^y} - 1 \equiv 0 \mod q$$

- assume x, y differ (chance $\geq 50\%$), wlog $x < y$

$$(a^k)^{2^x} - 1 \equiv 0 \mod p \qquad (a^k)^{2^x} - 1 \not\equiv 0 \mod q$$

$$\implies \gcd\left((a^k)^{2^x} - 1, n\right) = p$$

- try out all $x \rightsquigarrow$ success

Groups

Definition

A group is a structure $\mathcal{G} = (G, \circ, *^{-1}, 1)$ such that

- $\circ$ is associative
- $\forall g \in G \,.\, 1 \circ g = g = g \circ 1$ (neutral element)
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- $(\mathbb{Z}, +)$, $(\mathbb{Z}_n, +)$
- $(\mathbb{Z}_n^*, \cdot)$ all numbers coprime to n
- S_n: the group of permutations of n elements
- point addition on elliptic curves ($\rightsquigarrow$ later section)

Some more Algebra

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Lemma (Properties of order)

Let G be a group, $g \in G$

- $o(g) \mid |G|$ *(element order divides group order),* $g^{|G|} = 1$
- *If* $g^n = 1$, *then* $o(g) \mid n$.

Definition

Let G be a group, $g \in G$. If $o(g) = |G|$, then G is called cyclic, and g is called generator. Equivalently: $G = \langle g \rangle = \{g^n : n \in \mathbb{Z}\}$.

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Lemma

The multiplicative group of every finite field is cyclic.

In particular for prime p there is some $g < p$ such that $\mathbb{Z}_p^* = \langle g \rangle$.

Notation: $ed - 1 = x \cdot \varphi(n) = k \cdot 2^s$

Proof (cont.)

$$\mathbb{Z}_n \cong \mathbb{Z}_p \times \mathbb{Z}_q \qquad \text{CRT}$$

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Proof (cont.)

Recall $a^k \cong (g^y, h^z) \in \mathbb{Z}_p^* \times \mathbb{Z}_q^*$

$\mathrm{M}_2\left(|\mathbb{Z}_p^*|\right)$: How much squaring is irreversible in $\mathbb{Z}_p^*$?

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Calm Down

Recap

What we did so far:

- public key (n, e)
- private key (n, d)
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Next Steps:

- goal: find original plaintext in other words: focus on OW-CPA or OW-CCA
- exploit properties of RSA

RSA-Problem

Definition (RSA-Problem)

Given $n, e, m^e \bmod n$, find m (i.e. the e-th modular root).

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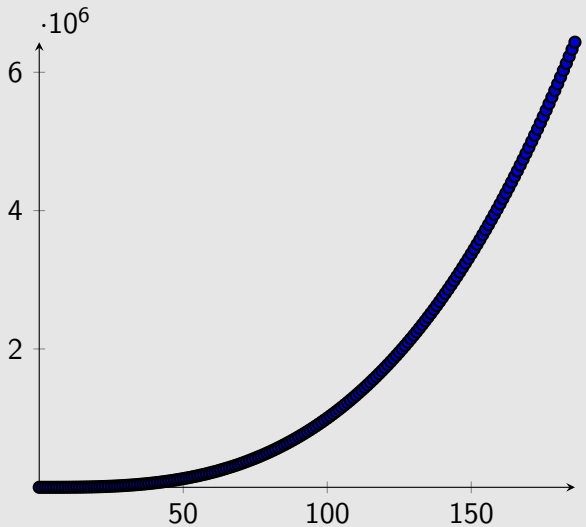

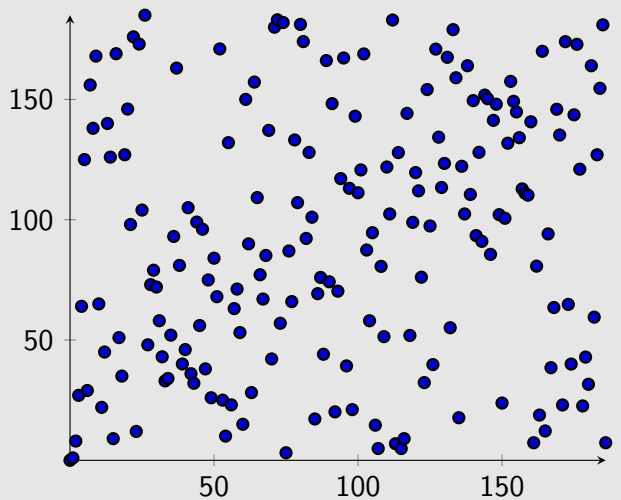

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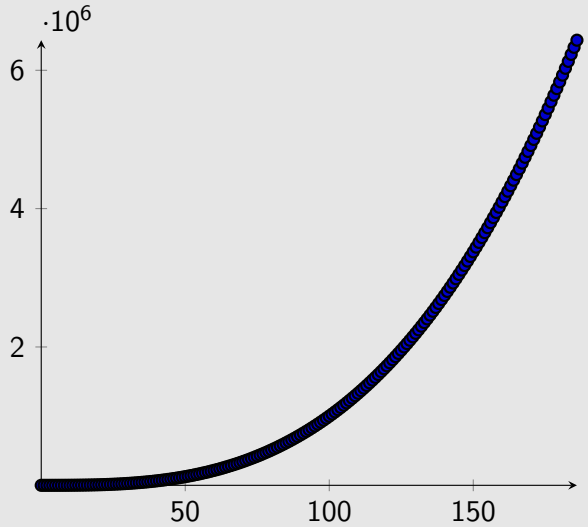

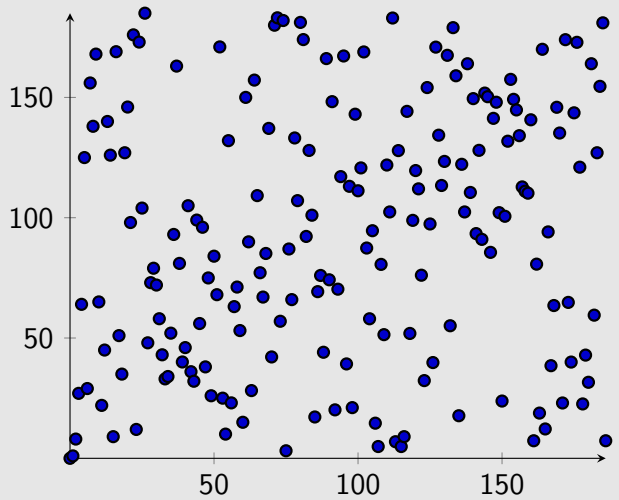

Difference

In $\mathbb{Z}$ we have an order and monotonicity, allows e.g. bisection.

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Clearly, we have the reduction

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i.e. they are equivalent in deterministic polynomial time. For the first reduction, the converse is open.

$$\text{finding } d \overset{?}{\leq_p} \text{RSA-problem}$$

Insecure Special Cases

What can go wrong?

In general, RSA is secure but take care if:

e **is small:** several attacks, find m

same n**, but different** e**:** find m

m **is very small:** find m

d **is small:** find d, thus everything

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There is a standard, avoiding all/most of these.

Public-Key Cryptography Standard (PKCS)

PKCS #1 covers RSA, currently in version 2.2

`https://tools.ietf.org/html/rfc8017a`

Small Public Exponent e

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Use asymmetric crypto to exchange key, then use (faster) symmetric encryption
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Example

Key $(10\,720\,441, 3)$, i.e. n has 24 Bit, message $m = 102$

$$m^3 \bmod n = 1\,061\,208 \bmod 10\,720\,441 = 1\,061\,208$$

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Encrypt: $m' := m \cdot r$ for fixed $r \in \mathbb{Z}_n^*$; $c = (m')^e \mod n$
Decrypt: $m = r^{-1} \cdot (c)^d \mod n$

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Artificially enlarge message to $m' = \mathrm{Pad}(m)$, such that $m' \approx n$.

Toy Example 1: multiply with fixed, known number

Encrypt: $m' := m \cdot r$ for fixed $r \in \mathbb{Z}_n^*$; $c = (m')^e \mod n$
Decrypt: $m = r^{-1} \cdot (c)^d \mod n$

Break

RSA is multiplicative

If we know $\mathrm{enc}(m)$, then we also know $\mathrm{enc}(x \cdot m)$ for every x.

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If we know $\mathrm{enc}(m)$, then we also know $\mathrm{enc}(x \cdot m)$ for every x.
Reduce to previous case, put

$$c' := \left(r^{-1}\right)^e \cdot c = \left(r^{-1}\right)^e \cdot (m')^e = \left(r^{-1} m'\right)^e = \left(r^{-1} r m\right)^e = m^e$$

So we know m^e, if m small, decrypt as before

Padding

Toy Example 2: Fill with 0s

Putting $m' = m \,||\, 0 \ldots 0$

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Toy Example 3: Concatenate

concatenate m with itself: $m' = m \,||\, \ldots \,||\, m$
But mathematically, that is just

$$m' = m \cdot 1 \underbrace{0 \ldots 01}_{\lceil \log m \rceil} 0 \ldots 01 \ldots 01_2$$

- guess length of m: $\log m < \log n$, i.e. small, we can test all
- we know $10 \ldots 010 \ldots 01 \ldots 01_2$ (i.e. in binary)
- break like Version 1

Coppersmith

Theorem (Coppersmith, 1996)

Let $f \in \mathbb{Z}[x]$ normalised, $e = \deg f$. Then we can compute all $x_0 \in \mathbb{Z}$ with $f(x_0) \equiv 0 \mod n$ and $|x_0| \leq \sqrt[e]{n}$ in polynomial time.

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polynomial time $\neq$ efficient

Theorem (from Nina Jekel, Bsc-thesis, 2017)

Let $f \in \mathbb{Z}[x]$ normalised, $\deg f = e$. Assume we have an upper bound for our roots

$$X \leq \frac{1}{2} n^{\frac{1}{e} - \varepsilon}$$

for some $\varepsilon > 0$. Then the running time of Coppersmith is in

$$\mathcal{O}\left(\frac{e^9}{\varepsilon^5} \log n\right)$$

Coppersmith

Proof idea.

Transform into lattice problem, apply LLL-algorithm to reduce base
Way(!) too involved for this course. □

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Corollary (Application in RSA)

If m has (significantly) fewer than $\log(n)/e$ bits, and we have any fixed padding, we can compute m.

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Corollary (Application in RSA)

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In Sagemath implemented as `f.small_roots()`, (but has issues)
Alternatively: CTF-writeup from github

> *If you want an implementation of a crypto algorithm, write a crypto CTF challenge that needs it and read writeups.*
>
> *(ubuntor)*

Coppersmith – Application

Example (PWN-CTF 2018, Whistle)

Padding PKCS#1 v1.5 (RFC 2313, Nov 1993),
but applied padding for private-key-operation (i.e. for $m^d \bmod n$):

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$n \sim 4096$ bit, $e = 3$, $m \sim 128$ bit AES key, so padding is

$$\underbrace{0 \ldots 0}_{15} \underbrace{1 \ldots\ldots 1}_{\text{rest}} \underbrace{0 \ldots 0}_{8} \underbrace{0 \ldots 0}_{128}$$

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m has much fewer than $\log(n)/e \approx 1365$ bits $\rightsquigarrow$ Coppersmith finds m

Coppersmith Failure

Example (NSUCrypto 2019, Problem 3)

We know p, q have 500 bits. Given $n = pq$ and

$$h = 3^{2019}p^2 + 5^{2019}q^2 \mod \underbrace{n^2 + 8 \cdot 2019}_{=:N}$$

Find p, q.

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Find p, q.
Multiply with p^2, use $n^2 = p^2q^2$ and rewrite into

$$0 \equiv p^4 - \left(h \cdot \left(3^{2019}\right)^{-1}\right) p^2 + 5^{2019} \cdot \left(3^{2019}\right)^{-1} n^2 \mod N$$

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But ε too small $\rightsquigarrow$ takes too long.
Likewise if $q < p$.

Håstad Broadcast

Lemma

If a message is encrypted with the same exponent e but e different moduli n_i, we can recover the message.

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Proof.

If some $\gcd(n_i, n_j) \neq 1$, we found a prime factor. ✓
So wlog system of congruences with coprime n_i

$$c_i \equiv m^e \mod n_i \qquad i = 1, \ldots, e$$

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Put $x = m^e$ and solve via CRT. Unique solution $m^e \bmod \prod n_i$

$$m < n_i \implies m^e < \prod n_i \implies m^e \bmod \prod n_i = m^e$$

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then just compute root in $\mathbb{Z}$ (as before). □

Example

Same message m, $e = 3$, $c_i = m^e \bmod n_i$:

$$c_1 = 533 \qquad n_1 = 551$$
$$c_2 = 333 \qquad n_2 = 943$$
$$c_3 = 357 \qquad n_3 = 527$$

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CRT yields

$$\begin{aligned} m^3 &\equiv 1061208 \mod 273825511 \\ \implies m^3 &= 1061208 \\ \implies m &= 102 \end{aligned}$$

General Håstad Broadcast

Theorem

Let n_i be coprime. Assume we modify some base message via $m_i = f_i(m)$ for $i = 1, \ldots, k$ for known polynomials f_i. If

$$k \geq e \cdot \max\{\deg f_i : i = 1, \ldots, k\},$$

then we can recover m from the f_i and $c_i = m_i^e \bmod n_i$.

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- Special case: $f_i = \mathrm{id}$ is original Håstad.
- Typical padding $f_i(m) = 2^* \cdot m + *$ or $f(m) = 2^* \cdot * + m$ so often $\deg f_i = 1$

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Corollary

Any fixed padding scheme becomes dangerous, given enough messages. Use randomised padding.

Proof.

- put $g_i(x) = f_i(x)^e - c_i$, so all $g_i(m) \equiv 0 \mod n_i$,
  Note: $\deg g_i = e \cdot \deg f_i \leq k$

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$$g(x) := \sum_{i=1}^{k} T_i \cdot g_i(x)$$

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$$m < \min_i n_i < \left(\prod n_i\right)^{\frac{1}{k}} \leq \left(\prod n_i\right)^{\frac{1}{\deg g}}$$

- so we find m via Coppersmith

Polynomial Rings

Polynomials

A (univariate) polynomial is an expression of the form

$$f = \sum_{k=0}^{D} a_k x^k$$

We can add/subtract/multiply (as long as can do so with the a_k).
$\rightsquigarrow$ polynomials form a ring.
Coefficients $a_k \in R$, the ring of polynomials (in x) is denoted $R[x]$.

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Special Properties of $\mathbb{C}[x]$

- We can do polynomial division (with remainder).
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- Every polynomial splits into linear factors (Vieta/Viète).

Don't want Complex Numbers $\rightsquigarrow$ What holds over $\mathbb{Z}_n$?

Polynomial Division

Let $g, h \in \mathbb{Z}_n[x]$ given as

$$g = \sum_{k=0}^{D_1} g_k x^k \qquad h = \sum_{k=0}^{D_2} h_k x^k$$

with $D_1 \geq D_2$.

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Then first step of division is

$$g = \left(g_{D_1} \cdot h_{D_2}^{-1} \cdot x^{D_1 - D_2} \right) \cdot h + \text{Rem}_1$$

where $\deg(\text{Rem}_1) < \deg g$.

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where $\deg(\text{Rem}_1) < \deg g$. Continue with Rem_1 and h.

We always just multiply with inverse of leading coefficient of h.

As long as this exists, we can perform polynomial division.

In $\mathbb{Z}_n[x]$ we mostly can do division with remainder
(behaves like Euclidean Ring)

- for polynomial division, we must divide by coefficients
- i.e. must be able to invert elements
- if it fails, we found a divisor of n
- have solved the problem otherwise

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Euclidean Algorithm for polynomials

- (Extended) Euclidean Algorithm works for polynomials
- we can compute the gcd of two polynomials

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Linear Factors

Let $f \in \mathbb{Z}_n[x]$ and $f(x_0) = 0$. Then there is $g \in \mathbb{Z}_n[x]$ with $f = (x - x_0)g$, which we can compute via polynomial division.

Franklin-Reiter-Related-Message-Attack

Theorem

If two messages are related via $m_2 = f(m_1)$ for some known polynomial f, we often can recover them from $c_i = m_i^e \bmod n$. The time is $\mathcal{O}\left((e \cdot \deg f)^2\right)$ arithmetic operations.
If f is linear and $e = 3$ the attack is guaranteed to work.

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If f is linear and $e = 3$ the attack is guaranteed to work.

Proof.

Define polynomials

$$g(x) = x^e - c_1 \qquad h(x) = f(x)^e - c_2$$
$$\implies g(m_1) = h(m_1) = 0$$
$$\implies (x - m_1) \mid \gcd(g, h)$$

Mostly gcd is linear, if $e = 3$ and $\deg f = 1$, this is guaranteed. □

Example

- message: "Diary entry ??: Today I investigated [secret stuff]."
- ?? are consecutive numbers,
- assume only last digit changed: $f(x) = x + 2^{37 \cdot 8}$ (count bytes)

$$g(x) = x^3 - c_1 \qquad h(x) = (x + 2^{296})^3 - c_2$$

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Calling Euclidean Algo:

$$r_1 = h - g = 3 \cdot 2^{592} x^2 + 3 \cdot 2^{296} x + c_1 - c_2 \qquad \text{cancel } x^3$$

$$r_2 = g - \left(3 \cdot 2^{592}\right)^{-1} x r_1 - * \cdot r_1 = k(x - m_1) \qquad \text{cancel } x^3, x^2$$

for some $k \in \mathbb{Z}_n$.

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If inverting $3 \cdot 2^{592}$ or k fails, then $\gcd(*, n) \in \{p, q\}$.
So we get m_1 (and also m_2).

But there's an even less artificial scenario ...

Coppersmith Short Pad

"Why 256 Bit padding is not enough for $e = 3$."

Theorem

Let $R \leq \log(n)/e^2$, and $m_i = m \cdot 2^R + r_i$ for $i = 1, 2$. Then we can (probably) recover the message from the cipher $c_i = m_i^e \bmod n$. This always works for $e = 3$.

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Scenario

- intercept handshake
- receiver won't send ACK
- handshake is sent again, but with different random padding

Starting to break it

Define polynomials

$$g(x, y) = x^e - c_1 \qquad h(x, y) = (x + y)^e - c_2$$

- See y as parameter and x as actual variable.
- If $y = r_2 - r_1$, then common root
  $g(2^R m + r_1, y) = h(2^R m + r_1, y) = 0$.

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Algebra knows something about this:
Search Engine "polynomials common root" $\rightsquigarrow$ resultants

Lemma

Let R comm. ring with 1. If $g, h \in R[x]$, then the resultant $\mathrm{res}(g, h) = 0$ *iff they have a common root.*

Resultants

For $e = 3$ we have

$$\operatorname{res}(g, h) = \det \begin{pmatrix} 1 & 0 & 0 & -c_1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -c_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -c_1 \\ 1 & 3y & 3y^2 & y^3 - c_2 & 0 & 0 \\ 0 & 1 & 3y & 3y^2 & y^3 - c_2 & 0 \\ 0 & 0 & 1 & 3y & 3y^2 & y^3 - c_2 \end{pmatrix}$$

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- only last e rows contain parameter y
- maximal power y^e in each of them
- in total maximal power $(y^e)^e = y^{e^2}$
- Resultant is polynomial in y of degree $e^2 (= 9)$

Coppersmith + Franklin-Reiter

- $\text{res}(g, h) \in \mathbb{Z}_n[y]$ of degree e^2
- Assumptions: $y = r_2 - r_1 < 2^R$, with $R \leq \log(n)/e^2$
- hence $y < \sqrt[e^2]{n}$, so find via Coppersmith

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But now we have Franklin Reiter with linear f:

- Relation: $m_2 = m_1 + y =: f(m_1)$
- (Try to) Recover via

$$x - m_1 = \gcd\left(x^e - c_1, (x + y)^e - c_2\right)$$

- original message $m = m_1 // 2^R = (m_1 >> R)$

Coppersmith + Franklin-Reiter

- $\text{res}(g, h) \in \mathbb{Z}_n[y]$ of degree e^2
- Assumptions: $y = r_2 - r_1 < 2^R$, with $R \leq \log(n)/e^2$
- hence $y < \sqrt[e^2]{n}$, so find via Coppersmith

But now we have Franklin Reiter with linear f:

- Relation: $m_2 = m_1 + y =: f(m_1)$
- (Try to) Recover via

$$x - m_1 = \gcd\left(x^e - c_1, (x + y)^e - c_2\right)$$

- original message $m = m_1 // 2^R = (m_1 >> R)$

Formally, R is not given, but for $n \sim 4096$ bits, we have $R \leq 455$, so just bruteforce.

Very Small Message m

Brute Force via Meet-in-the-Middle:
Assume $m = m_1 m_2$ in $\mathbb{Z}$ where $m_i \leq 2^{b_i}$
Rewrite:

$$c = m^e \mod n \implies c m_1^{-e} \equiv m_2^e \mod n$$

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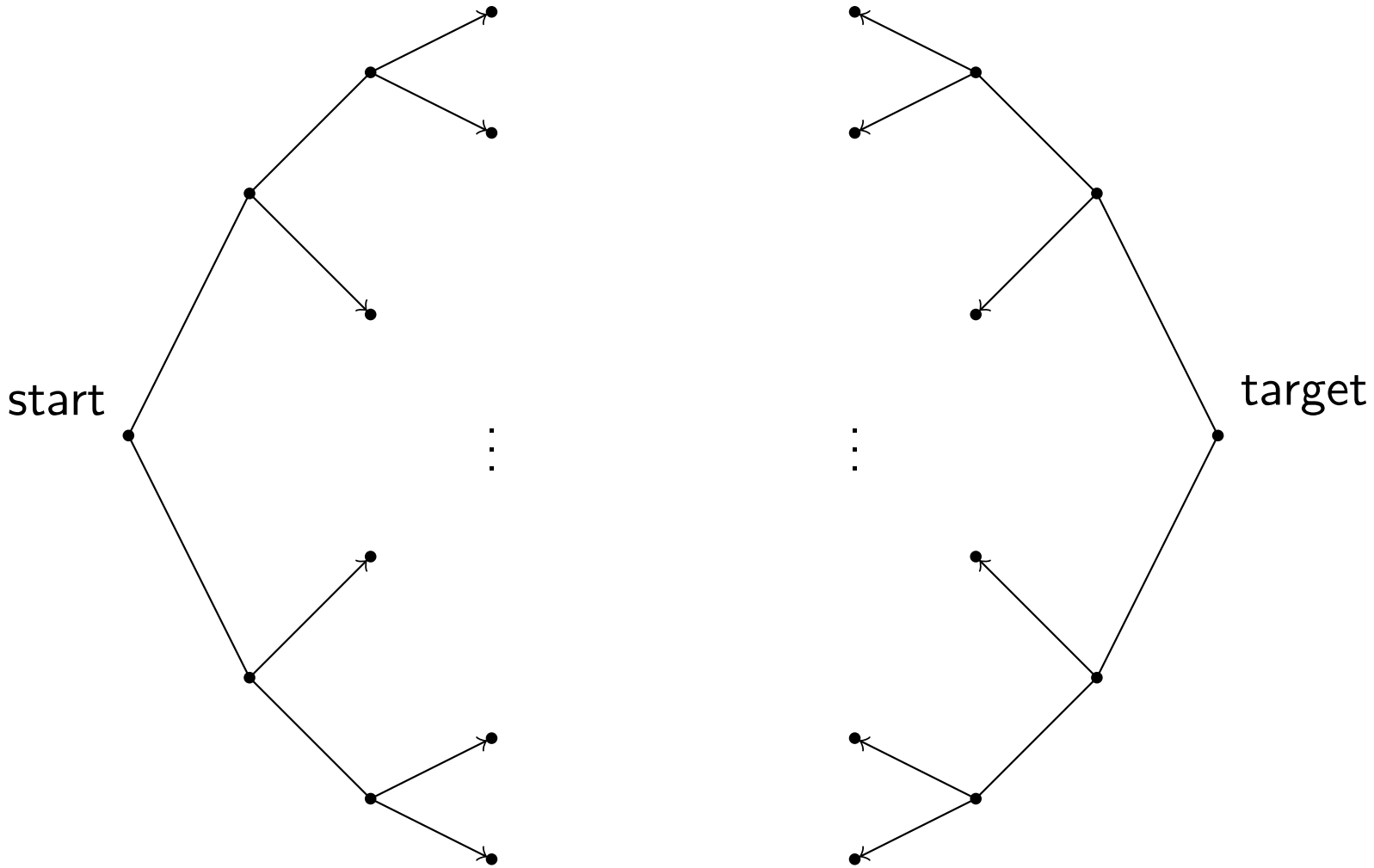
Assume $m = m_1 m_2$ in $\mathbb{Z}$ where $m_i \leq 2^{b_i}$
Rewrite:

$$c = m^e \mod n \implies cm_1^{-e} \equiv m_2^e \mod n$$

Strategy:

- List $cm_1^{-e} \mod n$ for all $m_1 \leq 2^{b_1} \rightsquigarrow$ (parallel write, not trivial)
- Look up $m_2^e \mod n$ for all $m_2 \leq 2^{b_2} \rightsquigarrow$ (parallel, only read)
- Search for collision

Meet in the Middle



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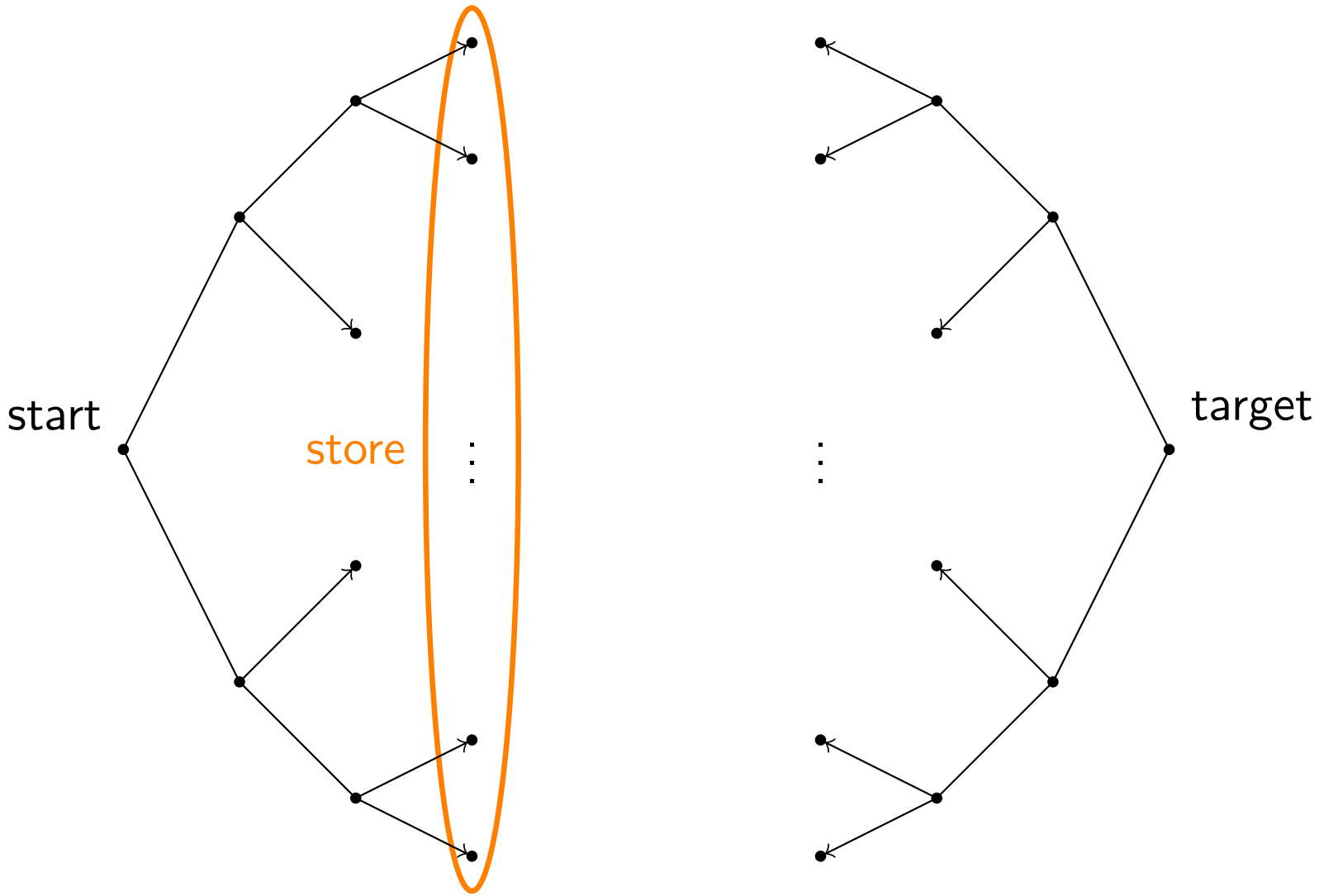

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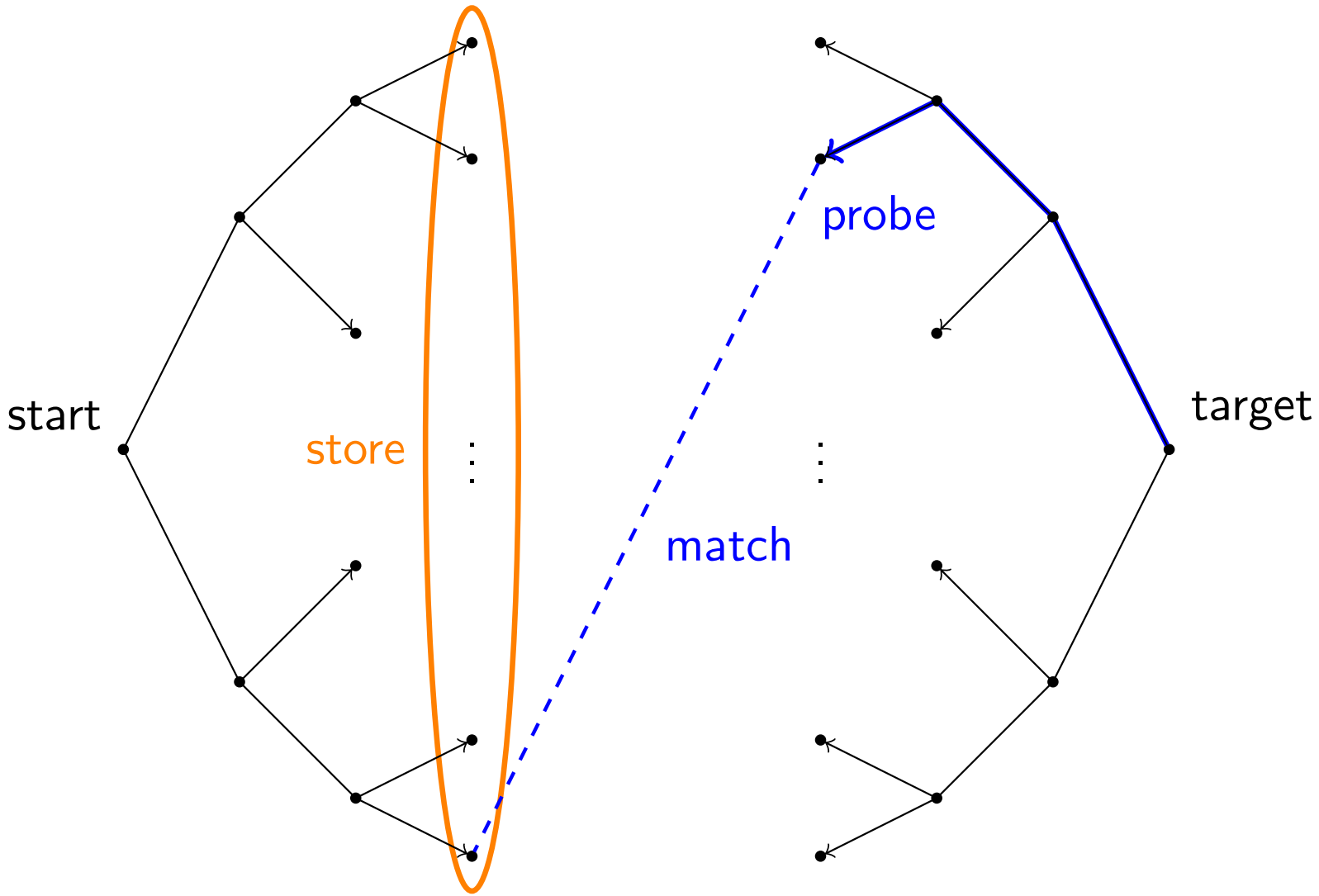

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Analysis (assume $b_1 < b_2$):

Time: $\mathcal{O}(2^{b_1} + 2^{b_2}) = \mathcal{O}\left(2^{b_2}\right)$ exp. and $\mathcal{O}(2^{b_1})$ inv.

Memory: $\mathcal{O}\left(2^{b_1} \log n\right)$

Total: if all goes well $b_1 = b_2$, so $2^{b_1} \approx \sqrt{m}$
$\mathcal{O}(\sqrt{m} \cdot \text{poly})$ time and space

Compare: Brute Force $\mathcal{O}(m \cdot \text{poly})$ time, but $\mathcal{O}(\log n)$ space
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The probability that a 64 bit number splits into two equally large parts lies around 18%.

Common Modulus

"What if n_i are not coprime, but same?"

Theorem

If we have keys (n, e_1) *and* (n, e_2) *with* $\gcd(e_1, e_2) = 1$*, then we can read every message sent to both.*

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- Compute m via

$$\left(c_1^{-1}\right)^{|s|} c_2^t \equiv \left(\left(m^{e_1}\right)^{-1}\right)^{|s|} \left(m^{e_2}\right)^t \equiv m^{se_1+te_2} \equiv m \mod n$$

Recap – What can go wrong?

In general, RSA is secure but take care if:

- ✓ e **is small:** several attacks, find m
- ✓ m **is very small:** find m
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other attacks completely break key

Small Private Exponent d

Theorem (Wiener, 1989)

Assume $q < p < 2q$, $e < \varphi(n)$ and $d < \frac{1}{3}n^{\frac{1}{4}}$. Then we can compute d from (n, e) in $\mathcal{O}\left(\log(n)^2\right)$ arithmetic steps.

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Example

Let $n \sim 4096$ bit, choose $d \sim 1000$ bit and put $e = d^{-1} \bmod \varphi(n)$.
$\rightsquigarrow$ already unsafe

Consequences

- Decrypting "always" takes rather long

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proof uses continued fractions

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Approximate large fractions by short fractions (in terms of bit size)

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Example (Euclidean Algo gcd(67, 24))

$$67 = 2 \cdot 24 + 19 \qquad 5 = 1 \cdot 4 + 1$$
$$24 = 1 \cdot 19 + 5 \qquad 4 = 4 \cdot 1 + 0$$
$$19 = 3 \cdot 5 + 4$$

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yields representation

$$\frac{67}{24} = 2 + \frac{19}{24} = 2 + \frac{1}{\frac{24}{19}} = 2 + \frac{1}{1 + \frac{5}{19}} = 2 + \frac{1}{1 + \frac{1}{3 + \frac{4}{5}}}$$
$$= 2 + \frac{1}{1 + \frac{1}{3 + \frac{1}{1 + \frac{1}{4}}}} =: [2; 1, 3, 1, 4]$$

divide – swap – repeat

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Observations

- difference alternates sign
- absolute value of difference decreases ↑
- enumerator and denominator increase ↑

Continued Fractions in General

- Input $\frac{a}{b} \in \mathbb{Q}$, ($a > b$ else we start with $[0; \ldots]$)
- Euclid: start with $r_{-1} = a$, $r_0 = b$
  recursion: $r_{k-1} = z_k r_k + r_{k+1}$
- n-th convergent: $[z_0; z_1, \ldots, z_n] = \frac{p_n}{q_n}$

$$p_k = z_k p_{k-1} + p_{k-2} \qquad p_{-1} = 1 \qquad p_{-2} = 0$$
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Remark

Generalised idea also works for $x \in \mathbb{R}$:

$$x_0 = x \qquad z_k = \lfloor x_k \rfloor \qquad x_{k+1} = \frac{1}{x_k - \lfloor x_k \rfloor}$$

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But we are mainly interested in Rationals and finite fractions.

Bonus Slide

Can be infinite

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}} =: [1; 1, 1, 1, 1, \ldots]$$

yields $x = 1 + \frac{1}{x}$,

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Theorem

The infinite, periodic continued fractions correspond to solutions of quadratic equations, i.e. algebraic numbers of degree 2.

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- φ has n-th convergent $\frac{F_{n+2}}{F_{n+1}}$, with Fibonacci numbers $F_0 = 0$
- φ is worst number to approximate,
  as Fibonacci numbers are worst case for Euclidean Algorithm

Properties of Continued Fractions

Improving: each step improves approximation

Alternating: even $\rightarrow$ smaller, odd $\rightarrow$ larger value

$$\frac{p_{2i}}{q_{2i}} < \frac{p_{2(i+1)}}{q_{2(i+1)}} \leq x \leq \frac{p_{2(i+1)+1}}{q_{2(i+1)+1}} < \frac{p_{2i+1}}{q_{2i+1}}$$

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If $\frac{p}{q}$ is a better approximation, then $q > q_i$.

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"only" good approximation: $\left| x - \frac{p}{q} \right| < \frac{1}{2q^2} \implies \frac{p}{q}$ is a cont.frac.

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Theorem (Wiener, 1989, slightly generalised)

Assume $q < p < aq$, $e < \varphi(n)$ *and* $d < \frac{1}{\sqrt{2(a+1)}} n^{\frac{1}{4}}$. *Then we can compute* d *from* (n, e) *in* $\mathcal{O}\left(\log(n)^2\right)$ *arithmetic steps.*

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Proof Idea.

- Idea: Approximate $\frac{e}{n}$ with cont.frac.
- We have $ed - k\varphi(n) = 1$ for some unknown $k, d, \varphi(n)$
- have $\varphi(n) \approx n$, slightly smaller, hence $\frac{e}{n} \approx \frac{e}{\varphi(n)}$
- estimate error $\left|\frac{e}{n} - \frac{k}{d}\right| < \ldots < \frac{1}{2d^2}$

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- have $\varphi(n) \approx n$, slightly smaller, hence $\frac{e}{n} \approx \frac{e}{\varphi(n)}$
- estimate error $\left|\frac{e}{n} - \frac{k}{d}\right| < \ldots < \frac{1}{2d^2}$
- $\implies$ $\frac{k}{d}$ is a cont.frac. of $\frac{e}{n}$
- compute all continued fractions $\rightsquigarrow$ list of $\log n$ candidates
  a) check decoding: $2^{ed} \bmod n \stackrel{?}{=} 2$
  b) try to factor n, note we also have k, thus $\varphi(n)$ □

Proof for Wiener attack.

Error from $\varphi(n)$ to n:

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Error between fractions:

$$\begin{aligned}\left|\frac{e}{n} - \frac{k}{d}\right| &= \left|\frac{ed - k\varphi(n) - kn + k\varphi(n)}{nd}\right| \\ &= \left|\frac{1 - k(n - \varphi(n))}{nd}\right| < \frac{(a+1)k\sqrt{n}}{nd} = \frac{(a+1)k}{d\sqrt{n}}\end{aligned}$$

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$$k\varphi(n) = ed - 1 \text{ and } e < \varphi(n) \implies k < d$$
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hence $\frac{k}{d}$ is a continued fraction of $\frac{e}{n}$ □

Example (Wiener Attack)

- assume given public key

$$n = 389033 \qquad e = 332383$$

- calculate continued fractions

$$\begin{aligned}
\frac{332383}{389033} &= \frac{1}{1+\frac{56650}{332383}} && \rightsquigarrow 1 \\
&= \frac{1}{1+\frac{1}{5+\frac{49133}{56650}}} && \rightsquigarrow \frac{5}{6} \\
&= \frac{1}{1+\frac{1}{5+\frac{1}{1+\frac{7517}{49133}}}} && \rightsquigarrow \frac{6}{7}
\end{aligned}$$

- checking $e \cdot 7 - 1 \bmod 6 = 0$ and $2^{e \cdot 7} \bmod n = 2$
- hence $d = 7$

Outlook on Wiener's Attack

Extension to Wiener's Attack

- via lattice methods breakable for $d < n^{0.292}$
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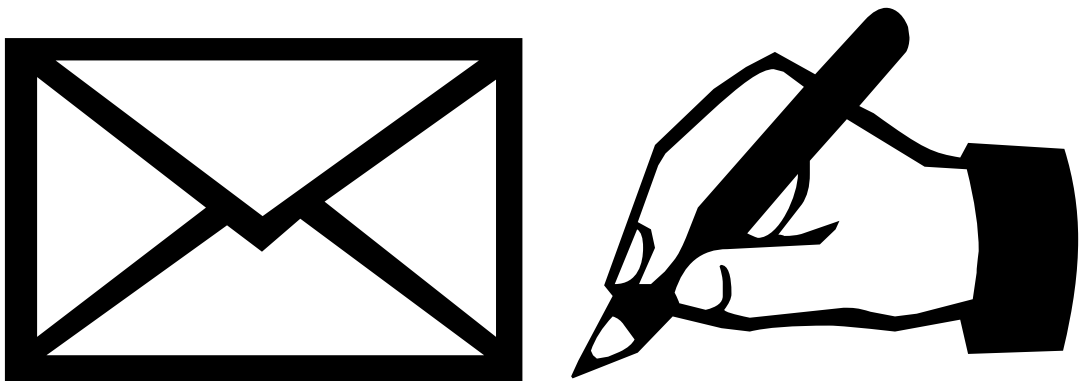
Possible Countermeasures

- put $e' = e + * \cdot \varphi(n)$, destroys assumption $e < \varphi(n)$
- optimised decryption: make $d_p = d \bmod p - 1$ and $d_q = d \bmod q - 1$ small-ish
  can factor n in $\mathcal{O}\left(\min\left(\sqrt{d_p}, \sqrt{d_q}\right)\right)$

But ongoing research, so security unsure.

Digital Signatures



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- Authentication: sender only has to convince recipient
- Signature: recipient can also convince "judge"
- must depend both on sender and message

  if not message: use old signature from other message
  if not sender: recipient can forge

Desired Property

often Encryption/Decryption commute

$$\mathrm{enc}\left(\mathrm{dec}\left(m\right)\right) = \mathrm{dec}\left(\mathrm{enc}\left(m\right)\right)$$

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Basic Idea (RSA, '77)

- Alice sends to Bob:
$$s = m^{d_A} \bmod n_A$$
- Bob gets s, checks with Alice's public key:
$$m = s^{e_A} \bmod n_A$$
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Problem

- What to check the message against?
- This setting is flawed!

Mathematical Model

Definition (Signature System)

A signature system is a quintuple $(P, S, K, \text{sign}, \text{vrfy})$ where

- P is the set of all plaintexts
- S is the set of all signatures
- K is the set of all keys
- $\text{sign} : P \times K \rightsquigarrow S$ is the signature relation
  (not necessarily a map)
- $\text{vrfy} : P \times S \times K \to \{0, 1\}$ is the verification function

$$\text{vrfy}(m, s, k) = \begin{cases} 1 & : s \in \text{sign}(m, k) \text{ i.e. possible outcome} \\ 0 & : \text{else} \end{cases}$$

- sign, vrfy are efficiently computable

Correctness: $\forall m \in \mathcal{M}, \forall k \in \mathcal{K} \,.\, \text{vrfy}(m, \text{sign}(m, k_{\text{priv}}), k_{\text{pub}}) = 1$

Observations

- We must be able to reject messages.
- Signature + message must contain redundancy.
- If message derived from signature, redundancy must be in message.

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Improved Plain-RSA signature

- Alice computes $s := \text{sign}\,(m, (n, d)) = m^d \bmod n$
- Send (m, s) to Bob
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How does Bob get (n, e)?

- want to guard against manipulation of message
- transmitted public key could have been changed
- Public Key Infrastructure (PKI): topic of its own

Example

- Alice's key is $(n, e, d) = (1073, 17, 593)$.
- She want to send $m = 123$.
- Compute $s = 123^{593} \bmod 1073 = 219$.
- Bob gets $(m, s) = (123, 219)$ and knows (n, e).
- Bob checks $123 \stackrel{?}{=} 219^e \bmod n$
- They match, so Bob accepts the message.

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RSA-specific problems

- Every number $s < n$ is a valid signature for some $m < n$.
- Plain-RSA is multiplicative: If (m_1, s_1) and (m_2, s_2) are valid pairs, then $(m_1 m_2, s_1 s_2)$ also is valid.

$$m_1 = s_1^e \qquad m_2 = s_2^e \implies m_1 m_2 = (s_1 s_2)^e$$

Signature Oracle Attack

Assumptions

- Assume we sign with plain-RSA
- want to forge signature for message m
- Have access to online oracle, that signs any $m' \neq m$ (or some restricted subset)

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Attack

- factor $m = m_1 \dots m_k \bmod n$ such that all m_i accepted by oracle (not necessarily prime factors)
  e.g. pick some $m_1 < n$ and put $m_2 := m \cdot m_1^{-1} \bmod n$
- get $s_i = m_i^d \bmod n$ for $i = 1, \dots, k$
- have signature $s = \prod s_i$ for m

Example (Signature Oracle Attack)

- public key $(3084396941, 5)$
- want to forge signature for `flag`
- factor `flag`: $5 \cdot 499 \cdot 688729$
- ask signature oracle

$$s_1 = \text{sign}(5) \qquad s_2 = \text{sign}(499) \qquad s_3 = \text{sign}(688729)$$

- send $s := s_1 \cdot s_2 \cdot s_3$

Remark

- *possibly additional tricks with 0-bytes if in C*

Attack Scenarios

What does Eve know?

No message: just public key

Signatures: Eve has some message-signature pairs (m_i, s_i)

e.g. observing traffic

Chosen message: Eve can choose messages m_i to be signed

e.g. impersonating authentication server

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What is a success?

Total Break: find private key

Universal Forgeability: forge signature for every message

Selective Forgeability: forge signature for m given by Alice

Existential Forgeability: forge signature for m chosen by Eve

Goal

Strongest Security

EUF-CMA Existential Unforgeability under Chosen message Attack:

- Eve may request signatures s_i for $m_1, \ldots, m_k$
- forges signature s for some $m \notin \{m_1, \ldots, m_k\}$

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Plain RSA fails:

- EUF with no message
- Universal Unforgeability (UUF) under CMA

Public Key Cryptography Standard

OAEP – Optimal Asymmetric Encryption Padding

"How to do it right."

- part of PKCS #1, version 2.2,
- RFC 8017, October 2012, last update Nov. 2016

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Parameters

- hash function $h : \text{Byte}^* \to \text{Byte}^{\text{hLen}}$
  - recommended: SHA-224, SHA-256, SHA-384, SHA-512, SHA-512/224, and SHA-512/256 (i.e. SHA-2)
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  - SHA-3 was too fresh, unclear why not included in update
- mask generation function $\text{MGF} : (\text{seed}, \ell) \mapsto \text{Byte}^{\ell}$

  $T \leftarrow$ empty string
  for $c = 0$ **to** $\lceil \ell / \text{hLen} \rceil - 1$ **do**
  $T \leftarrow T \,||\, h(\text{seed} \,||\, c)$

OAEP-Encryption

Encryption

- (n, e) public RSA key
- m message, $\|m\| \leq \|n\| - 2\text{hLen} - 2$
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function ENCRYPT(m, L)

DB $\leftarrow h(L) \,||\, 00 \ldots 0001 \,||\, m$ ▷ data block

Seed $\leftarrow$ random seed of length hLen

mDB $\leftarrow$ MGF(Seed) $\oplus$ DB ▷ masked DB

mSeed $\leftarrow$ Seed $\oplus$ MGF(mDB) ▷ masked seed

EM $\leftarrow 00 \,||\, \text{mSeed} \,||\, \text{mDB}$ ▷ encoded message, EM $< n$

return $\text{EM}^e \bmod n$

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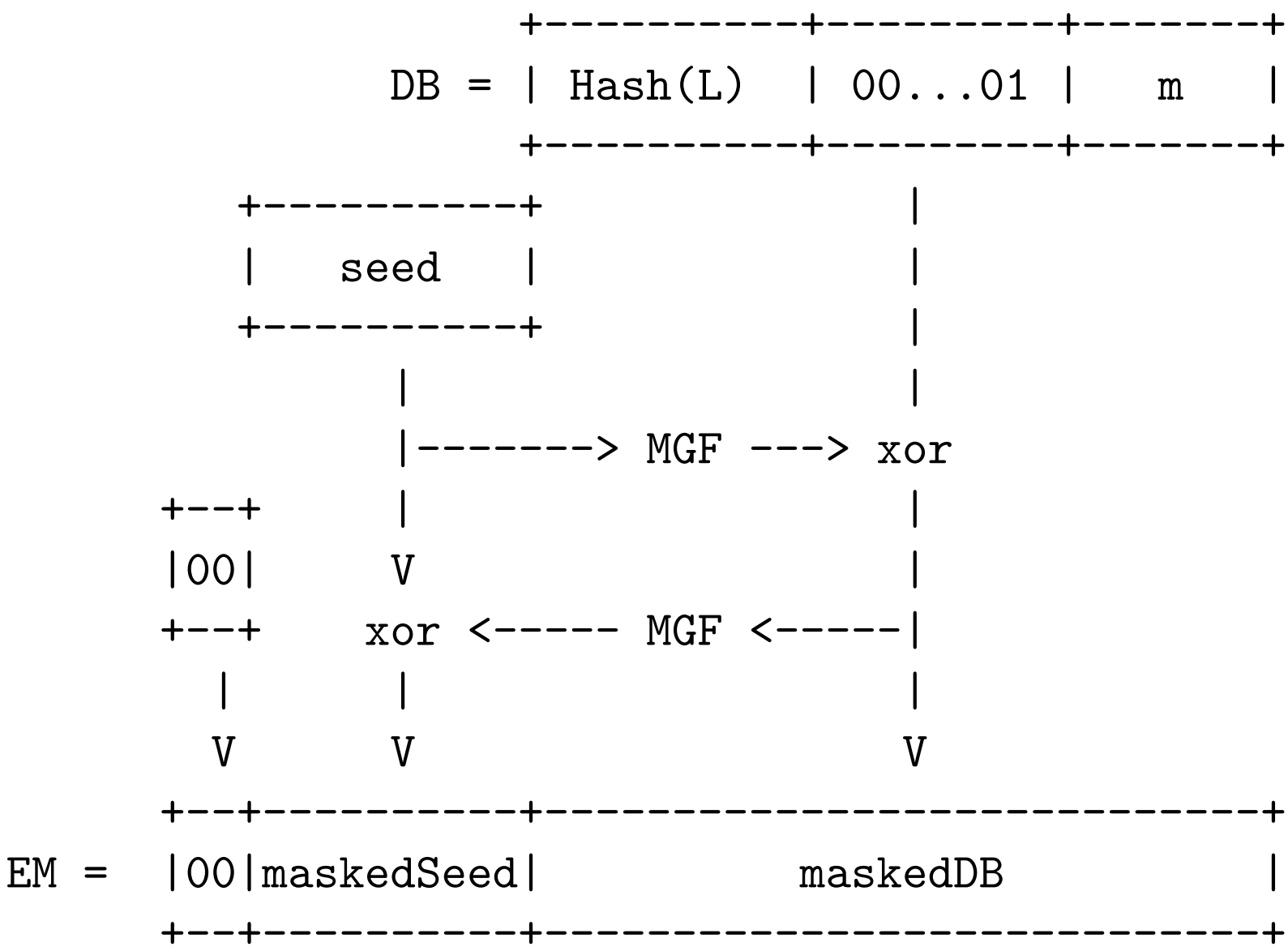
- Payload m: 50% - 89% of cipher, $\geq$ 1000 Bit
- more than enough for AES key
- continue with symmetric encryption


DB = | Hash(L) | 00...01 | m |
seed
MGF
xor
MGF
xor
EM = |00|maskedSeed| maskedDB |

RSASSA-PSS – Idea

Naming

SSA Signature Scheme with Appendix

PSS Probabilistic Signature Scheme

EMSA Encoding Methods for Signatures with Appendix

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- encode message with EMSA-PSS: $\mathrm{EM} = \mathrm{encode}(m)$
- apply RSA primitive/plain-RSA: $s = \mathrm{EM}^d \bmod n$

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- check consistency with EMSA-PSS-VERIFY

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- check consistency with EMSA-PSS-VERIFY

- encoding uses $\mathrm{hash}(m)$ instead of m
- can sign arbitrarily long message (document)

RSASSA-PSS – Details

Arguments for Encoding

- m message to be signed
- h hash function
- MGF mask generation function
- sLen salt length (bytes), mostly hash length or 0
- L desired output length, $\geq \|h(*)\| + \text{sLen} + 2$

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Verification

- $m' = 00 \ldots 00 \,||\, h(m) \,||\, \text{salt}$ with 8 Zero-bytes
- $\text{DB} = 00 \ldots 0001 \,||\, \text{salt}$ of length $L - \|h(*)\| - 1$
- mask DB with $M(h(m'))$
- output $\text{EM} = \text{maskedDB} \,||\, h(m') \,||\, 0xbc$

RSASSA-PSS – Details

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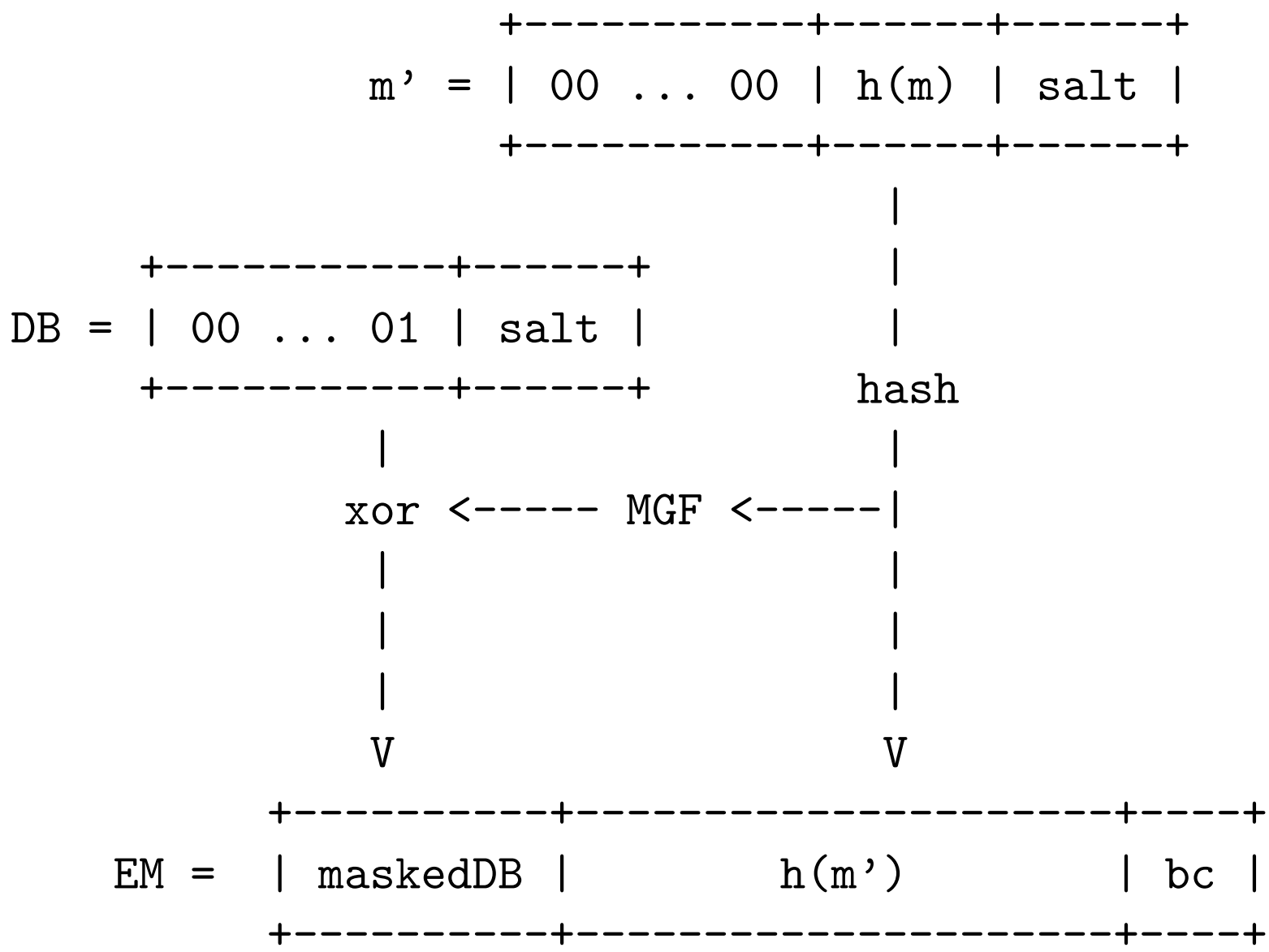
Decode

- split EM by length to get the parts maskedDB$'$, H'
- with H' unmask to get DB'
- know salt length, so get salt$'$
- construct $m' = 00 \ldots 00 \,||\, h(m) \,||\, \text{salt}'$
- check $H' = h(m')$,
- if yes (and all hardcoded bytes correct), accept


m' = | 00 ... 00 | h(m) | salt |
DB = | 00 ... 01 | salt |
hash
xor <----- MGF <-----
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RSASSA-PSS/Signing

- L desired output length, so $L = \|n\|$
- one hash, one salt, two fixed bytes
- only restriction $\|n\| \geq \text{hLen} + \text{sLen} + 2$
- no restriction on m, as hashed anyway

Why so complicated

- PKCS #1 v1.5 was easier, for encryption we have

$$00 \,||\, 02 \,||\, \text{random} \,||\, 00 \,||\, m_0$$

- broken 1998 by Bleichenbacher
- SSL 3.0 (from '96) $\rightarrow$ TLS 1.0 (in '99)

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PKCS #1 v1.5 only for compatibility, should be avoided if possible.

skip details

Bleichenbacher's Attack – Decrypt

- let B bound on $m := (\text{random} \,||\, 00 \,||\, m_0)$
- input c, get the information whether $2B \leq c^d \bmod n \leq 3B$
- given $c_0 = m^e \bmod n$, find s_i such that $c_0(s_i)^e$ is accepted
- M_i set of intervals, one contains m, $M_0 = \{[2B, 3B]\}$
  finished if $M_* = \{[m, m]\}$, i.e. one singleton

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  finished if $M_* = \{[m, m]\}$, i.e. one singleton
- s_i is conform if $2B \leq ms_i \bmod n < 3B$
- assume $m \in [a, b]$

$$2B \leq ms_i - rn \leq 3B - 1 \qquad \text{for some } r \in \mathbb{N}$$

$$\xRightarrow{a \leq m \leq b} as_i - (3B - 1) \leq rn \leq bs_i - 2B \qquad \text{inductive bounds}$$

some candidates for r, for each

$$\frac{2B + rn}{s_i} \leq m \leq \frac{3B - 1 + rn}{s_i}$$

$i = 1$: smallest $s_1 \geq n/(3B)$ s.t. conform

$|M_{i-1}| > 1$: smallest $s_i > s_{i-1}$ s.t. conform

$|M_{i-1}| = 1$: find smallest r_i, then s_i with

$$r_i \geq 2\frac{bs_{i-1} - 2B}{n} \qquad \frac{2B + r_i n}{b} \leq s_i \leq \frac{3B + r_i n}{a}$$

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- combine both old and new bounds

$$M_i = \left\{ \left[\max\left(a, \frac{2B + rn}{s_i}\right), \min\left(b, \frac{3B - 1 + rn}{s_i}\right) \right] : [a, b] \in M_i, \frac{as_i - 3B + 1}{n} \leq r \leq \frac{bs_i - 2B}{n} \right\}$$

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$|M_{i-1}| > 1$: smallest $s_i > s_{i-1}$ s.t. conform

$|M_{i-1}| = 1$: find smallest r_i, then s_i with

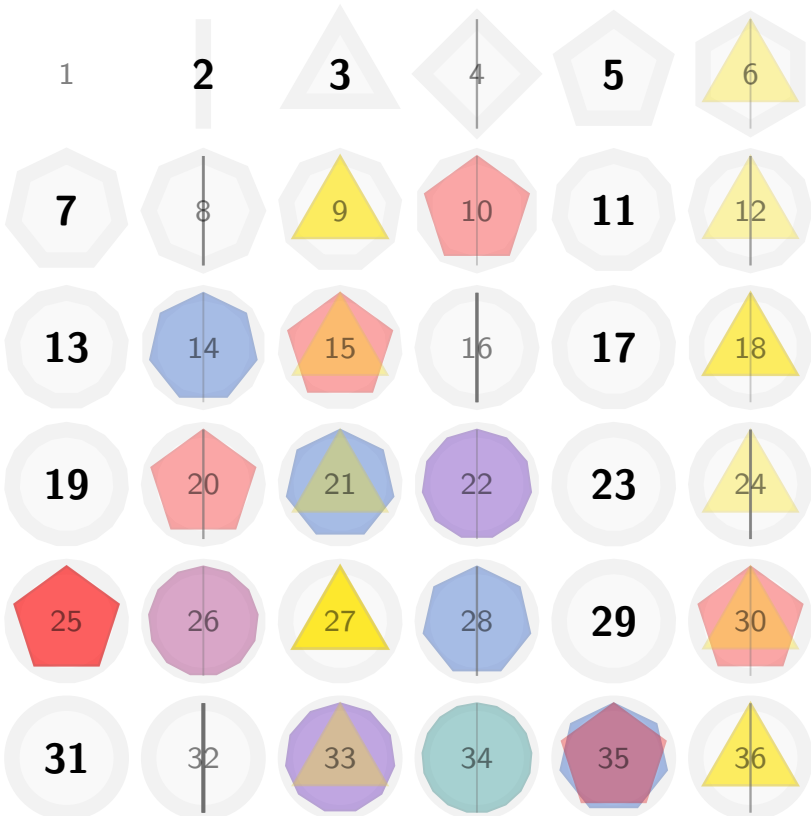
$$r_i \geq 2\frac{bs_{i-1} - 2B}{n} \qquad \frac{2B + r_i n}{b} \leq s_i \leq \frac{3B + r_i n}{a}$$

- combine both old and new bounds

$$M_i = \left\{ \left[\max\left(a, \frac{2B + rn}{s_i}\right), \min\left(b, \frac{3B - 1 + rn}{s_i}\right) \right] : [a, b] \in M_i, \frac{as_i - 3B + 1}{n} \leq r \leq \frac{bs_i - 2B}{n} \right\}$$

- probability analysis to get expected number of attempts
- experiments on 1024 bit key: between 300k and 2M
- allows practical attacks on SSL 3.0

Primality Tests



Sieve of Erathosthenes – Thanks to Todd Lehmann on texoverflow

Primality Testing

- RSA needs big primes
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- chance is good (Prime Number Theorem)
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Time must be polynomial in $\|n\| \approx \log n$.

- exact checkers are too slow, even though polynomial $\sim \|n\|^{6+\varepsilon}$
- use probabilistic method
  (chance of wrong answer $\sim$ chance of guessing key)

Naive Test

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function PRIME(n)
    for $i = 2, \ldots, \lfloor \sqrt{n} \rfloor$ **do**
        if $n \bmod i = 0$ **then**
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- obviously works correctly
- time $\mathcal{O}^*(\sqrt{n}) = \mathcal{O}^*\left(2^{\frac{1}{2}\log n}\right)$, i.e. exponential
  ($\mathcal{O}^*$ means, we leave out polynomial factors)

Fermat Test

Lemma

If p is prime, and $\gcd(a, p) = 1$, *then* $a^{p-1} \equiv 1 \mod p$.

Only implication! No "if and only if"!

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- Let $n = 97$, $a = 68$. Have $a^{n-1} \mod n = 1$, so n passes.

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  But $a = 19 \implies a^{n-1} \mod n = 64 \implies n$ is not prime.
- Let $n = 561$, then n passes for every a, but $n = 3 \cdot 11 \cdot 17$.

Fermat Test – Analysis

- quick: need $\mathcal{O}(\|n\|)$ arithmetic operations per run,
- run t times, with different a
- correct answer if n prime
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Fermat test has no success guarantee > 0.

Miller-Rabin Test

- Developed by Artjuhov ('67), Miller ('76'), Rabin ('80).
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Idea

- if n prime, then $x^2 \equiv 1 \mod n$ only has solutions $x = \pm 1$
- in Fermat a^{n-1} is an even power
- taking roots, we should arrive at -1
- for odd powers, we cannot compute root
  (find root equivalent to factoring),
  so we must stop

Miller-Rabin Test

Miller-Rabin-Test

function MILLER-RABIN(n)

pick random $a < n$

if $\gcd(a, n) \neq 1$ **then return** False

write $n - 1 = 2^s \cdot k$, for k odd

if $a^k \equiv 1 \mod n$ **then return** True

for $i = 0, \ldots, s - 1$ **do**

if $(a^k)^{2^i} \equiv -1 \mod n$ **then return** True

return False

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  return False

- If n not prime, $\leq \frac{\varphi(n)}{4}$ choices of base a give false answer
- run test t times, takes $\mathcal{O}(t \cdot \|n\|)$ arithmetic operations
- reject n if one iteration fails
- $\rightsquigarrow$ error chance $\leq \left(\frac{1}{4}\right)^t$

Example Miller-Rabin

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- Let $n = 91$
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  - $a^{45} \equiv 90 \equiv -1 \mod 91$
  - hence accepted (for now)

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- write $n - 1 = 90 = 45 \cdot 2^1$
- pick base $a = 38$
  - $a^{45} \equiv 90 \equiv -1 \mod 91$
  - hence accepted (for now)
- pick base $a = 23$
  - $a^{45} \mod 91 = 64 \neq \pm 1$
  - $\left(a^{45}\right)^2 \mod 91 = 1$
  - hence composite

Prime and Prejudice (2018)

- usually, testing few small numbers suffices
- many implementation fix(ed?) t or bases a_i
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Failure Chance Against Adversary

OpenSSL 1.1.1-pre6 fix $t = 2$ for $\log n \geq 1300$,
failure chance $\frac{1}{16}$

GNU GMP bases a_i depend deterministically on n,
100% failure for $t \leq 15$

LibTomMath $t \leq 256$, use first t primes as bases,
100% failure

Creating Adversarial Input

Counting false witnesses

- let $S(n)$ be how many bases pass test for (composite) n
- so far, we have upper bound $S(n) \leq \frac{\varphi(n)}{4}$ for false witnesses
- Can we reach this bound?

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Theorem (Monier, '80)

Assume we write

$$n = 2^s \cdot k + 1 = \prod_{i=1}^{m} p_i^{e_i}$$

with primes $p_i = 2^{s_i} k_i + 1$ *and* k, k_i *odd. Then*

$$S(n) = \left(\prod \gcd(k, k_i)\right) \left(\frac{2^{\min(s_i)\cdot m} - 1}{2^m - 1} + 1\right)$$

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Corollary

Let x odd with with $2x + 1$ and $4x + 1$ prime. Then $n = (2x + 1)(4x + 1)$ achieves the worst error chance for Miller-Rabin.

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apply formula.

- $p_1 = 2x+1$, $p_2 = 4x+1$, so $k_1 = k_2 = x$, $s_1 = 1$, $s_2 = 2$
- $n = 8x^2 + 6x + 1$, so $s = 1$ and $k = 4x^2 + 3x$
- $\gcd(k, k_i) = x$, hence $S(n) = 2x^2 = \frac{\varphi(n)}{4}$ □

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Construction: guess x and check primality

Miller-Rabin – Wrap-Up

Do's

- stick to the pseudo-code
- use random bases
- t rounds give $2t$ bit security level

Miller-Rabin – Wrap-Up

Do's

- stick to the pseudo-code
- use random bases
- t rounds give $2t$ bit security level

Don't's

- small number of rounds t: can efficiently create adversarial input
- fixed bases: can create input with guaranteed false answer, procedure more involved, but feasible

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Idea for AKS

- Let $a, n \in \mathbb{N}$. We have $(x+a)^n \equiv x^n + a^n$ in $\mathbb{Z}_n[x]$ iff n is prime.
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- reduce this modulo smaller polynomial
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- Takes time $\mathcal{O}\left(\|n\|^{6+\varepsilon}\right)$, too much

Sieve of Erathosthenes

- not really a primality test
- good method to generate all primes $p \leq n$

function ERATHOSTHENES(n)
  create array $a_i = 1$ for $i \leq n$
  $i \leftarrow 2$
  while $i^2 \leq n$ **do**
    if $a_i = 1$ **then**
      for $j = 2, \ldots, \lfloor n/i \rfloor$ **do**
        $a_{i \cdot j} \leftarrow 0$
  return $\{p : a_p = 1\}$

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- on PC, for $n = 2^{30}$ about 6 seconds (with some optimisations)
- running time $\mathcal{O}(n \log \log n)$, space $\mathcal{O}(n)$
- only use, if array fits into RAM!

Primality Tests – Overview

Fermat: easy, fast, can have one-sided errors, fails for some numbers

Miller-Rabin: Method of choice

- as fast as Fermat,
- also one-sided error
- repeated, independent calls: error $\searrow 0$
- in most crypto-libraries,
  but many implementations were (are?) vulnerable to
  malicious input $\rightsquigarrow$ "Prime and Prejudice"

AKS: no error, but long running time

Erathosthenes: no test, but a method to generate all primes $\leq n$
only recommended for $n <$ RAM

Prime Generation – Revisited

Optimisation

- chance of primality is $\sim 1/\log n$
- improve factor by ruling out small primes as divisors
  e.g. prime odd, better $p = 6 \cdot k \pm 1, \ldots$

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How not to do it:

RSAlib by Infineon

- enumerate primes p_i for $i = 1, 2, \ldots$
- put $M := \prod_{i \leq s} p_i$ (primorial),
  for $s = 39, 71, 126, 225$, depending on key-size
- choose random k, a and put $p = kM + (65537^a \bmod M)$

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- choose random k, a and put $p = kM + (65537^a \bmod M)$
- p not divisible by any of the small primes
- increase the chance of p to be prime

RSAlib – First Concern

Numbers for 2048 Bit Key – 1024 Bit Prime

- $s = 126$, so $M = 2 \cdot 3 \cdot \ldots \cdot 701 \sim 971$ Bit
- leaves only $k \sim 53$ Bit
- $\varphi(M) \sim 968$ Bit, but $o_{\mathbb{Z}_M^*}(65537) \sim 255$ Bit,
- i.e. 713 Bits entropy lost!

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512 - 960	39	62	154
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1984 - 3936	126	255	713
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but it gets worse

ROCA – Return of Coppersmith Attack

Theorem (Coppersmith)

Let $p = \sum a_{ij} x^i y^j \in \mathbb{Z}[x, y]$ irreducible; X, Y bounds for solutions. Put $W := \max\{|a_{ij}| \cdot X^i Y^j : i, j\}$ and $\delta = \max(\deg_x(p), \deg_y(p))$. Assume $XY < W^{2/(3\delta)}$. Then we can find integer root (x_0, y_0) with $|x_0| < X$, $|y_0| < Y$, if it exists.

Find "small" integer roots in 2 variables.

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- $a + b = \log_{65537}(n \bmod M)$,
  easy to compute, since M has only small primes
  (compute for each prime, compose with CRT)
- guess a, yields b, then compute k, ℓ via Coppersmith

ROCA – Application

Assume 2048 bit key $\rightsquigarrow$ 1024 bit prime

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Then check $XY < W^{2/3}$, which holds (by far).
Hence, Coppersmith finds solution k, ℓ, i.e. the primes

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- less unknown bits for a, but more unknown bits for k, ℓ
- Coppersmith takes longer, but much less attempts
- for each key-length find optimal trade-off
- 2048 bit takes ca. 35 CPU-years, cost ~ 1240 € (rough guess)
  $\rightsquigarrow$ feasible for private person
- easily in parallel

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  generate p that is for sure not divisible by $2, 3, 5, 7, 11, \ldots, p_s$
- create remainders $a_i < p_i$, use CRT on

$$x \equiv a_i \mod p_i \qquad \text{for } i = 1, 2, \ldots, s$$

- increasing $s \rightsquigarrow$ higher chance for prime, but also more time

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  generate p that is for sure not divisible by $2, 3, 5, 7, 11, \ldots, p_s$
- create remainders $a_i < p_i$, use CRT on

$$x \equiv a_i \mod p_i \qquad \text{for } i = 1, 2, \ldots, s$$

- increasing $s \rightsquigarrow$ higher chance for prime, but also more time

Exercise

Analyse Effort and speedup of this idea: theory and practice
Warning: don't have too much hope

Theory

- as before use primorial $M := \prod_{i \leq s} p_i$
- prime candidate $kM + a$ where a is solution of CRT
- thus $a \in \mathbb{Z}_M^*$ random, instead of random from $\mathbb{Z}_M$
- increase chance by factor $M/\varphi(M)$, practically ≤ 12

$$\frac{M}{\varphi(M)} = \prod \frac{p_i}{p_i - 1} = \prod \left(1 + \frac{1}{p_i - 1}\right) > \sum \frac{1}{p_i} \sim \ln \ln p_s$$

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- begin primality test with trial division
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- long part is Miller-Rabin on actual prime
- less random bits, but barely any speed gain

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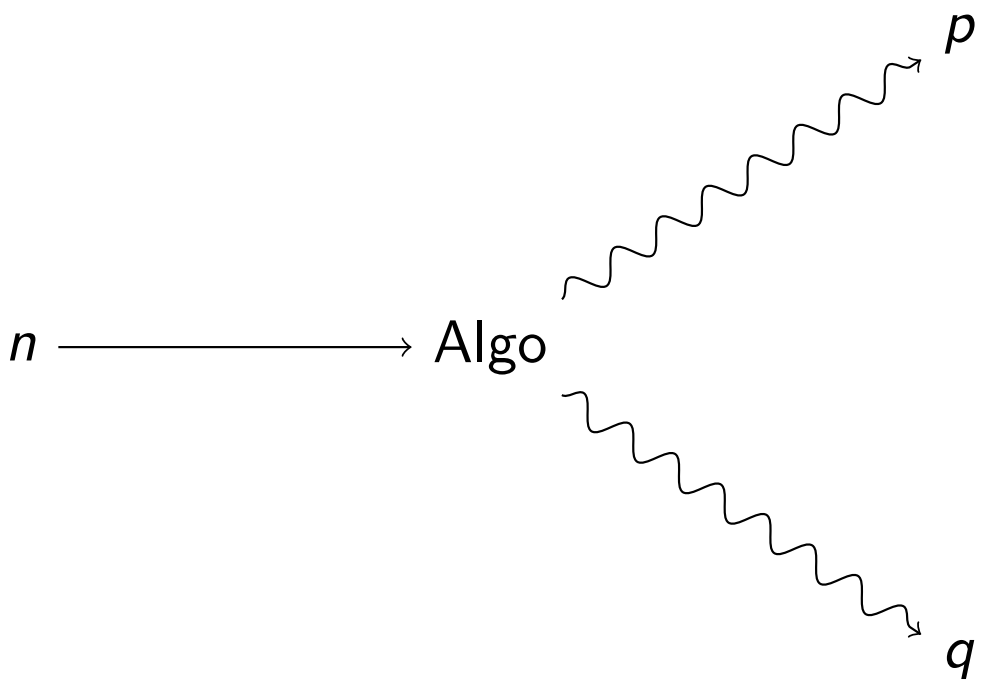
$\rightsquigarrow$ picking odd random number works well enough to find prime

Factoring



The Factorisation Problem

Task

Given $n \in \mathbb{N}$, find a non-trivial divisor d of n.

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Decision problem

Given $n, U \in \mathbb{N}$, does n have a prime divisor p with $p \leq U$?

- Factoring is neither known to be in P nor known to be NP-complete.
- problem lies in NP $\cap$ coNP, hence most likely not NP-complete
  - prime factor $p \leq U$ serves as witness
  - factorisation with all $p_i > U$ serves as non-witness

Fermat Factorisation

Method

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$$n = pq = \left(\frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2$$

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Pseudocode

$m \leftarrow \lceil \sqrt{n} \rceil$
for $i \in \mathbb{N}$ **do**
    $\Delta_i = \sqrt{(m+i)^2 - n}$
    if $\Delta_i \in \mathbb{N}$ **then**
        return $p \leftarrow m + i - \Delta_i$

Fermat Factorisation – Analysis

$$\Delta_i = \sqrt{(\lceil \sqrt{n} \rceil + i)^2 - n}$$

$$n = pq = \left(\frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2$$

- wlog $p > q$, then $\Delta_i = \frac{p-q}{2}$ in the end
- reach this when $m + i = \frac{p+q}{2}$,

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- in total: works well if p, q nearly same in upper half

Fermat Factorisation – Example

Example

Let $n = 583$, thus $m = 25$

$$
\begin{aligned}
i &= 0 & \Delta_i^2 &= 2 \cdot 3 \cdot 7 \\
i &= 1 & \Delta_i^2 &= 3 \cdot 31 \\
i &= 2 & \Delta_i^2 &= 2 \cdot 73 \\
i &= 3 & \Delta_i^2 &= 3 \cdot 67 \\
i &= 4 & \Delta_i^2 &= 2 \cdot 3 \cdot 43 \\
i &= 5 & \Delta_i^2 &= 317 \\
i &= 6 & \Delta_i^2 &= 2 \cdot 3^3 \cdot 7 \\
i &= 7 & \Delta_i^2 &= 3^2 \cdot 7^2
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$\Delta_7 = 21 \rightsquigarrow p = 25 + 7 - 21 = 11$ and $q = 25 + 7 + 21 = 53$

Quadratic Sieve

Idea

- construct $a^2 \equiv b^2 \bmod n$ from steps of Fermat-factorisation
  i.e. $a^2 - b^2 = k \cdot n$ instead of $a^2 - b^2 = n$
- take them as collection of congruences
- combine some, to get squares on both sides
- $\gcd(a \pm b, n)$ divisor with probability $\geq \frac{1}{2}$

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- details, "why" it works, too complicated for this lecture
  but "how" is okay
- good for up to 100 decimal digits
- used for RSA-129 (from 1977), solved in 1994
- running time

$$\mathcal{O} * \left(\exp \left(\sqrt{\log n \cdot \log \log n} \right) \right)$$

Quadratic Sieve – Example

Example

Let $n = 583 = 11 \cdot 53$, use Fermat

$$i = 0 \qquad m + i = 25 \qquad \Delta_i^2 = 2 \cdot 3 \cdot 7$$

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obtain $\gcd(25 \cdot 31 - 2 \cdot 9 \cdot 7, 583) = 11$

Quadratic Sieve – Digging Deeper

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Solution: factor into small primes

- more general approach
- try to factor the Δ_i into small primes
- regard exponent modulo 2 (square or not)
- solve linear equation system modulo 2 to find combination

Factoring Through p_k-smooth Numbers

Let p_k be the k-th prime.

Definition

An integer is p_k-smooth, if all its prime divisors are $\leq p_k$.

- use Sieve of Erathosthenes!

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Idea (Morrison & Brillhart '75; Dixon '81)

- search for numbers a such that ($a^2 \bmod n$) is p_k-smooth
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Idea also used by Schnorr in his recent (failed) attempt at factoring. Instead of the first k primes, we may use any set of fixed primes.

- assume we have $k+1$ such numbers $a_0, \ldots, a_k$ with

$$(a_j^2 \bmod n) = \prod_{i=1}^{k} p_i^{d_{ij}}$$

- define the matrix $\boldsymbol{D} \in \{0,1\}^{(k+1)\times k}$ via $D_{ij} := d_{ij} \bmod 2$. regard it as row vectors
- non-trivial solution $\boldsymbol{t} \cdot \boldsymbol{D} = \boldsymbol{0} \mod 2 \quad \boldsymbol{t} \in \{0,1\}^{k+1}$

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- have $x^2 \equiv y^2 \mod n$ (mult. the entries with $t_j = 1$)
- 50% chance: $x \equiv y \bmod p$ and $x \equiv -y \bmod q$ (or vice versa)
- get p, q from $\gcd(x+y, n)$ or $\gcd(x-y, n)$

Quadratic Sieve – Algorithm

- put $m = \lceil \sqrt{n} \rceil$, empty matrix $\boldsymbol{D}$
- for $j = 0, 1, \ldots$ try to factor

$$(m+j)^2 - n = \prod p_i^{d_{ij}} \cdot \text{remainder}$$

- if remainder $= 1$:
  store $m + j$ and append $d_{*,j}$ mod 2 to matrix $\boldsymbol{D}$

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- do Gaussian elimination on a copy $\boldsymbol{D}'$
- break if $\boldsymbol{D}'$ has zero-row
- construct $x^2 \equiv y^2 \mod n$ as above
- $p := \gcd(x \pm y, n)$, if $p \in \{1, n\}$, try again

Example (Back to the Quadratic Sieve)

Let $n = 583$, and factor base $S = \{2, 3, 5, 7\}$

i	2	3	5	7	remainder	store
0	1	1	0	1	1	✓
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already have matrix of lower rank $\leadsto$ break

$$\boldsymbol{D} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Example (Back to the Quadratic Sieve, cont.)

- try to factor $(m+i)^2 - n$ by factors from S
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- hence, we get

$$(25 \cdot 31)^2 \equiv (2 \cdot 3^2 \cdot 7)^2 \mod 583$$

General Factorisation

Rough Steps

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$\rightsquigarrow$ time polynomial in $\|n\|$

Factoring – Overview

- Trial division only for small numbers, $\leq 2^{64}$
- checking power is feasible
- $\leq 10^{100}$ quadratic sieve
- beyond: Number Field Sieve, time roughly $\mathcal{O}^*(\exp(c\sqrt[3]{\log n}))$

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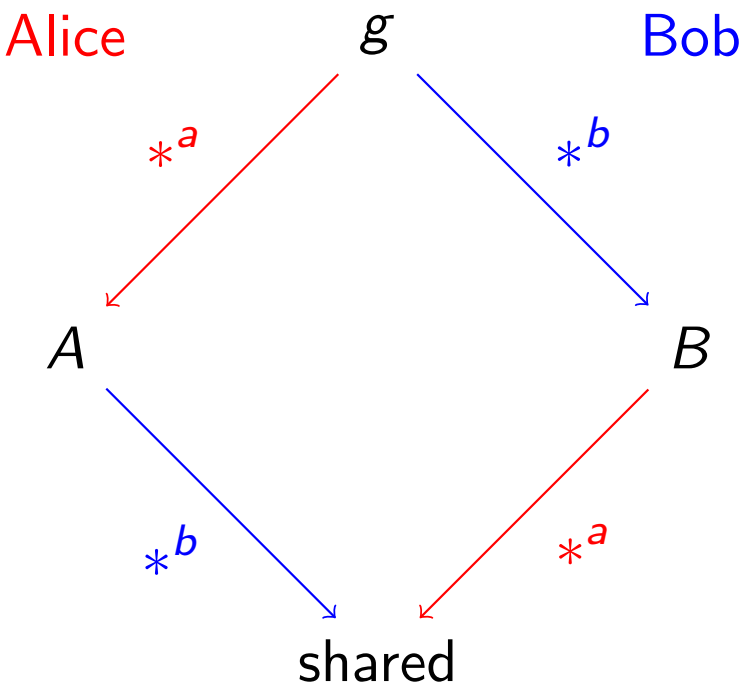
Implementations

- SymPy (slows down quickly)
- YAFU: quadratic Sieve
- cypari: number field sieve, easy from Python
- cado-nfs: fastest(?) number field sieve

Group Based Cryptography



Reminder Groups

What is a group?

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- have neutral element and inverse
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Example

- just think of $G = \mathbb{Z}_p^* = \{1, \ldots, p-1\}$ with mult. for some prime p
- neutral element 1, modular inverse

Lemma

There always is some $g \in \mathbb{Z}_p^*$ with $\langle g \rangle = \mathbb{Z}_p^*$.

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Exercise

If we can solve the DLP in $\mathbb{Z}_n$, we can also factor n.

Diffie-Hellman Key-Exchange (1976)

Overview

- first published idea of public key cryptography
- no crypto-system, but key exchange
- we do not encode messages (yet), but get a common key then e.g. continue with symm. encryption
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Method

- Alice chooses $a < o(g)$, computes $A = g^a$, sends A to Bob
- Bob chooses $b < o(g)$, computes $B = g^b$, sends B to Alice
- Alice computes key $K = B^a$
- Bob computed key $K = A^b$
- works, because $(g^a)^b = g^{ab} = g^{ba} = (g^b)^a$

Diffie-Hellman Key-Exchange – Example

Common, public agreement

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common secret $K = 5997$

ElGamal (1985)

Key Generation

- secret key: choose random $a < o(g)$
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- Encrypt: m message to be encrypted
  choose random $b < o(g)$,
  send $(B, c) = \left(g^b, m \cdot A^b\right)$
- Decrypt: get (B, c), compute $m = c \cdot (B^a)^{-1}$

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- in fact just "asynchronous" Diffie-Hellman,
- use secret from handshake as mask

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- Bob chooses $b = 5456$,
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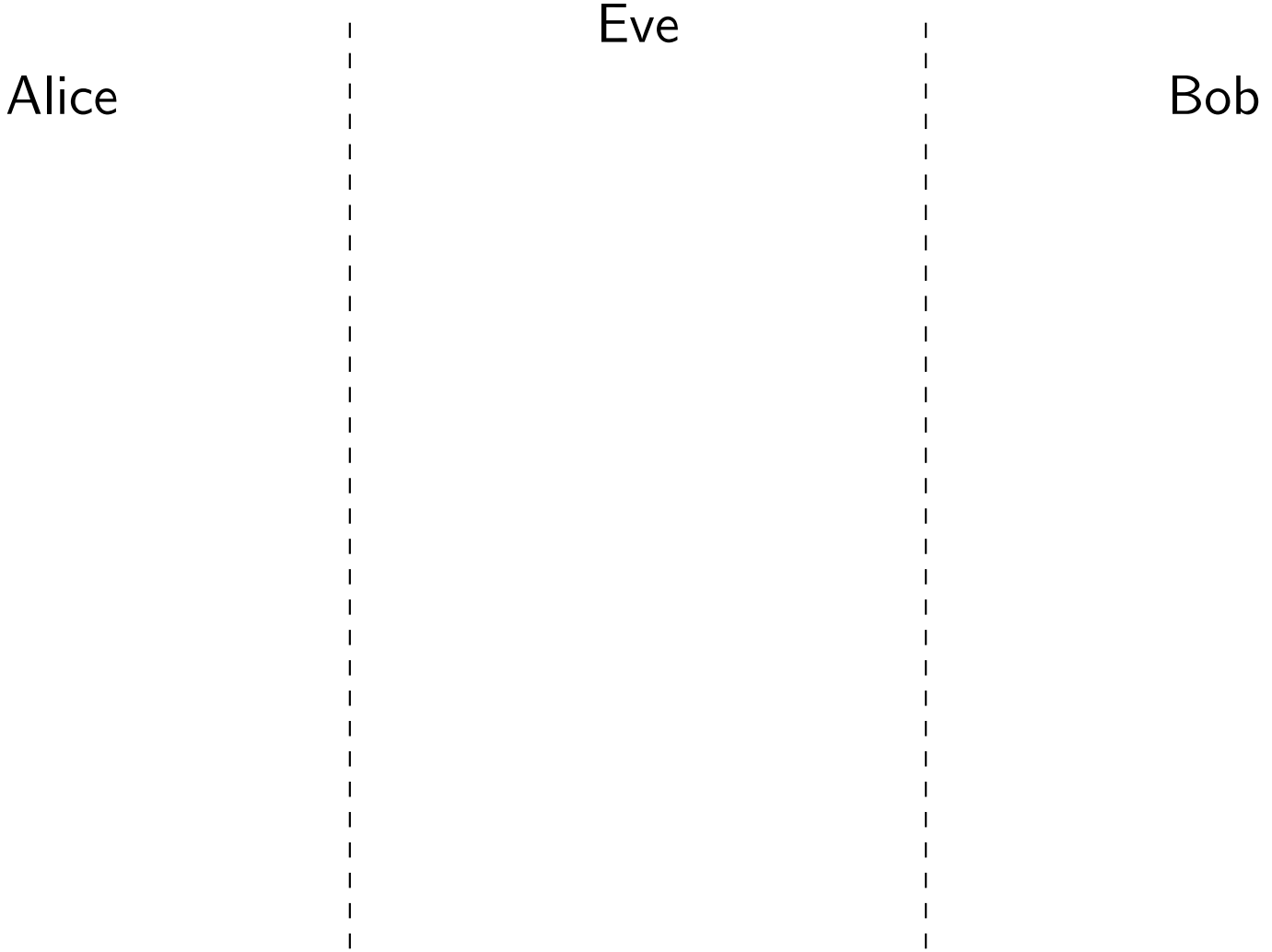
Alice decrypts:

- original message via $(B^a)^{-1} \cdot c = 102$

Diffie-Hellman from Eve's view

Public Knowledge: $g, G = \langle g \rangle$



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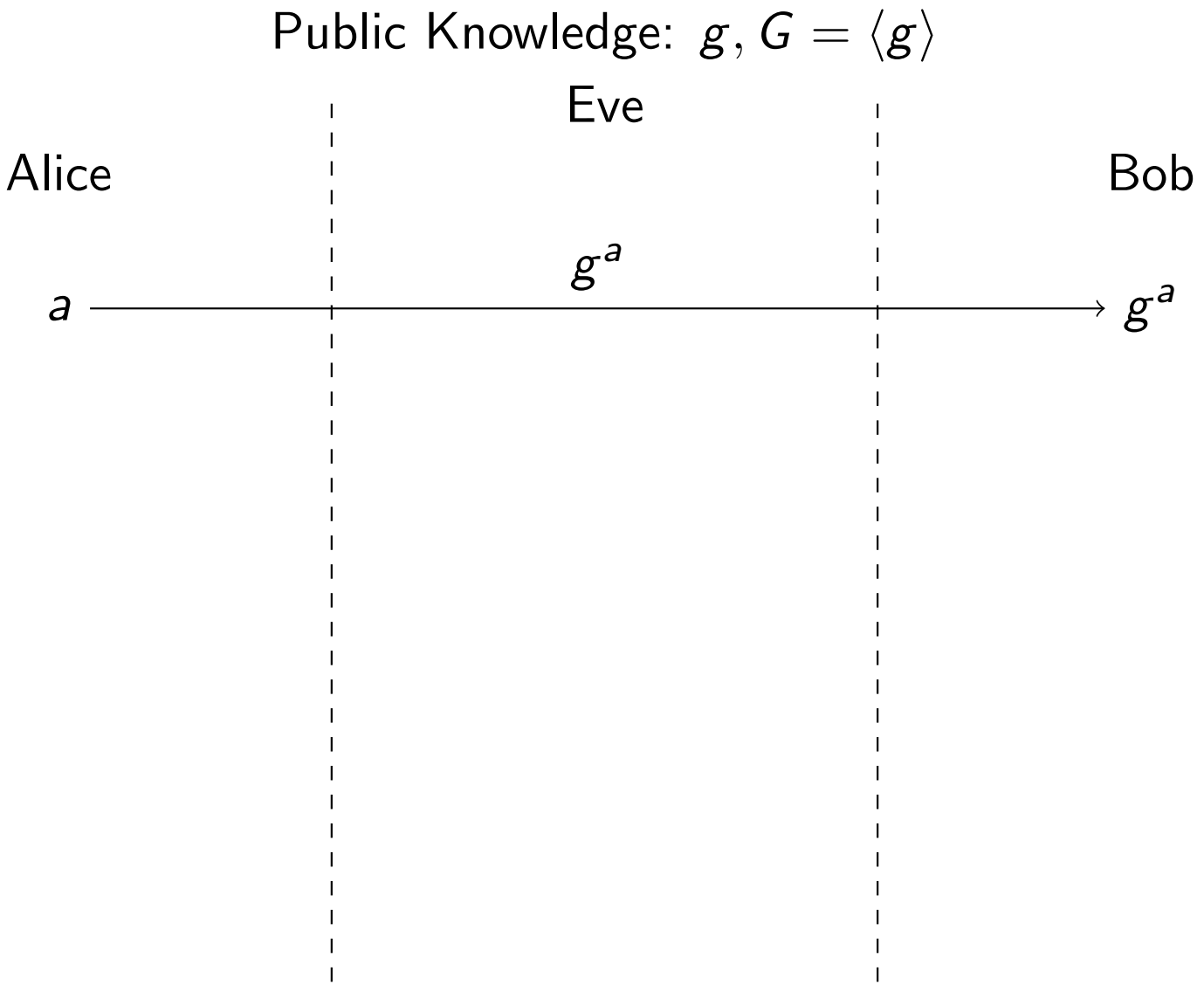

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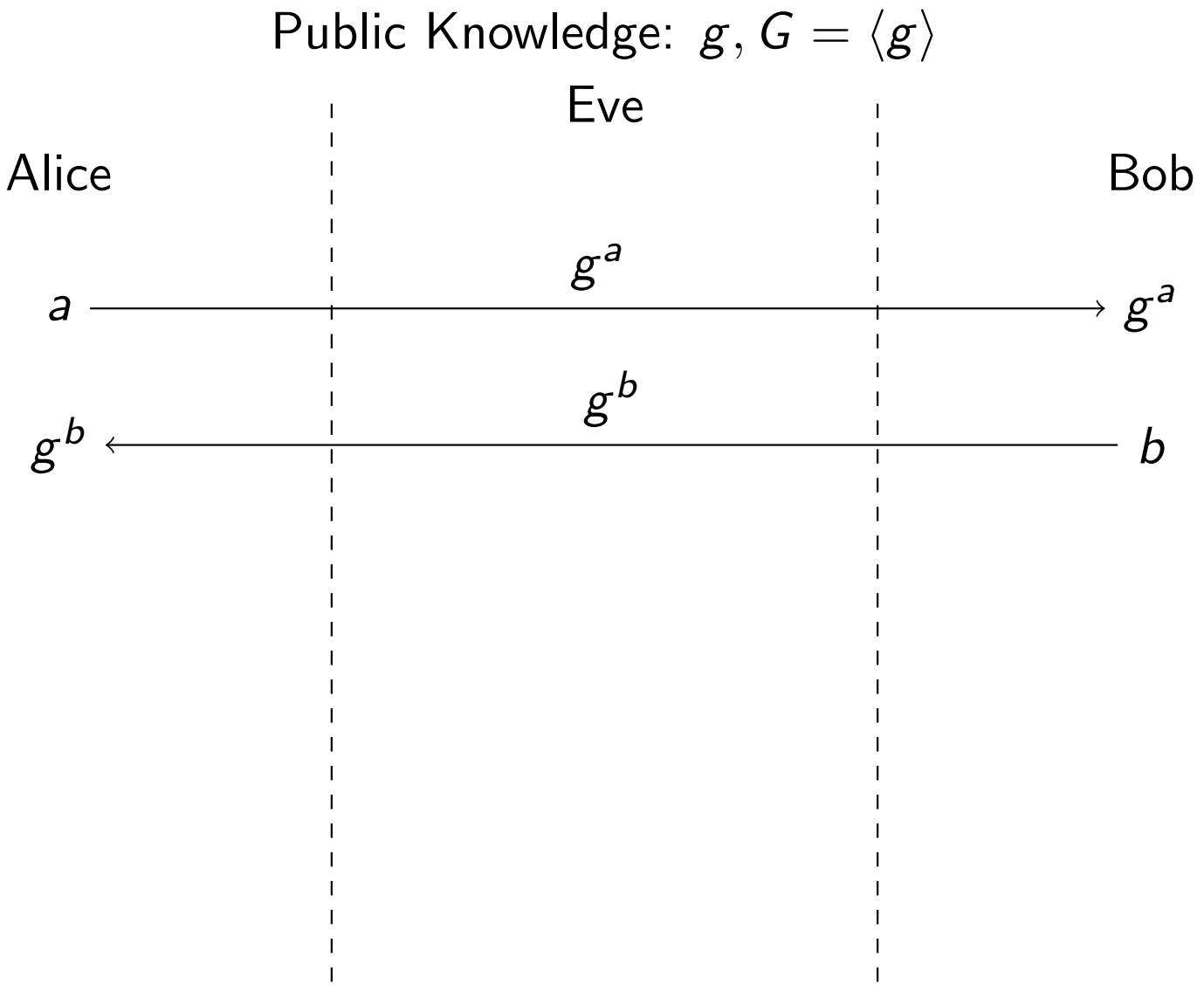

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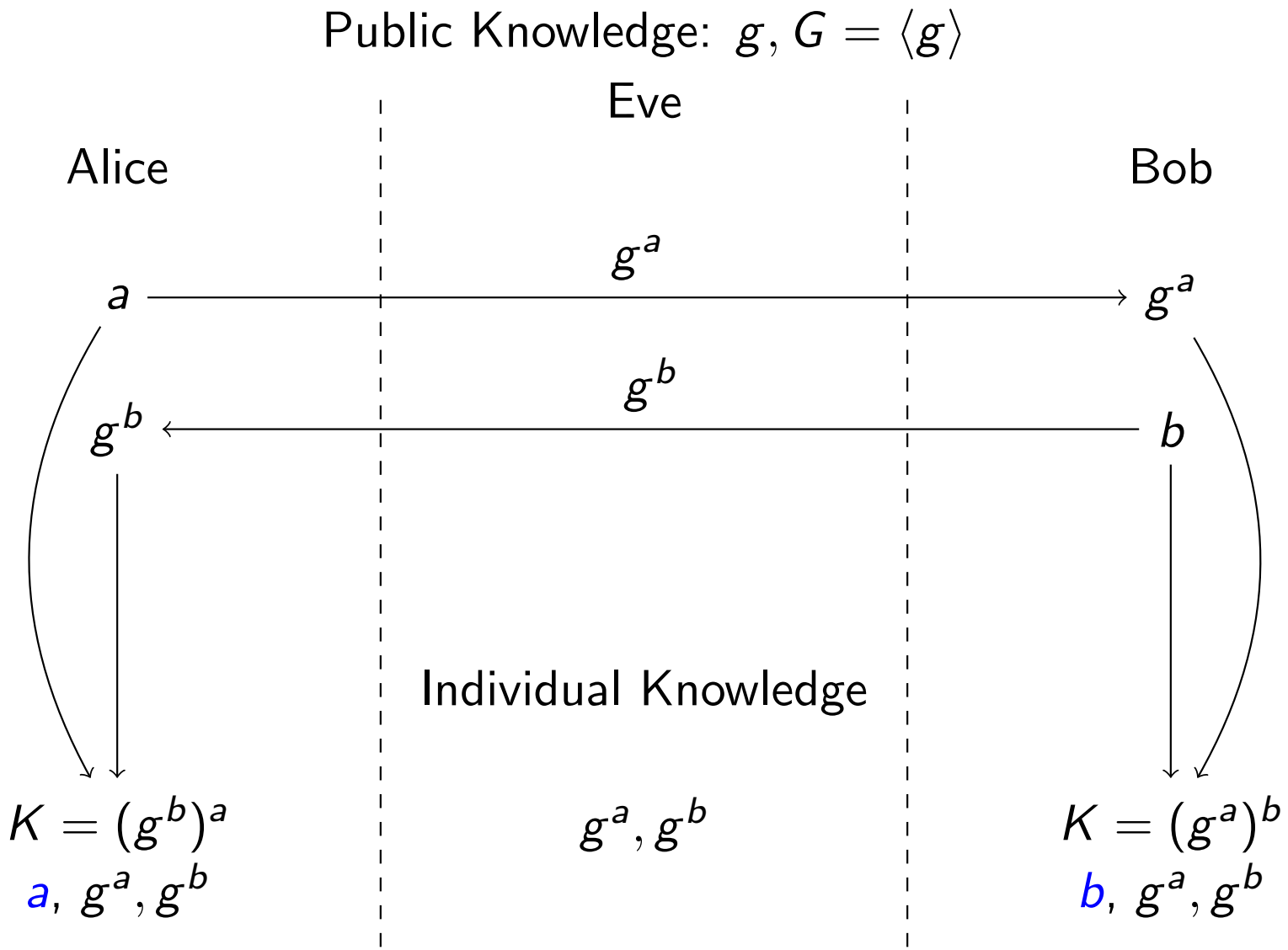

Attack Scenarios on Diffie-Hellman

Discrete Logarithm (DLP): given g, g^x, find x

find secret key

Computational DH (CDH): given g, g^a, g^b, find g^{ab}

find session key

Decisional DH (DDH): given g, g^a, g^b, h, decide $g^{ab} = h$

decide, which cipher belongs to message

recognise session key

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Attacks on DLP

- Generic Attacks
- Attacks that exploit properties of the group

Brute-Force

Brute-Force attack on DLP

Input:

g – generator of group

$y = g^x$ for unknown x

Output: x – discrete log

function $\mathrm{DLP}(g,y)$

for $x = 0$ **to** n **do**

if $g^x = y$ **then**

return x

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Analysis

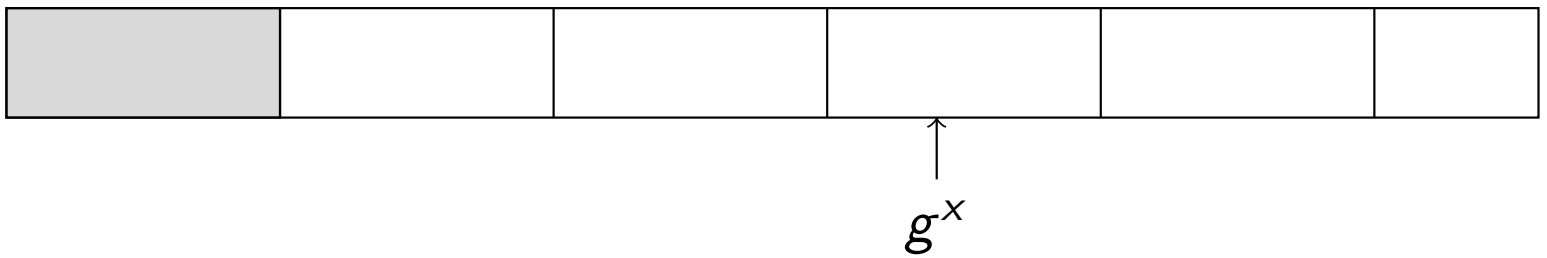
Let $n = o(g)$

Time: $\mathcal{O}(n)$ worst-case and expected

Space: $\mathcal{O}(1)$ numbers/group elements

which are technically of size $\mathcal{O}(\log n)$ each

Shanks Baby-Step-Giant-Step – Picture



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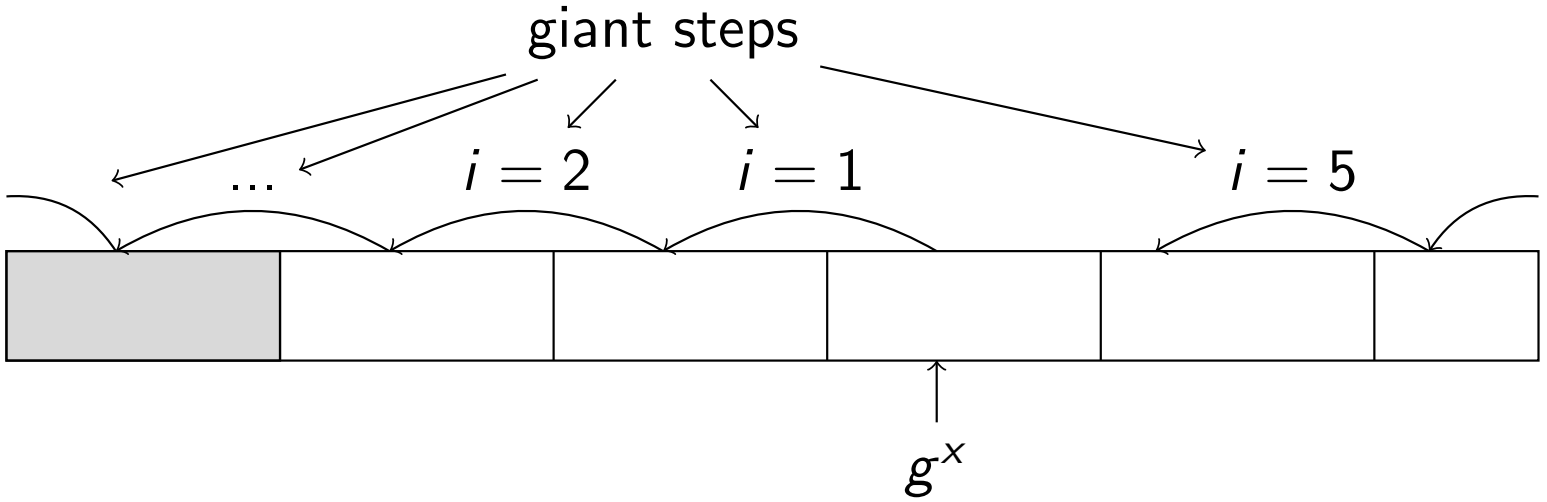

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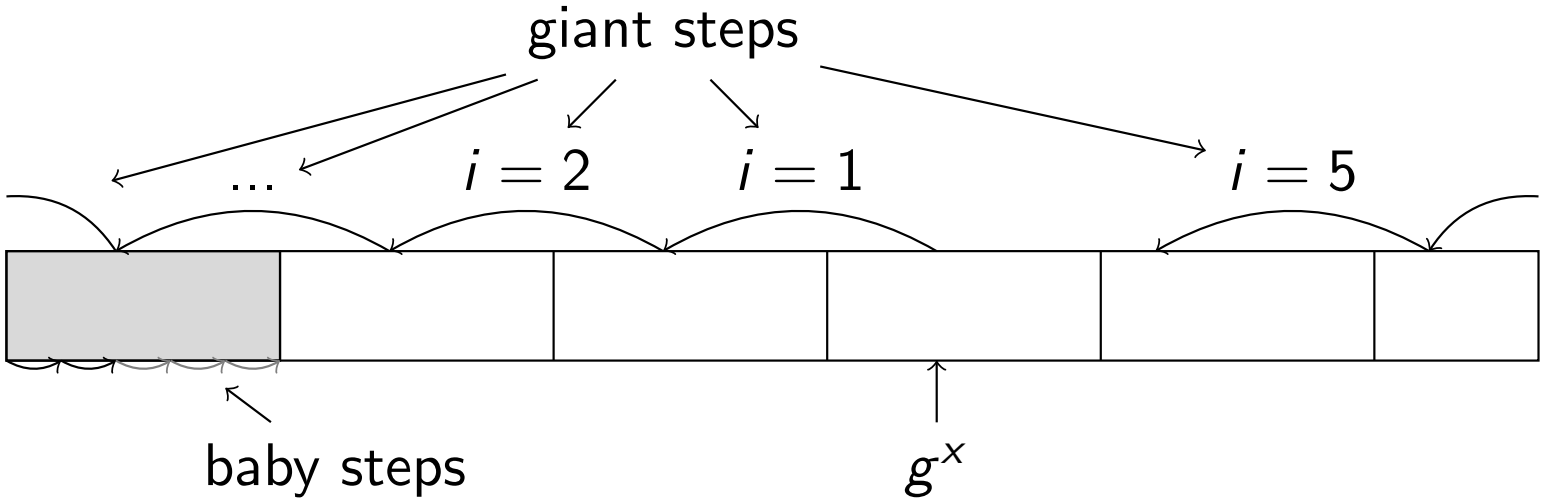

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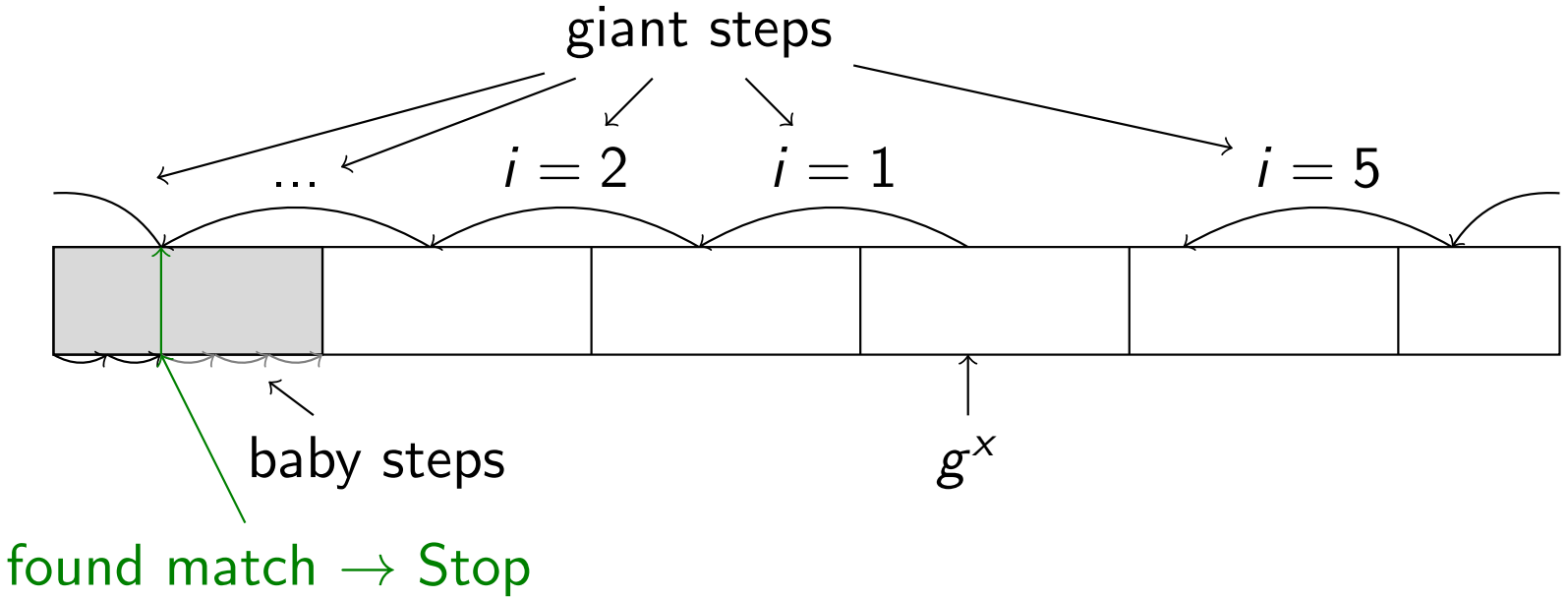

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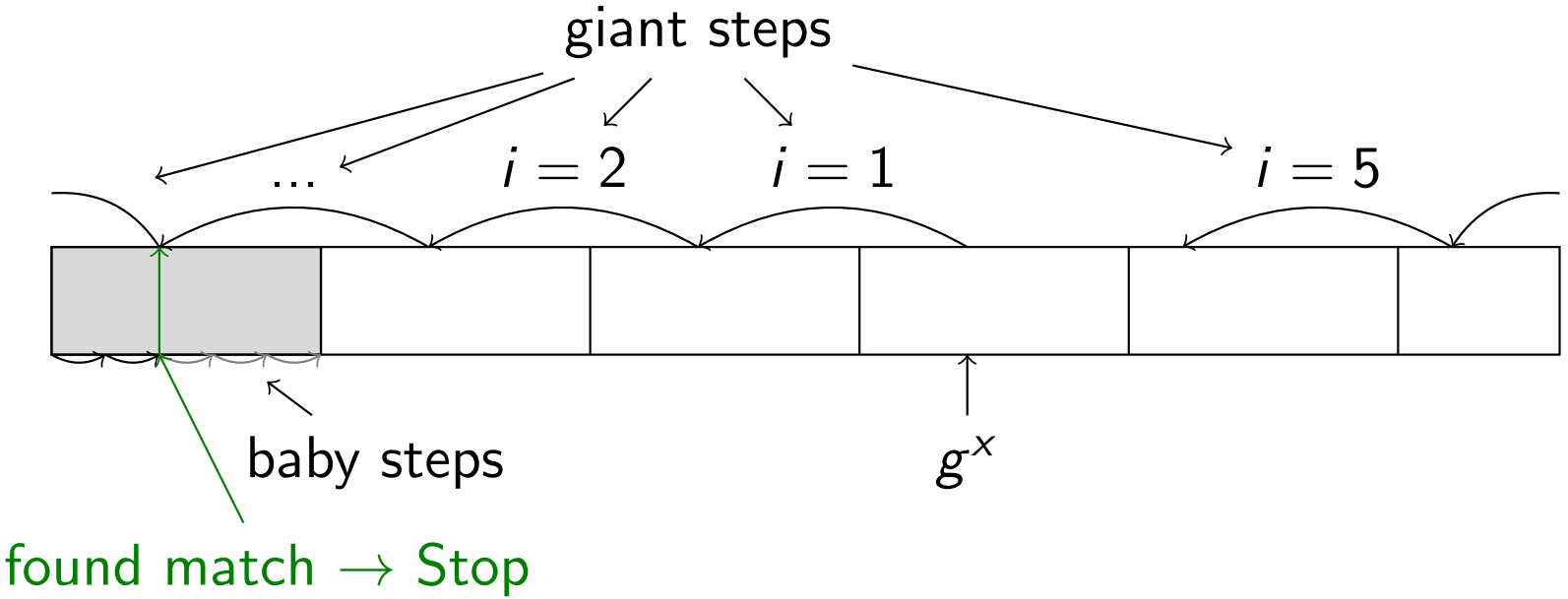

- store all giant steps
- can forget past baby steps
- some giant step will land in first (grey) block (but don't know which)
- some baby step will give a match

Baby-Step-Giant-Step – Formal

- Solve DLP: given g, g^x find x
- Let $n = o(g)$, pick giant-step size k
- secret x has unique representation $x = ki + j$ with $j < k$

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- compute powers via single steps, to improve speed
- compute once $s = \left(g^k\right)^{-1}$,
  then always "multiply" s in first loop
- always "multiply" g in second loop

Pohlig-Hellman Algorithm

Overview

- improve computation if factorisation of $n = o(g)$ is known
- solve problem for subgroups of prime power size
- compose with CRT
- Let p be largest prime divisor of n, running time

$$\mathcal{O}(\text{poly}(\|n\|) \cdot \sqrt{p})$$

- essential part of ROCA (see earlier)

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Protection

- ensure n has large prime divisor
- "safe prime" p: select p such that $\frac{p-1}{2}$ also is prime, if $G = \mathbb{Z}_p^*$, then $n = p - 1$, ensured $n = 2 \cdot p'$, Pohlig-Hellman does not help

Breaking Prime Powers (Hensel Lifting)

- assume $|G| = n = p^e$ and $y = g^x$
- write $x = \sum_{i<e} x_i p^i$ in base p, then find "digits"
- put $h = g^{p^{e-1}}$, of order p (note $h^p = g^{p^e} = 1$)
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$$y^{p^{e-1}} = \left(g^{x_0 + x_1 p + \ldots + x_{e-1} p^{e-1}}\right)^{p^{e-1}} = g^{x_0 p^{e-1} + p^e \cdot *} = h^{x_0}$$

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- find $x_0 = \log_h \left(y^{p^{e-1}}\right)$, e.g. via Shanks in $\mathcal{O}(\sqrt{p})$
- continue for $i = 1, \ldots, e-1$

$$\left(y \cdot g^{-(x_0 + \ldots + x_{i-1} p^{i-1})}\right)^{p^{e-i-1}} = g^{x_i p^{e-1} + p^e \cdot *} = h^{x_i}$$

Composing Solution

- Assume we have factorisation

$$n = |G| = p_1^{e_1} \cdot \ldots \cdot p_\ell^{e_\ell}$$

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$$n_i := n/p_i^{e_i} \qquad g_i := g^{n_i} \qquad y_i := y^{n_i}$$

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- running time $\mathcal{O}\left(\sum_i e_i(\log n + \sqrt{p_i})\right)$ group operations
  note: $\sum e_i \leq \log n$

Return of ROCA

Recall – Return of Coppersmith

- primorial $M = \prod p_i$ product of first primes
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- given $65537^{a+b} \bmod M$, find $a + b$

- work in $\mathbb{Z}_M^*$
  in fact just the subgroup generated by 65537
- group size is $\varphi(M) = \prod_i^s (p_i - 1)$
- each factor small $< p_s$, so only small prime factors
  can actually factor each factor by trial division

$\Longrightarrow$ Pohlig-Hellman works

Overview Generic Attacks for DLP

In group $G = \langle g \rangle$ of size n, given g, g^x find x

Presented Methods

Shanks: meet-in-the-middle, space and time $\mathcal{O}(\sqrt{n})$

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Other Methods

Pollard's Rho algorithm: probabilistic,
avoids large storage, time $\mathcal{O}(\sqrt{n})$

Pollard's Lambda/kangaroo algorithm: probabilistic,
if restricted to interval of size w, time $\mathcal{O}(\sqrt{w})$

DLP in Different Groups

Diffie-Hellman handshake is a template

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Common Examples of (Finite) Groups

- additive group $(\mathbb{Z}_n, +)$
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- symmetric group S_n (permutations)
- invertible matrices $GL(n, p^k) = \left\{ M \in \mathrm{GF}(p^k)^{n \times n} : \det M \neq 0 \right\}$

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Do not yield significant cryptographic advantage over RSA.

Elliptic Curves

- regard some curve in dimension 2
- define an "addition" for the points of that curve
- $\rightsquigarrow$ new kind of group (actually since end of 19th cent.)
- make everything discrete and finite

Additive Groups

Additive Group $(\mathbb{Z}_n, +)$

- group exponent is just multiple

$$y := g^x = \underbrace{g + \ldots + g}_{x\text{-times}} = x \cdot g$$

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- actually have isomorphism $\varphi : \langle g \rangle \cong (\mathbb{Z}_{o(g)}, +)$, with $\varphi(g) = 1$, in general $\varphi(g^k) = k$ no matter which choice of g and G
- translate from any group into $(\mathbb{Z}_n, +)$?
- but finding isomorphism is the DLP

Symmetric Group S_n

- breakable (bit like Pohlig-Hellman): regard cycles!

$$g = c_1 \circ c_2 \circ \ldots \circ c_k$$

- disjoint cycles independent: $g^x = c_1^x \circ \ldots \circ c_k^x$

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- let ℓ_i be cycle lengths, then have $x \bmod \ell_i$ for all i
- compose with (generalised) CRT modulo the lcm
- result is bounded by

$$o(g) \leq \ell_1 \cdot \ldots \cdot \ell_k \overset{\text{AM-GM}}{\leq} \left(\frac{\ell_1 + \ldots + \ell_k}{k}\right)^k = \left(\frac{n}{k}\right)^k \leq e^{n/e}$$

bit size $\|o(g)\| \in \mathcal{O}(n)$, for input size $\mathcal{O}(n \log n)$

Symmetric Group S_n – Example

Example (DLP in S_n)

take base element

$$g = (1, 7)(2, 6, 8)(3, 5, 4, 9, 10)$$

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yields system

$$a \equiv 1 \mod 2 \qquad a \equiv 2 \mod 3 \qquad a \equiv 2 \mod 5$$

with solution $a = 17$

Invertible Matrices

General Linear Group

$$\mathrm{GL}(n, p^k) = \left\{ M \in \mathrm{GF}(p^k)^{n \times n} : \det M \neq 0 \right\}$$

$\mathrm{GF}(q)$ field with q elements (not $\mathbb{Z}_q$ if q is proper prime power)

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We can transfer the DLP in $\mathrm{GL}(n, p^k)$ *to the DLP in* $\mathrm{GF}(p^{kn})$.

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- transfer t group with $p^{nk} - 1$ elements
- attack that one like $\mathbb{Z}_p$
- computation with matrices more expensive

$\Longrightarrow$ matrix has no advantage over $\mathrm{GF}(p^{nk})$

DLP in $\mathbb{Z}_p$

Task: given $g, n = o(g), y = g^x \mod p$, find x

Attack by Index Calculus

- pick up ideas from quadratic sieve
- let $p_1, \ldots, p_k$ be primes that can be written as $p_i = g^* \mod p$
- find powers g^r with

$$g^r \cdot y \equiv p_1^{e_1} \ldots p_k^{e_k} \mod p$$

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- enough of them give linear equation system

$$\log_g y \equiv -r + e_1 \log_g p_1 + \ldots + e_k log_g p_k \mod n$$

variables $\log_g y$ and the $\log_g p_i \rightsquigarrow$ solve

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- find powers g^r with

$$g^r \cdot y \equiv p_1^{e_1} \ldots p_k^{e_k} \mod p$$

- enough of them give linear equation system

$$\log_g y \equiv -r + e_1 \log_g p_1 + \ldots + e_k log_g p_k \mod n$$

variables $\log_g y$ and the $\log_g p_i \rightsquigarrow$ solve

running time like factoring

DLP in $\mathbb{Z}_p$

Setup

- prime n such that $p = 2n + 1$ is prime (Sophie-Germain prime)
- work in $\mathbb{Z}_p$, has order $\varphi(p) = p - 1 = 2n$
- pick random $g \neq 1$ until $g^n = 1$ (chance $\approx \frac{1}{2}$)
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- best protection against Pohlig-Hellman
- same bit size as RSA for given security level
- Alice/Bob have two large exponentiations per handshake
- $\rightsquigarrow$ no advantage over RSA in that aspect
  (though better for "perfect forward secrecy")

Elliptic Curves

Definition

- let K be a finite field, $2 \neq 0 \neq 3$; e.g. $K = \mathbb{Z}_p$
- let $a, b \in K$ be parameters with $4a^3 + 27b^2 \neq 0$ (discriminant), needed to avoid degenerate case (curve behaves nicely)
- then the elliptic curve over K (in Weierstrass form) is

$$E(K) := \left\{(x, y) \in K^2 : y^2 = x^3 + ax + b\right\} \cup \{\infty\}$$

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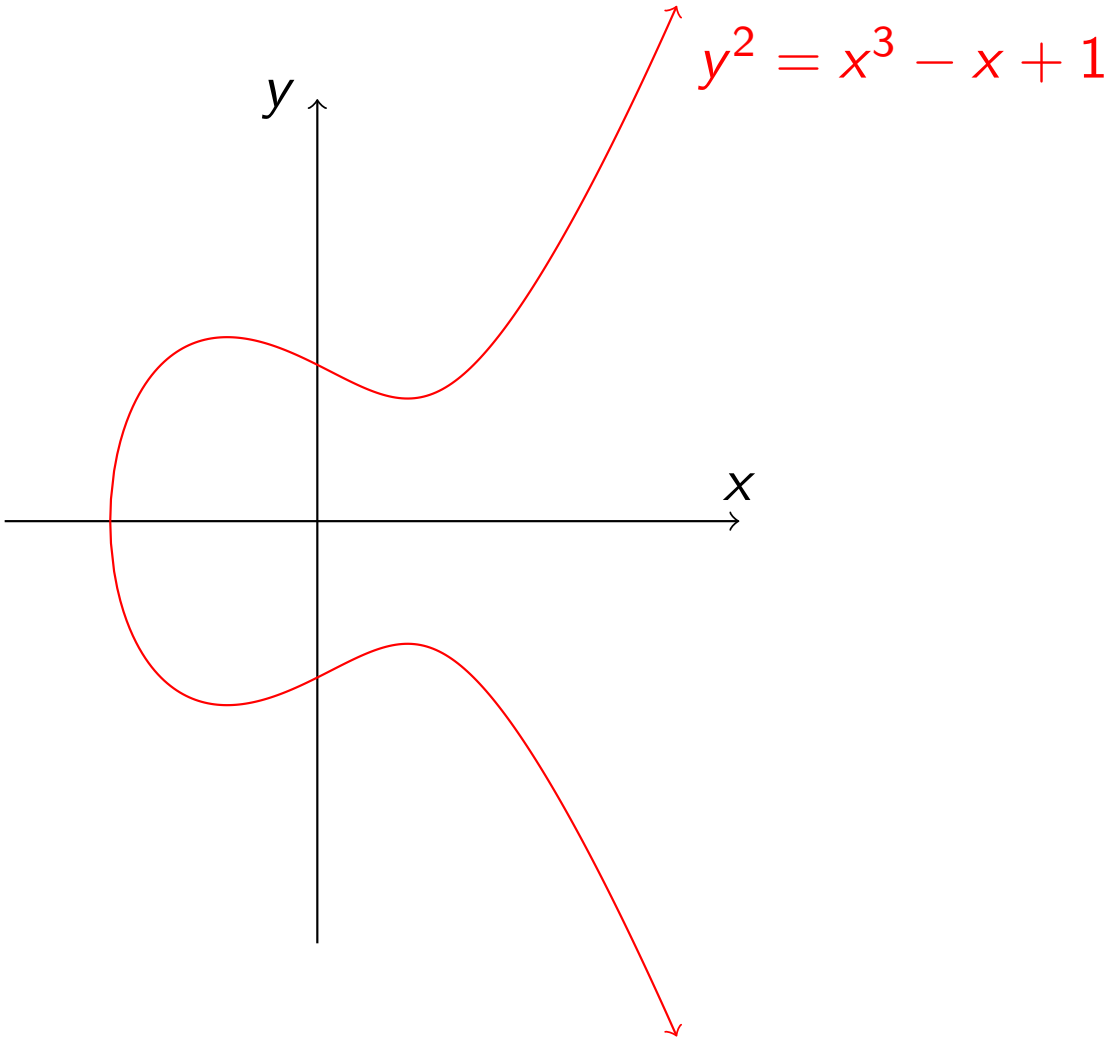
Remark

- *often used in projective coordinates, i.e. in K^3*
  no inversion in addition $\rightsquigarrow$ speed-up
- *alternative form: Montgomery curve*
  different formulas for addition

Elliptic Curves as Group



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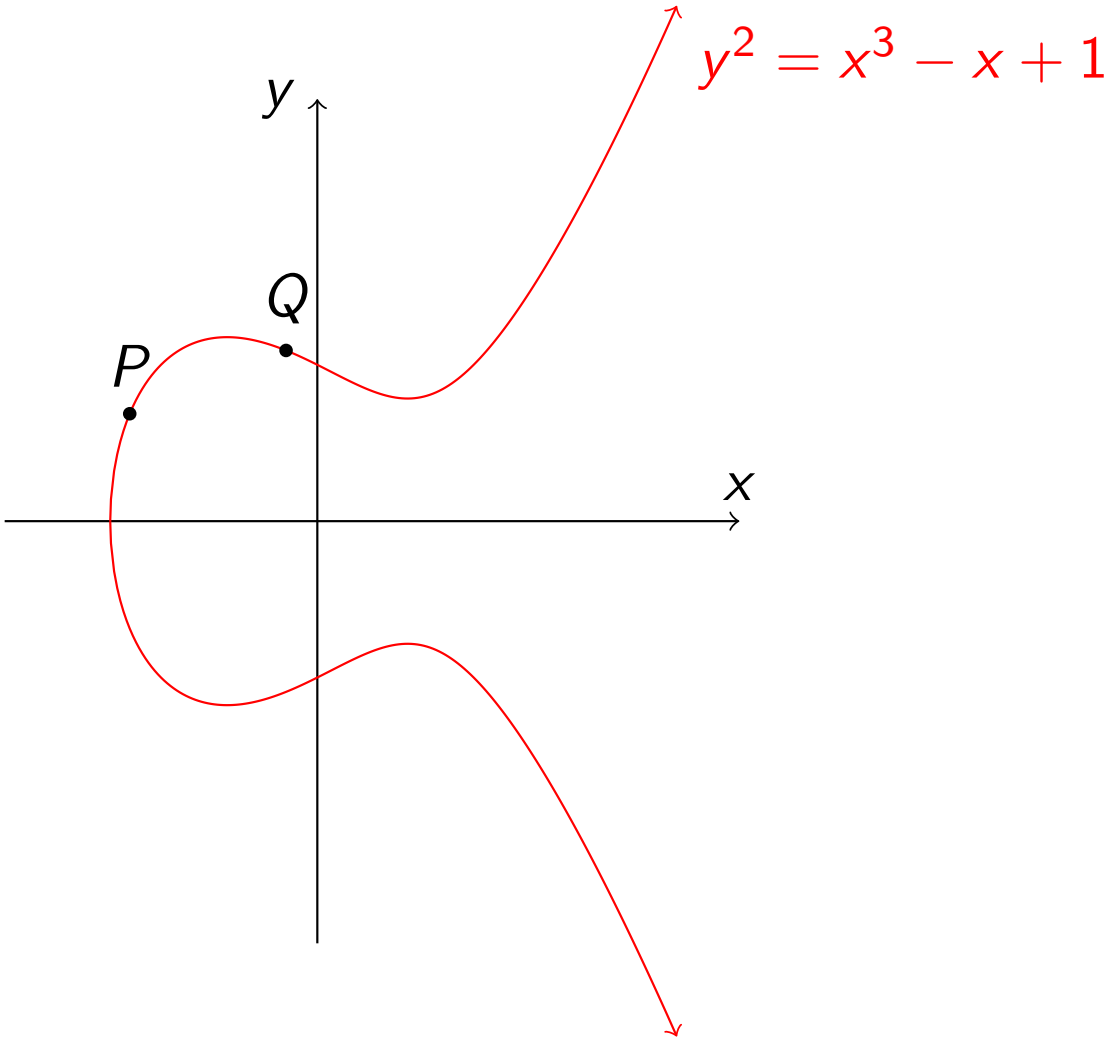

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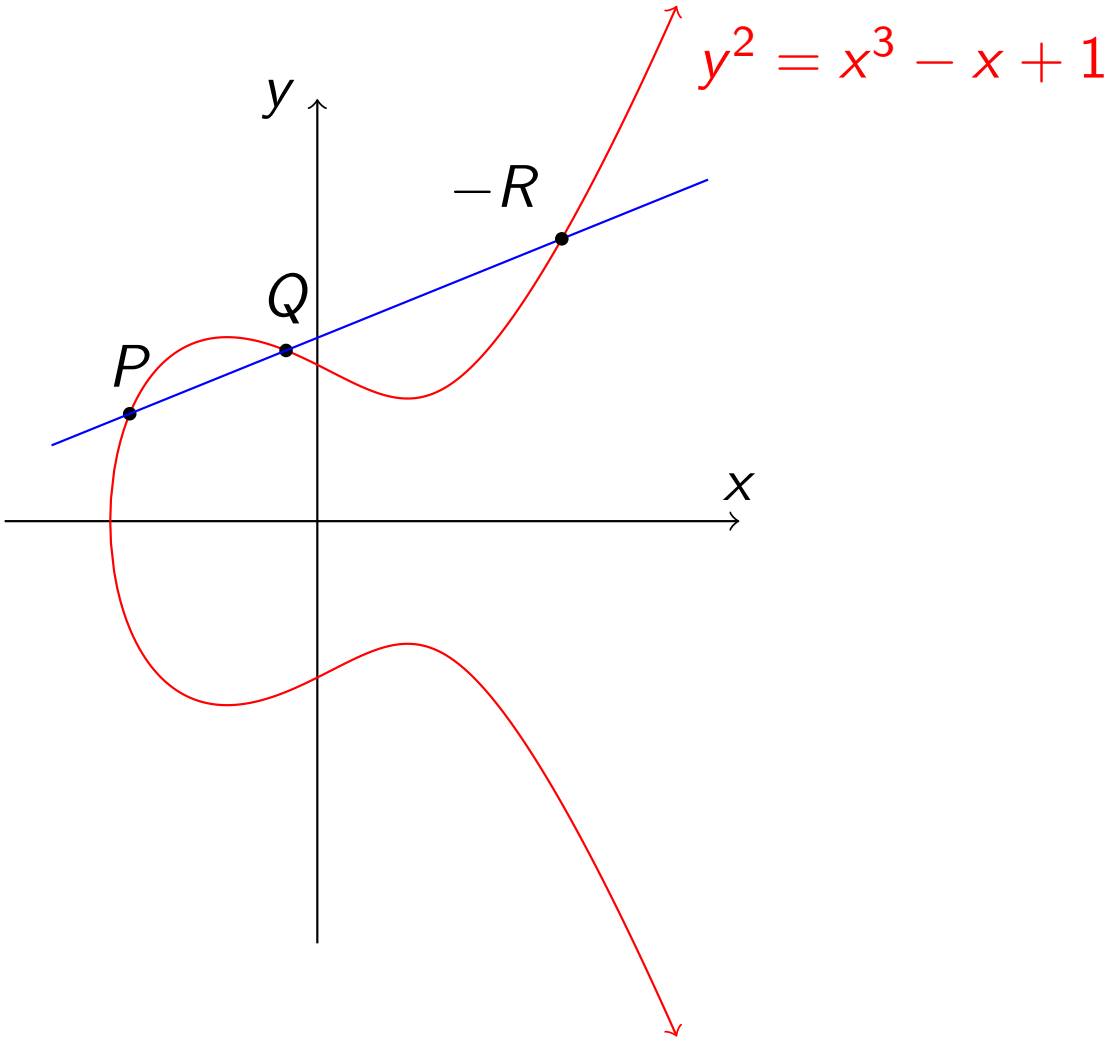

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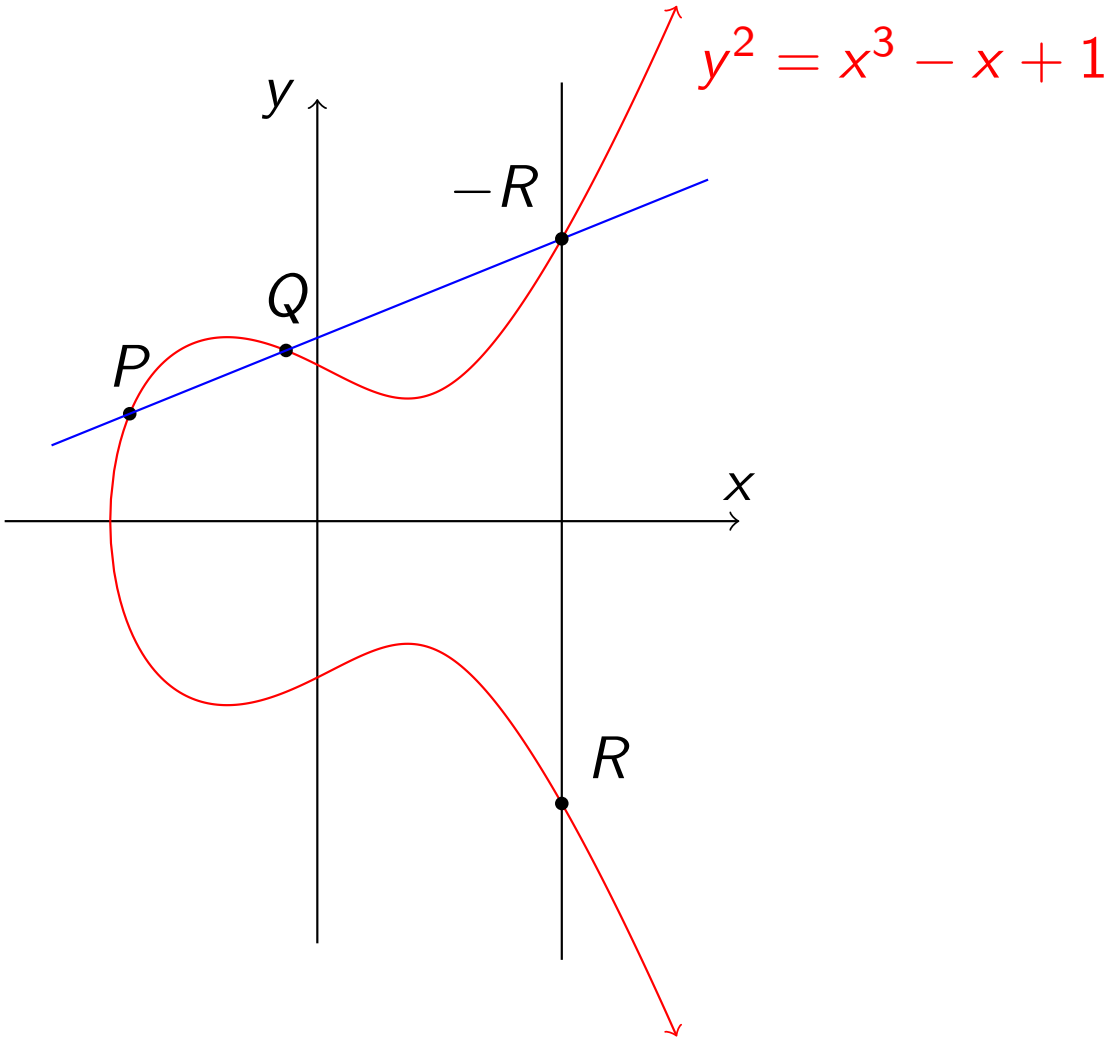

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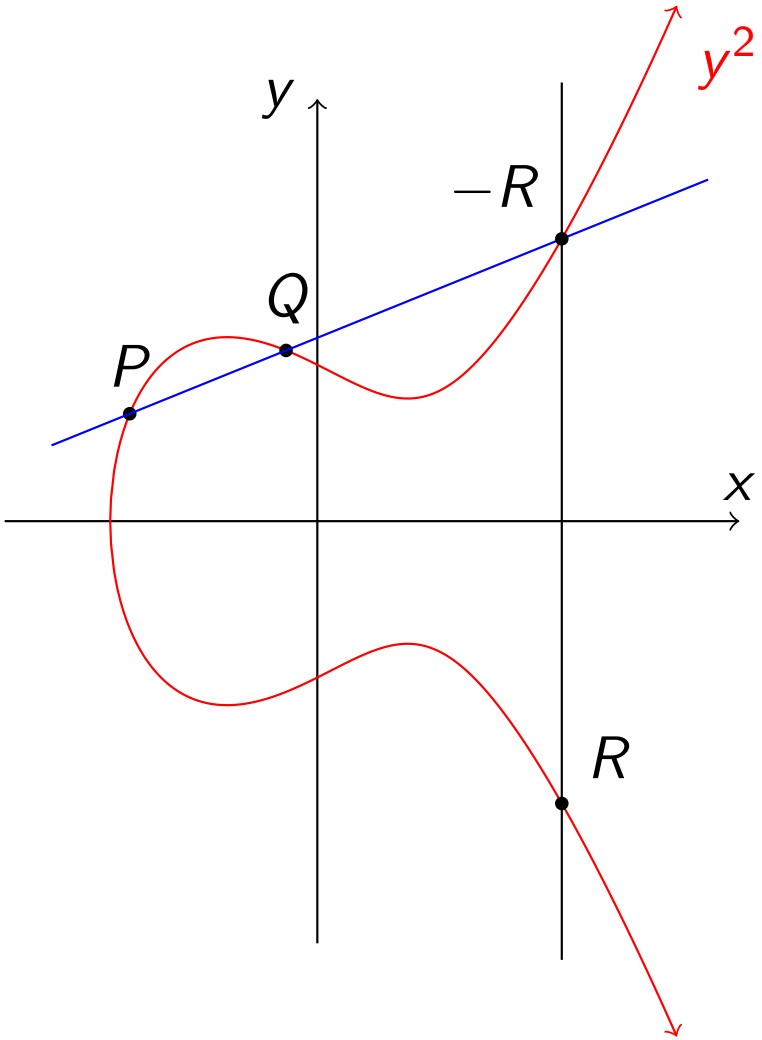

Point "Addition": $R = P \cdot Q$

- draw line through P and Q
- get third intersection with curve $\rightsquigarrow R^{-1}$
  (ensured by discriminant)
- mirror on x-axis

Case $R = P \cdot P$:

- use tangent instead

Case $P_x = Q_x$, $P_y \neq Q_y$:

- $R = \infty$

Formulas for point addition

Neutral element: $P \cdot \infty = P$ for all P

Inverse: $P_x = Q_x$ but $P_y = -Q_y$, then $P \cdot Q = \infty$

General case: $R = P \cdot Q$

$$\lambda = \begin{cases} \frac{Q_y - P_y}{Q_x - P_x} & : P \neq Q \\ \frac{3P_x^2 + a}{2P_y} & : P = Q \end{cases}$$

$$R_x = \lambda^2 - P_x - Q_x$$

$$R_y = \lambda(P_x - R_x) - P_y$$

formulas work in every field (as long as $2 \neq 0 \neq 3$)

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formulas work in every field (as long as $2 \neq 0 \neq 3$)

in fact, they even work for points not on the curve

Faulty Curve Injection

get Alice's secret key in ElGamal

Condition

- Chosen Cipher Attack
- Alice does not check whether $B \in E$

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- computations don't use b
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- works for $P, Q \notin E$

Idea

- Eve sends $X \notin E$ instead of $B \in E$
- can extract Alice's secret exponent

Faulty Curve Injection – Attack

Attack: given $A = g^a$, find a

- Eve picks random point $B' \in K^2$
- gives point on new curve $E' : y^2 = x^3 + ax + b'$
- try until order $o(B')$ has only small prime divisors

  chance is good enough, offline search

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  - chance is good enough, offline search
- send (B'^{-1}, ∞) (recall: ∞ is neutral element)
  - usually would send $(B, c) = (g^b, m \cdot A^b)$
- decryption: $m = \infty \cdot (B'^{-1})^{-a} = B'^a$
  - usually would be $m = c \cdot B^{-a}$
- use Pohlig-Hellman to compute a

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Remark

- could have used real message
- since we know c, m, we always get B'^a

Open Questions

Question

- How do we get an elliptic curve?
- Which base element do we pick?
- What is the group size?
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Constructing an Elliptic Curve

- choose random prime p of chosen bit size, work in $\mathbb{Z}_p$
- choose random $a, b \in \mathbb{Z}_p \rightsquigarrow$ defines curve
- compute size of group, want prime order
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Do once $\rightsquigarrow$ just pick some standard curve

Counting Points

Theorem (Hasse, 1933)

Let $K = \mathbb{Z}_p$. For the size $|E|$ of the curve, we have the bound

$$||E| - (p+1)| \leq 2\sqrt{p}$$

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Counting Points

let $M(k)$ denote complexity of multiplication in k digits

Baby-Step-Giant-Step: $\mathcal{O}(\sqrt[4]{p})$ group operations

Schoof: time $\mathcal{O}\left(\|p\|^2 M(\|p\|^3)/\log\|p\|\right) \approx \mathcal{O}\left(\|p\|^5\right)$

Schoof-Elkies-Atkin: time $\mathcal{O}\left(\|p\|^2 M(\|p\|^2)/\log\|p\|\right) \approx \mathcal{O}\left(\|p\|^4\right)$

significant improvement: $p^{\frac{1}{4}} \rightsquigarrow \text{poly}(\log p)$

slow, but feasible

Ideas Behind Methods

Baby-Step-Giant-Step

- $|E|$ lies in interval $\left[p+1 \pm 2\sqrt{p}\right]$ of size $4\sqrt{p}$
- pick random $P \in E$: pick random x, until $x^3 + ax + b$ is square (50% chance), compute y as root
- if only single k in interval with $P^k = \infty$, then $|E| = k$
- else try new P, chance sufficiently good
- reduce time via meet-in-the-middle

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Schoof (with a lot of Galois theory)

- find $|E|$ mod q_i for some primes q_i
- until $\prod q_i > 4\sqrt{p}$
- then $|E|$ is CRT solution in interval $p+1 \pm 2\sqrt{p}$

Find Base Element

Design Goal

subgroup $\langle g \rangle = G \leq E$ of prime order $|G| = p$ with $\|p\| \approx \|n\|$

- computational effort grows with n
- want high security level (large p) with low effort (small n)

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- trial division by the first few primes
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- compute size $n := |E|$
- trial division by the first few primes
- if remainder p not prime, start again
- Algebra: for prime $p \mid n$, there is g with $o(g) = p$

  actually $p - 1$ many
- try random g, chance $\approx p/n$

Computational Effort in Key Generation

Construct Group

group generation involves

- point counting, $\mathcal{O}^*\left(\|p\|^4\right)$
  feasible, but may need several attempts $\rightsquigarrow$ long time
- factoring: only trial division $\rightsquigarrow$ fast
- find generator: good chance $\rightsquigarrow$ fast

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Key Observation

- Alice and Bob use the same curve
- same curve for everyone
- expensive computations have to be done only once

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Individual Part

- create random number $r < p$, compute $g^r \rightsquigarrow$ easy

Optimisation in ECDH

Speed Up Computations

- frequently have to compute P^k
- use square-and-multiply (double-and-add),
  $\mathcal{O}(\log k)$ operations
- negation cheap: also use subtraction
  $k = *0, 1, 1, \ldots, 1, 1, 0*$ becomes $*1, 0, \ldots, 0, -1, 0*$
  some doubling + 1 subtraction
  worst case: $\frac{3}{2}k$ ops. (instead of $2k$)

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Side Channel Attacks

- varying time leaks information
- mostly aim for constant time
- even at the prize of longer time

Other Insecure Special Cases

Standard curves are also tested against other attacks.

Multiplicative Transfer

- let $\ell = o(g)$ with $\gcd(\ell, p) = 1$, k minimal with $\ell \mid p^k - 1$
- can transfer DLP to $\left(\mathrm{GF}(p^k)^*, \cdot\right)$, subexponential solutions

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Final Take-Away

Just use a given curve, maybe not from NIST.

Encryption with Elliptic Curves – Overview

Encryption

- choose one of the standard curves
  - all in every standard library $\rightsquigarrow$ no effort
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But what about signatures?

ECDSA – Elliptic Curve Digital Signature Algorithm

Setting

- subgroup $\langle g \rangle$ of prime size n in an elliptic curve
- $a < n$ – secret key
- $A = g^a$ – public key
- hash – some hash function, e.g. SHA
- msg – message to be signed

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Signature (ignoring edge cases)

- random $k < n$, compute $(x, y) = g^k$
- $r = x \bmod n$
- $s = k^{-1}(\text{hash}(\text{msg}) + r \cdot a) \bmod n$
- signature (r, s)

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Verification

- receive signature (r, s) and message msg
- compute $u = \text{hash}(\text{msg}) \cdot s^{-1} \bmod n$ and $v = rs^{-1} \bmod n$
- compute $(x', y') = g^u \cdot A^v$ in the curve
- accept if $r \equiv x' \bmod n$

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Correctness

plugging in the supposed values:

$$\begin{aligned}(x', y') &= g^u \cdot A^v \\ &= \left(g^{\text{hash}(\text{msg})} \cdot g^{ra}\right)^{s^{-1}} \\ &= g^k = (x, y)\end{aligned}$$

Psychic Paper

- $(x, y) = g^k$ and $r = x \bmod n$
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Why edge cases are important

- above pseudo code is vulnerable
- some implementations say $0^{-1} \bmod n = 0$
- also ∞ not treated correctly, but as with zero
- then signature $(0, 0)$ always accepted

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Fun Fact

vuln named after psychic paper in Doctor Who

Sony's failure with the PS3

- fixed value k (instead of random)
- for two messages m, m' get signatures (r, s) and (r, s')

$$s - s' = k^{-1}(\text{hash}(m) + ra - \text{hash}(m') - ra)$$

$$\implies k = \frac{\text{hash}(m) - \text{hash}(m')}{s - s'}$$

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Countermesaure Without Randomness – RFC 6979

- generate k from msg and a
  iterated use of HMAC (hash, concatenate, xor)
- k still is unique for every message

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- public key is g^a, hence also based on DLP
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Lesson Learned

- even large corporations/libraries fail
- edge cases are important in adversarial setting
- follow the pseudo code

Post Quantum Cryptography

What to do, if Eve has a quantum computer and I don't.

Current Situation

When Quantum Computers Arrive

- quantum computers can solve factoring and DLP
- both RSA and every DH scheme get broken

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New Crypto Schemes

- base crypto scheme on NP-hard problem, hard on average
- most common candidates:
  - lattice problems
  - multivariate polynomials
  - problems from coding theory

Lattice

Definition

Given a base of vectors $B = \{v_1, \ldots, v_n\}$, their lattice is

$$L(B) = \text{span}_{\mathbb{Z}}(B) = B\mathbb{Z}^n = \left\{ \sum_{i=1}^{n} a_i v_i : a_i \in \mathbb{Z} \right\}$$

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Properties

- bases A, B create same lattice if $A = UB$ for some $U \in \mathbb{Z}^{n \times n}$ with $\det U = \pm 1$ (U is unimodular)
- isomorphism $L \cong \mathbb{Z}^n$ for every lattice
  but isomorphism destroys angles and distances

Lattice Problems

Shortest Vector Problem (SVP)

- given L, find a vector $v \in L \setminus \{\mathbf{0}\}$ with $\|v\|$ minimal

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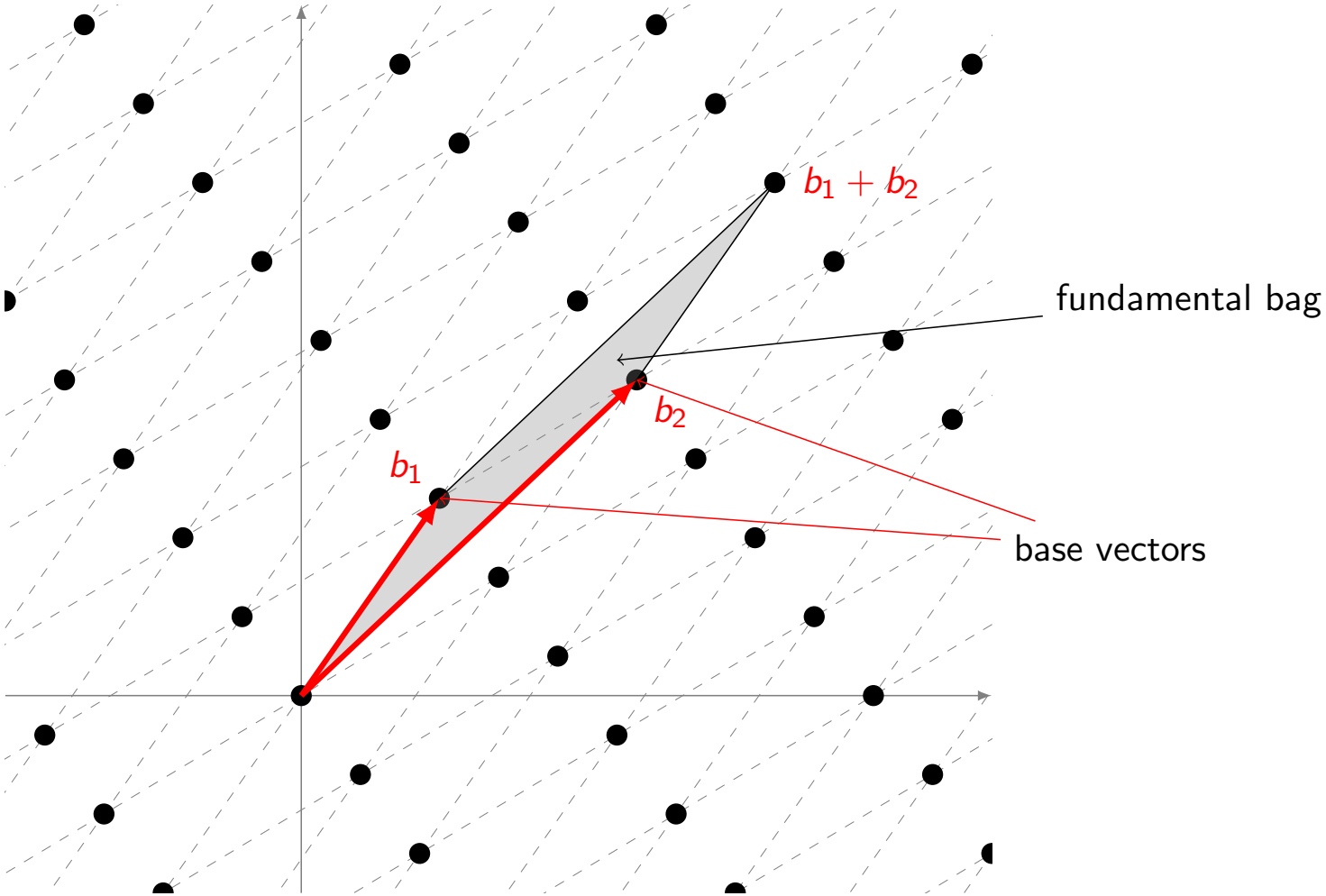
Hardness

- SVP NP-hard under randomised reduction
- CVP is NP-complete
- short base makes SVP and CVP significantly easier

Lattices



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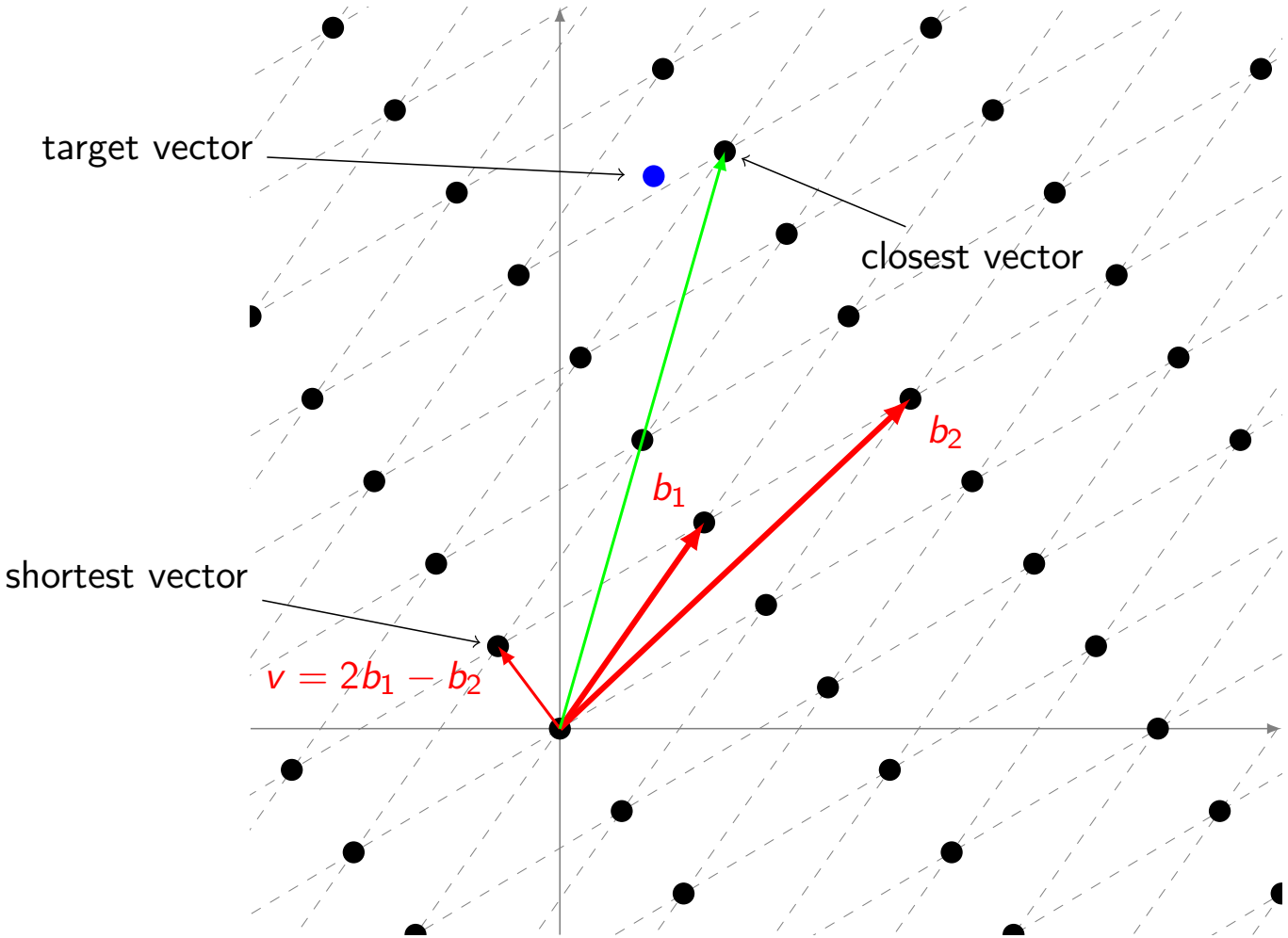

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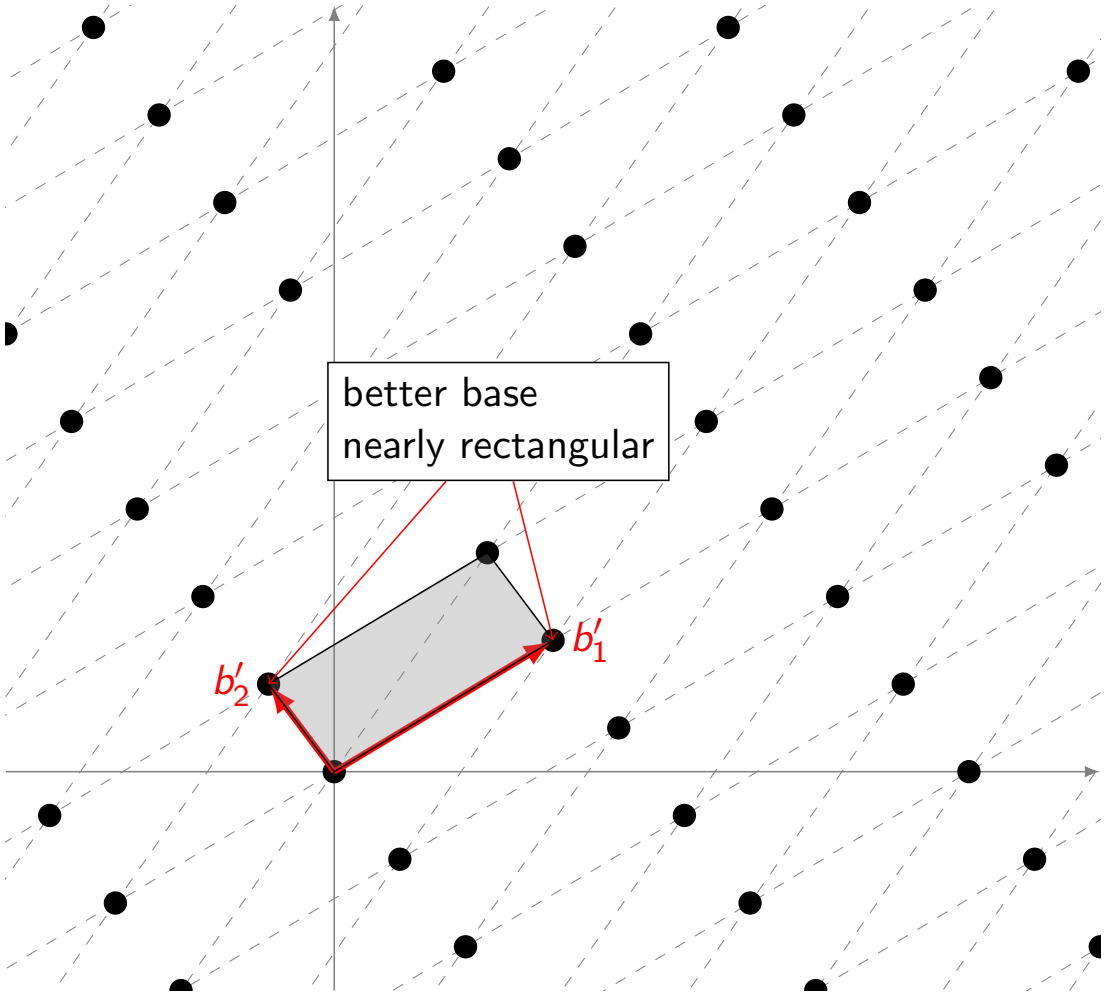

Heuristic Solutions

CVP – Babai's Roundoff

- lattice $L = L(B)$
- given $u \in \mathbb{Z}^n$, find closest $v \in L$
- solve linear equation system $Bx = u$ in $\mathbb{Q}$
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$$B = \begin{pmatrix} 6 & 10 \\ 7 & 12 \end{pmatrix} \qquad u = \begin{pmatrix} 3.8 \\ 4.1 \end{pmatrix} \qquad \Longrightarrow \; x = \begin{pmatrix} 2.3 \\ -1 \end{pmatrix}$$

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SBP/SVP — LLL Algorithm

- LLL algorithm gives reduced lattice
- shortest base vector can differ from optimum by exponential factor

From Lattices to Cryptography

Tasks

- math problem $\rightarrow$ crypto scheme/key exchange
- $\mathbb{Z}^n$ is unbounded
  - want something finite
  - what changes if we add some mod p?
- how to create "always hard instances"?
- actual parameters?

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What can go wrong?

current research

NTRU – (n-th Degree Truncated Polynomial Ring)

Overview

- proposed in 1997, relatively mature
- feasible key size, still unbroken, NIST post-quantum candidate

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- coprime numbers p, q; standard $p = 3$, $q = 2^*$
- sets $\mathcal{L}_f, \mathcal{L}_g, \mathcal{L}_r, \mathcal{L}_m \subseteq \mathbb{Z}[x]$ of polynomials with "small" coefficients, usually coeff.s $\{-1, 0, 1\}$

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Warning

- Not all parameter sets work!
- notion of "correct" parameters, details later

see modulo as $\mathbb{Z}_p = \{\lfloor -p/2 \rfloor, \ldots, 0, \ldots, \lfloor p/2 \rfloor\}$

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- pick random $f, g \in R$ with small coefficients, $f \in \mathcal{L}_f, g \in \mathcal{L}_g$
- let $f_q = f^{-1} \bmod q$ and $f_p = f^{-1} \bmod p$
  solve linear equation systems; fail $\rightsquigarrow$ new f
- public key: $h := p \cdot f_q \cdot g \bmod q$
- secret key: f, f_p (g, f_q not needed any more)

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Decryption

- $a = f \cdot c \bmod q$
- $m = f_p \cdot a \bmod p$

Why/When NTRU works?

- In decryption

$$a = f \cdot c \bmod q = f(rh + m) \bmod q = f(rpf_q g + m) \bmod q$$
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- hence $q > 2n(p+1)$ is a "correct" choice

NTRU and Lattices

Calculation mod $x^n - 1$ allows for special translation.

Translate polynomials into lattices

- polynomial = vector of its coefficients, also as matrix

$$v = \sum_{k=0}^{n-1} v_k x^k \cong \begin{pmatrix} v_0 \\ v_1 \\ \vdots \end{pmatrix} \cong \begin{pmatrix} v_0 & v_{n-1} & \cdots & v_1 \\ v_1 & v_0 & \cdots & v_2 \\ & & \ddots & \\ v_{n-1} & v_{n-2} & \cdots & v_0 \end{pmatrix}$$

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- adding polynomials = adding vectors = adding matrices
- multiplication of polynomials f, g:

$$\text{Matrix}(f) \cdot \text{Vector}(g) = \text{Vector}(f \cdot g)$$

Hence, we are in the realm of lattices.

NTRU and Lattice Problems

Break Key

f, g only have small entries

$$(f, g) \in \mathcal{L}\left(\begin{pmatrix} I_n & 0 \\ p^{-1}h & qI_n \end{pmatrix}\right) \subseteq \mathbb{Z}^{2n}$$

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Find Message

r, m only have small entries

$$(r, c - m) \in \mathcal{L}\left(\begin{pmatrix} I_n & 0 \\ h & qI_n \end{pmatrix}\right) \subseteq \mathbb{Z}^{2n}$$

$r, m \in \{-1, 0, 1\}^n$, so we look for a vector close to $(0, c)$ $\rightsquigarrow$ CVP

NTRU – Improvements

Selecting Polynomials

- additionally restrict polynomials,
  - $\mathcal{T}$ ternary polynomial, coefficients $\{-1, 0, 1\}$, degree $\leq n - 2$
  - $\mathcal{T}(d)$: additionally $\frac{d}{2}$ coeff.s 1, $\frac{d}{2}$ coeff.s -1, else 0
- let $f, r \in \mathcal{T}$ and $g, m \in \mathcal{T}(q/8 - 2)$ with $p = 3$, then

$$|a_i| \leq p \cdot (q/8 - 2) + q/8 - 2 = q/2 - 8 < \frac{q}{2}$$

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NTRU-HPS – Recommended Values

- $n = 501$ and $q = 2048$
- $n = 677$ and $q = 2048$
- $n = 821$ and $q = 4096$

good speed with high security, keys and cipher 900-1600 byte

NTRU – Summary

- basic form: public key cryptosystem (i.e. en-/decrypt)
- submitted version generates session keys
- based on other mathematical problem
  - shortest vector: break key
  - closest vector: find message
- believed to be quantum resistant
- faster than RSA/ECDH
- public keys larger than RSA
- only recently greater focus $\rightsquigarrow$ less researched

skip multivariate crypto

Multivariate Cryptography

Problem MQ – Multivariate Quadratic

Given: finite field K, polynomials $f_i \in K[x_1, \ldots, x_n]$ of degree 2

Task: find $x \in K^n$ with $f_i(x) = 0$ for all i

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Example

take formula $\varphi = (x_1 \wedge x_2 \wedge \neg x_3) \vee (\neg x_2 \wedge x_3)$
to find satisfying assignment, solve system

$$y_1 = x_1 \cdot x_2 \qquad y_2 = y_1 \cdot (1 - x_3)$$
$$y_3 = (1 - x_2) \cdot x_3 \qquad 1 = y_2 + y_3 - y_2 \cdot y_3$$

Turning MQ into Cryptography

Basic Idea

additional secret information allows to solve hard problem

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finding root equivalent to

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translate into cryptography

Keys: $P = (f_1, \ldots, f_m)$ – public key, P^{-1} – secret key

Encryption: y – cipher, x – message

Signing y – message, x – signature

Key Generation

- pick easily "invertible" polynomial system F
- pick two invertible affine (linear + shift) maps S, T
- public key $P = T \circ F \circ S$ (meaning $x \to T \to F \to S \rightsquigarrow P(x)$)
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Encryption – several schemes outdated/broken!

Encrypt: message m, cipher $c = P(m)$

Decrypt: retrieve $m = P^{-1}(c)$

Multivariate Signatures

Key Observation

Signature just has to be some valid preimage under P.

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- never mix (multiply) oil with oil, then structure

$$y_i = \sum_{j,k} \gamma_{ijk} x_j a_k + \sum_{j,k} \lambda_{ijk} a_j a_k + \sum_j \xi_{ij} x_j + \sum_j \xi'_{ij} a_j + \delta_i$$

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- fix random values for vinegar a_j
- solve linear equation system to get x_j
- yields preimage $(\boldsymbol{x}, \boldsymbol{a})$ for $\boldsymbol{y} = (y_1, \ldots, y_n)$

Selecting Parameters

Broken Cases

- initially $n = v$ (balanced), broken by Kipnis and Shamir in 1998 also works for $v \approx n$
- for $v \geq n^2$ and char $K = 2$, finding solution is feasible

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- problem: key size $\mathcal{O}(m^3 \log q)$

Example Scheme – Rainbow

Rainbow

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- uses multivariante quadratic polynomials
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recently broken

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- signatures
  - lattice: Dilithium, Falcon
  - MQ: ~~Rainbow~~
- some alternative candidates (in part of other classes)
  worse in: security/ time/ communication size
- trade-off between sizes of public key, secret key, signature/cipher
  but also time and power consumption

Security Enhancements

What is lacking?

- many crypto primitives focus on OW-CPA
- preferred security IND-CCA2
- PKCS#1 already does that, but RSA-specific
- general transformation of weaker scheme into IND-CCA2

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Solution – Fujisaki-Okamoto-Transformation

- generic transformation
- essentially a hybrid system (PKC + AES)
- need hash and symmetric encryption
- transform OW-CPA into IND-CCA2

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Scenario

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TLS 1.3 does this (TLS $\leq$1.2: optionally), Signal-protocol as well

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Public Key Infrastructure

- If Eve controls network, how to avoid man-in-the-middle?
- How does Bob know, Alice's key was not changed?
- part of "Grundlagen der Rechnersicherheit"

Outlook

Some topics we left out:

Secret sharing

- split one secret across n people
- secret can be recovered if $\geq k$ people pair up
- e.g. via polynomial interpolation

Public Key Infrastructure

- If Eve controls network, how to avoid man-in-the-middle?
- How does Bob know, Alice's key was not changed?
- part of "Grundlagen der Rechnersicherheit"

Zero Knowledge Proofs

- Alice shows, she knows secret, without revealing secret

Larger Practical Use Cases

Signal-Protocol

- double ratchet method
- handling asynchronous communications and group chats

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Generally, field with lot of ongoing research...

The End ...

Happy Hacking!

I hope you had fun.
Maybe see you at some CTF ;)