Spectral Theory

First, we will define the spectrum of an operator and related notions. Then we list some properties concerning the spectrum (without proving them).

Definition. Let E be a normed space and $T \in L(E)$.
The *spectrum* $\sigma(T)$ is defined as

$$\sigma(T) = \{\lambda \in \mathbb{C} : (\lambda \operatorname{Id} - T) \text{ is not invertible in } L(E)\}$$

and the elements of $\sigma(T)$ are called *spectral values* of T.
The *resolvent set* $\rho(T)$ is defined as

$$\rho(T) = \mathbb{C} \setminus \sigma(T)$$

and the elements of $\rho(T)$ are called *regular values* of T.
Also, $\lambda \in \mathbb{C}$ is called an *eigenvalue* and $x \in E$ is called an *eigenvector* associated with λ, if

$$Tx = \lambda x\,.$$

The linear space of all eigenvectors associated with λ is called *eigenspace* and is denoted by

$$E(\lambda) = \{x \in E : Tx = \lambda x\} = \ker(\lambda \operatorname{Id} - T)\,.$$

Obviously, each eigenvalue is also an spectral value.
For each $E \neq \{0\}$ and T we have:

- $\sigma(T)$ is closed (thus $\rho(T)$ is open) and
- $|\lambda| \leq \|T\|$ for each $\lambda \in \sigma(T)$. (hence $\sigma(T)$ is compact)

If further E is a Banach space, we have

- $\sigma(T) \neq \emptyset$ (Gelfand-Mazur) and
- $\sup\limits_{\lambda \in \sigma(T)} |\lambda| = \lim\limits_{n \to \infty} \|T^n\|^{\frac{1}{n}}$.
- Also, concerning the resolvent: If $|\lambda| > \|T\|$, then $(\lambda \operatorname{Id} - T)^{-1} = \sum\limits_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}}$ and $\|(\lambda \operatorname{Id} - T)^{-1}\| \leq \big(|\lambda| + \|T\|\big)^{-1}$.

Now, we consider compact operators $K \in \mathcal{K}(E)$ instead of an arbitrary $T \in L(E)$. The following properties hold for each normed space E and $K \in \mathcal{K}(E)$:

- Each spectral value $0 \neq \lambda \in \sigma(K)$ is an eigenvalue of K.
- The set of eigenvalues of K is at most countable and can only accumulate to zero. (thus the same holds for $\sigma(K)$)
- The eigenspace $E(\lambda)$ for any eigenvalue λ is finite-dimensional.
- If $\dim E = \infty$, then $0 \in \sigma(K)$.

We now consider a Hilbert space $\mathcal{H}$ and a self-adjoint operator $T \in L(\mathcal{H})$. Then we have:

- Each eigenvalue of T is real.
- The eigenspaces with respect to two different eigenvalues are orthogonal.

Finally, we consider compact, self adjoint operators K on a Hilbert space $\mathcal{H}$. For those, we have the following properties:

- Either $\|K\|$ or $-\|K\|$ is an eigenvalue of K.
- There exist a non-increasing sequence $(\lambda_n)_{n=1}^{N}$, $N \in \mathbb{N} \cup \{\infty\}$, (if $N = \infty$, then (λ_n) converges to zero) of all eigenvalues $\lambda_n \neq 0$ of K and an othonormal system (x_n) of eigenvectors x_n associated with λ_n, such that the number of appearences of an eigenvalue λ in (λ_n) is equal to the dimension of $E(\lambda)$ and there holds

  $$Kx = \sum_n \lambda_n \langle x, x_n \rangle x_n$$

  for each $x \in \mathcal{H}$. (Spectral Theorem)

Note: If $K = 0$, it has no eigenvalues except zero, i.e. the Spectral Theorem does only hold for $K \neq 0$.