The spaces $L^p(I)$ for $1 \leq p < \infty$ and $p = \infty$

Definition 1. Let Ω be a set. A set Σ of subsets of Ω is called *σ-algebra*, if

(i) $\emptyset \in \Sigma$, $\Omega \in \Sigma$,

(ii) $A \in \Sigma \Rightarrow \Omega \setminus A \in \Sigma$ and

(iii) $A_1, A_2, \ldots \in \Sigma \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \Sigma$.

Example. $\{\emptyset, \Omega\}$ is the simplest σ-algebra.

Theorem 2. *For any set $\mathfrak{S} \subseteq \mathcal{P}(\Omega)$ of subsets of Ω, there is a "smallest" σ-algebra Σ_0 containing $\mathfrak{S}$, i.e.*

$$\exists! \sigma\text{-algebra } \Sigma_0 : \mathfrak{S} \subseteq \Sigma_0, \quad \Sigma \text{ is } \sigma\text{-algebra } \Rightarrow \Sigma_0 \subseteq \Sigma\,.$$

Σ_0 is called the σ-algebra generated by $\mathfrak{S}$.

Definition 3.

(1) The σ-algebra genereted by the open sets, $\{U \subseteq \mathbb{R}^n : U \text{ is open}\}$, is called the *Borel-$\sigma$-algebra* and is denoted by $\mathfrak{B}(\mathbb{R}^n)$.

(2) The *Lebesgue-measure* $\mu(A)$ of a set $A \in \mathfrak{B}(\mathbb{R}^n)$ is defined by

$$\mu(A) := |A| := \inf\left\{\sum_{i=1}^{m} |Q_i| : A \subseteq \bigcup_{i=1}^{m} Q_i,\ Q_i \text{ cuboids}\right\},$$

where $|Q_i| = |I_i^{(1)} \times \cdots \times I_i^{(n)}| = |I_i^{(1)}| \cdots |I_i^{(n)}|$, $|I| = |(a,b)| = b - a$.

(3) A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is called *measurable*, if the preimage of any open set $U \in \mathbb{R}^m$ is mearuable, i.e. $f^{-1}(U) \in \mathfrak{B}(\mathbb{R}^n)$.

(4) A function $f\colon \mathbb{R}^n \to \mathbb{R}$ of the form

$$f = \sum_{i=1}^{m} c_i \, \mathbb{1}_{A_i}$$

with $A_i \in \mathfrak{B}(\mathbb{R}^n)$ is called *simple*. If $c_i \geq 0$, we set

$$\int f \mathrm{d}x = \sum_{i=1}^{m} c_i \mu(A_i)\,.$$

Proposition. *If f, g are simple and non-negative functions, then*

$$f \leq g \Rightarrow \int f \mathrm{d}x \leq \int g \mathrm{d}x\,.$$

(5) If $f\colon \mathbb{R}^n \to [0,\infty)$ is measurable, then there is a sequence (f_n) of non-negative simple functions with $f_n \nearrow f$ (pointwise). We set

$$\int f \mathrm{d}x := \sup_{n \in \mathbb{N}} \int f_n \mathrm{d}x\,.$$

(6) A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called *(Lebesgue-)integrable*, if

$$\int |f| \mathrm{d}x < \infty\,.$$

Then we set

$$\int f \mathrm{d}x := \int f^+ \mathrm{d}x - \int f^- \mathrm{d}x\,,$$

where $f^+ := \max\{0, f\}$ and $f^- := \min\{0, -f\}$.

(7) A function $f\colon \mathbb{R}^n \to \mathbb{C}$ is integrable, if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are integrable.

Theorem 4. *Let f, g be integrable, $\alpha, \beta \in \mathbb{K}$. Then $\alpha f + \beta g$ is integrable and*

$$\int (\alpha f + \beta g) \mathrm{d}x = \alpha \int f \mathrm{d}x + \beta \int g \mathrm{d}x\,,$$

i.e. the integral is linear.
Also, the triangle inequality holds:

$$\left| \int f \mathrm{d}x \right| \leq \int |f| \mathrm{d}x\,.$$

Theorem 5 (Lebesgue)**.** *Let (f_n) be a sequence of integrable functions converging pointwise to some f. Assume there is an integrable function g, such that $|f_n| \leq g$ holds for all $n \in \mathbb{N}$. Then f is integrable and*

$$\int f \mathrm{d}x = \lim_{n\to\infty} \int f_n \mathrm{d}x\,.$$

Theorem 6 (Beppo Levi/monote convergence)**.** *Let (f_n) be a sequence of non-negative measurable functions with $f_n \nearrow f$ for some f. Then*

$$\int f \mathrm{d}x = \lim_{n\to\infty} \int f_n \mathrm{d}x\,.$$

We now define an equivalence relation by

$$\begin{aligned} f \sim g :&\Leftrightarrow \mu\left(\{x \in \mathbb{R}^n : f(x) \neq g(x)\}\right) = 0 \\ &\Leftrightarrow f = g \text{ almost everywhere (a.e.).} \end{aligned}$$

This yields equivalence classes $[f]$.

Definition 7. Let $1 \geq p < \infty$, $I \subseteq \mathbb{R}$ an interval. By $\mathcal{L}^p(I)$ we denote the space

$$\mathcal{L}^p(I) := \{f\colon I \to \mathbb{K} : |f|^p \text{ is integrable}\}$$

and we define

$$\|f\|_p^* := \left(\int_I |f|^p \mathrm{d}x\right)^{\frac{1}{p}}\,.$$

Using the equivalence classes defined above, we can obtain a normed space $L^p(I)$ while $\mathcal{L}^p(I)$ is a vector space with a semi-norm. In the following we will show that $\mathcal{L}^p(I)$ is a complete vector space with semi-norm and construct the Banach space $L^p(I)$ by using the quotient space.

Lemma 8. *The space $\mathcal{L}^p(I)$ is a vector space.*

Proof. For $f, g \in \mathcal{L}^p(I)$ and $\alpha \in \mathbb{K}$ we obviously have $\alpha f \in \mathcal{L}^p(I)$. Also, there holds

$$\begin{aligned}\int_I |f+g|^p \mathrm{d}x &\leq \int_I (|f|+|g|)^p \,\mathrm{d}x \leq \int_I (2\max\{|f|,|g|\})^p \,\mathrm{d}x \\ &= 2^p \int_I \max\{|f|^p, |g|^p\} \mathrm{d}x \leq 2^p \int_I (|f|^p + |g|^p)\,\mathrm{d}x < \infty\,.\end{aligned}$$

□

Definition 9. Let E be a vector space. A function $V\colon E \to [0,\infty)$ is called a *semi-norm* on E if

(i) $V(\alpha x) = |\alpha| V(x)$ for all $\alpha \in \mathbb{K}$, $x \in E$ and

(ii) $V(x+y) \leq V(x) + V(y)$ for all $x, y \in E$.

V is called a *norm* on E if further $V(x) = 0$ implies $x = 0$.

Satz. *The map $\|\cdot\|_p^*$ defines a semi-norm on $\mathcal{L}^p(I)$.*

For this, we need to show the triangle inequality, as $\|\alpha f\|_p^* = |\alpha| \|f\|_p^*$ is obvious. We first prove the Hölder-inequality.

Lemma 10 (Hölder's inequality)**.** *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $f \in \mathcal{L}^p(I)$ and all $g \in \mathcal{L}^q(I)$ we have*

$$\|fg\|_1^* \leq \|f\|_p^* \|g\|_q^*\,.$$

In particular, $fg \in \mathcal{L}^1(I)$.

Proof. The logarithm on $(0,\infty)$ is concave, in explicit

$$r\log(\sigma) + (1-r)\log(\tau) \leq \log(r\sigma + (1-r)\tau)$$

for all $\sigma, \tau > 0$ and $r \in [0,1]$. This yields

$$\sigma^r \tau^{1-r} \leq r\sigma + (1-r)\tau\,.$$

We set

$$A := \left(\|f\|_p^*\right)^p = \int_I |f|^p \mathrm{d}x \qquad\qquad B := \left(\|g\|_q^*\right)^q = \int_I |g|^q \mathrm{d}x\,.$$

By $r := \frac{1}{p}$ we obtain for $x \in I$

$$\left(\frac{|f(x)|^p}{A}\right)^{\frac{1}{p}} \left(\frac{|g(x)|^q}{B}\right)^{\frac{1}{q}} \leq \frac{1}{Ap}|f(x)|^p + \frac{1}{Bq}|g(x)|^q$$

and thus

$$\frac{1}{A^{1/p}B^{1/q}} \int_I |fg| \mathrm{d}x \leq \frac{1}{Ap} \underbrace{\int_I |f|^p \mathrm{d}x}_{=A} + \frac{1}{Bq} \underbrace{\int_I |g|^q \mathrm{d}x}_{=B} = 1\,.$$

□

Using this inequality we can now show the triangle inequality in $\mathcal{L}^p(I)$.

Lemma 11 (Minkowski's inequality)**.** *Let $1 \le p < \infty$ and $f, g \in \mathcal{L}^p(I)$. Then there holds*

$$\|f+g\|_p^* \le \|f\|_p^* + \|g\|_p^* .$$

Proof. For $p = 1$ we have

$$\int_I |f+g| \mathrm{d}x \le \int_I (|f|+|g|) \mathrm{d}x = \int_I |f| \mathrm{d}x + \int_I |g| \mathrm{d}x .$$

Now let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, by the Hölder's inequality

$$\begin{aligned}
\left(\|f+g\|_p^*\right)^p &= \int_I |f+g|^p \mathrm{d}x = \int_I |f+g||f+g|^{p-1} \mathrm{d}x \le \int_I |f||f+g|^{p-1} \mathrm{d}x + \int_I |g||f+g|^{p-1} \mathrm{d}x \\
&\le \left(\int_I |f|^p \mathrm{d}x\right)^{\frac{1}{p}} \left(\int_I |f+g|^{(p-1)q} \mathrm{d}x\right)^{\frac{1}{q}} + \left(\int_I |g|^p \mathrm{d}x\right)^{\frac{1}{p}} \left(\int_I |f+g|^{(p-1)q}\right)^{\frac{1}{q}} \\
&= \left(\|f\|_p^* + \|g\|_p^*\right) \underbrace{\left(\int_I |f+g|^p \mathrm{d}x\right)^{1-\frac{1}{p}}}_{=\left(\|f+g\|_p^*\right)^{p-1}} .
\end{aligned}$$

□

Remark. Note that $\|\cdot\|_p^*$ is not a norm. For this consider

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{Q} . \end{cases}$$

Then $\int_{[0,1]} |f|^p \mathrm{d}x = 0$, but $f \neq 0$.

Definition 12. Let E be a vector space with a semi-norm $\|\cdot\|^*$.

(i) $(x_n) \subset E$ is called a *Cauchy sequence* if for all $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$ we have $\|x_n - x_m\|^* < \varepsilon$.

(ii) $(x_n) \subset E$ *converges* to $x \in E$ if for all $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have $\|x_n - x\|^* < \varepsilon$.

(iii) $(E, \|\cdot\|^*)$ is called *complete* if each Cauchy sequence in E converges (in E).

Lemma 13. *Let E be a vector space with semi-norm $\|\cdot\|^*$. Then the following are equivalent:*

(i) $(E, \|\cdot\|^*)$ is complete.

(ii) Each absolutely convergent series in E converges, i.e.

$$\sum_{n=1}^{\infty} x_n$$

converges if

$$\sum_{n=1}^{\infty} \|x_n\|^* < \infty$$

converges.

Proof. Exercises. □

Theorem 14. *$(\mathcal{L}^p(I), \|\cdot\|_p^*)$ is complete.*

Proof. Let $(f_n) \subset \mathcal{L}^p(I)$ such that

$$a := \sum_{n=1}^{\infty} \|f_n\|_p^* < \infty .$$

Define $g_n, \tilde{g} \colon I \to \mathbb{R} \cup \{\infty\}$ by

$$g_n(x) := \sum_{i=1}^{n} |f_i(x)| \qquad \hat{g}(x) := \sum_{i=1}^{\infty} |f_i(x)| .$$

Then $g_n \in \mathcal{L}^p(I)$ and $\|g_n\|_p^* \le \sum_{i=1}^{n} \|f_i\|_p^* \le a$ for all $n \in \mathbb{N}$. Also by $g_n^p \nearrow \hat{g}^p$ Beppo Levi gives us

$$\int_I \hat{g}^p \mathrm{d}x = \lim_{n\to\infty} \int_I g_n^p \mathrm{d}x \le a^p < \infty . \tag{1}$$

Now, $N := \{x \in I : \hat{g}(x) = \infty\}$ has zero measure. Put

$$g(x) := \begin{cases} \hat{g}(x) & \text{if } x \notin N \\ 0 & \text{if } x \in N . \end{cases}$$

Thus,

$$g(x) = \sum_{i=1}^{\infty} |f_i(x)|$$

for $x \in N$ and hence

$$f(x) := \sum_{i=1}^{\infty} f_i(x)$$

for $x \notin N$ exists. Additionally, let $f(x) := 0$ for $x \in N$. We have $|f(x)| \le g(x)$ for $x \in I$. By (1) there is $g \in \mathcal{L}^p(I)$ and $f^p \le g^p$ and thus $f \in \mathcal{L}^p(I)$. Define

$$h_n(x) := \sum_{i=1}^{n-1} f_i(x) .$$

Then

$$|h_n - f|^p = \left| \sum_{i=n}^{\infty} f_i \right|^p \le \left(\sum_{i=n}^{\infty} |f_i| \right)^p \le g^p .$$

Since $|h_n - f| \to 0$, $n \to \infty$, we have

$$\int_I |h_n - f|^p \mathrm{d}x \to 0$$

which implies

$$\left\| f - \sum_{i=1}^{n} f_i \right\|_p^* \to 0 .$$

□

Thus, $(\mathcal{L}^p(I), \|\cdot\|_p^*)$ is a complete vector space.

Now we construct a Banach space $L^p(I)$.

Lemma 15. *Let E be a vector space with semi-norm $\|\cdot\|^*$. Then the following holds:*

(i) $F := \{x \in E : \|x\|^* = 0\}$ *is a subspace of* E.

(ii) $\|[x]\| := \|x\|^*$ *defines a norm on* E/F.

(iii) *If* $(E, \|\cdot\|^*)$ *is complete, then also* $(E/F, \|\cdot\|)$ *is complete.*

Proof. (i). For $x, y \in F$, $\lambda \in \mathbb{K}$ we have

$$\|\lambda x + y\|^* \leq \|\lambda x\|^* + \|y\|^* = |\lambda|\|x\|^* + \|y\|^* = \lambda \cdot 0 + 0 = 0\,.$$

(ii). $\|\cdot\|$ is well-defined, since for $x, y \in E$ with $x \sim y$, i.e. $x - y \in F$, we have

$$\Big|\|x\|^* - \|y\|^*\Big| \leq \|x - y\|^* = 0$$

and hence $\|x\|^* = \|y\|^*$. To show the norm properties, we have that

$$\|\lambda[x]\| = \|[\lambda x]\| = \|\lambda x\|^* = |\lambda|\|x\|^* = |\lambda|\|[x]\|$$

and

$$\|[x] + [y]\| = \|[x + y]\| = \|x + y\|^* \leq \|x\|^* + \|y\|^* = \|[x]\| + \|[y]\|\,.$$

(ii). Let $(E, \|\cdot\|^*)$ be complete and let $(x_n) \subset E$ be such that $([x_n]) \subset E/F$ be a Cauchy sequence. Since $\|[x_n] - [x_m]\| = \|x_n - x_m\|^*$, also (x_n) is a Cauchy sequence and therefore converges to some $x \in E$. We now have

$$\|[x_n] - [x]\| = \|[x_n - x]\| = \|x_n - x\| \xrightarrow{n\to\infty} 0$$

and thus $([x_n])$ converges to $[x]$. □

Definition 16. Now, for $1 \leq p < \infty$ we set

$$F_p(I) := \{f \in \mathcal{L}^p(I) : \|f\|_p^* = 0\} = \{f \in \mathcal{L}^p(I) : f = 0 \text{ a.e.}\}$$

and finally define

$$L^p(I) := \mathcal{L}^p(I)/F_p(I)\,.$$

By the previous Lemma, $L^p(I)$ is a Banach space.
Now, we consider $p = \infty$.

Definition 17. We set

$$\mathcal{L}^\infty(I) := \{f\colon I \to \mathbb{K} : f \text{ is measurable and } \exists N \in \mathfrak{B}(I) : \mu(N) = 0, f\big|_{I\setminus N} \text{ is bounded}\}$$

and further define

$$\|f\|_\infty^* := \operatorname*{ess\,sup}_{x\in I}|f(x)| = \inf_{\substack{N\in\mathfrak{B}(I)\\ \mu(N)=0}} \sup_{x\in I\setminus N} |f(x)| = \inf_{\substack{N\in\mathfrak{B}(I)\\ \mu(N)=0}} \left\|f\big|_{I\setminus N}\right\|\,.$$

Lemma 18. $\mathcal{L}^\infty(I)$ *is a vector space and* $\|\cdot\|_\infty^*$ *is a semi-norm on* $\mathcal{L}^\infty(I)$.

Proof. Let $f, g \in \mathcal{L}^\infty(I)$ and $\lambda \in \mathbb{K}$. Obviously, also $\lambda f \in \mathbb{K}$. Now let $N_f, N_g \in \mathfrak{B}(I)$ with $\mu(N_f) = \mu(N_g) = 0$, such that $f\big|_{I\setminus N_f}$ and $g\big|_{I\setminus N_g}$ are bounded. Then, for $N := N_f \cup N_g$, we have $\mu(N) = 0$ and $(f+g)\big|_{I\setminus N}$ is bounded, thus $f + g \in \mathcal{L}^\infty(I)$.
Also, we obviously have

$$\|\lambda f\|_\infty^* = |\lambda| \|f\|_\infty^* .$$

Now let $N \in \mathfrak{B}(I)$ with $\mu(N) = 0$. We have

$$\|f+g\|_\infty^* \leq \sup_{x\in I\setminus N} |(f+g)(x)| \leq \sup_{x\in I\setminus N} |f(x)| + \sup_{x\in I\setminus N_1} |g(x)| .$$

Taking the supremum over all $N \in \mathfrak{B}(I)$ with $\mu(N) = 0$ yields the claim. □

Lemma 19. *For each $\mathcal{L}^\infty(I)$ there exists $N \in \mathfrak{B}(I)$ with $\mu(N) = 0$ such that*

$$\|f\|_\infty^* = \left\| f\big|_{I\setminus N} \right\|_\infty .$$

Proof. For each $n \in \mathbb{N}$ there exists $N_n \in \mathfrak{B}(I)$ with $\mu(N_n) = 0$ such that

$$\|f\|_\infty^* + \frac{1}{n} \geq \left\| f\big|_{I\setminus N_n} \right\|_\infty .$$

Set

$$N := \bigcup_{i=1}^\infty N_i .$$

Then $N \in \mathfrak{B}(I)$, $\mu(N) = 0$ and

$$\|f\|_\infty^* \leq \left\| f\big|_{I\setminus N} \right\|_\infty \leq \left\| f\big|_{I\setminus N_n} \right\|_\infty \leq \|f\|_\infty^* + \tfrac{1}{n} .$$

Letting $n \to \infty$, the claim is proven. □

Theorem 20. $(\mathcal{L}^\infty(I), \|\cdot\|_\infty^*)$ *is complete.*

Proof. Let (f_n) be a Cauchy sequence in $(\mathcal{L}^\infty(I), \|\cdot\|_\infty^*)$. By the above Lemma there exists $N_{n,m} \in \mathfrak{B}(I)$ with $\mu(N_{n,m})$ such that

$$\|f_n - f_m\|_\infty^* = \left\| (f_n - f_m)\big|_{I\setminus N_{n,m}} \right\|_\infty .$$

Define

$$N := \bigcup_{n,m\in\mathbb{N}} N_{n,m} \in \mathfrak{B}(I) .$$

Then $\mu(N) = 0$ and

$$\left\| (f_n - f_m)\big|_{I\setminus N} \right\|_\infty \leq \left\| (f_n - f_m)\big|_{I\setminus N_{n,m}} \right\|_\infty$$

for all $n, m \in \mathbb{N}$. Hence $\left(f_n\big|_{I\setminus N}\right)$ is a Cauchy sequence in the space of bounded functions $B(I \setminus N)$. Therefore there exists some $f \in B(I \setminus N)$ with

$$\left\| (f_n - f)\big|_{I\setminus N} \right\|_\infty \xrightarrow{n\to\infty} 0 .$$

By setting $f(x) = 0$ on N, $f\colon I \to \mathbb{K}$ is measurable and bounded, thus $f \in \mathcal{L}^\infty(I)$ and

$$\|f_n - f\|_\infty^* \leq \left\| (f_n - f)\big|_{I\setminus N} \right\|_\infty \xrightarrow{n\to\infty} 0 .$$

□

Now, with

$$F_\infty(I) := \{f \in \mathcal{L}^\infty(I) : \|f\|_\infty^* = 0\}$$

we obtain the Banach space

$$L^\infty(I) := \mathcal{L}^\infty(I)/F_\infty(I)\,.$$

Theorem 21 (Hölder inequality)**.** *Let $1 \leq p, q \leq \infty$ such that $\frac{1}{p}+\frac{1}{q} = 1$ ($p = 1 \Rightarrow q = \infty$). Then for all $f \in L^p(I)$, $g \in L^q(I)$ we have $fg \in L^1(I)$ and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q\,.$$

Proof. We only deal with $p = \infty$. Let $f \in L^\infty(I)$, $g \in L^1(I)$ and let N be as in Lemma 19. There holds

$$\int_I |fg| \mathrm{d}x = \int_{I\setminus N} |f||g| \mathrm{d}x \leq \int_{I\setminus N} \|f\|_\infty |g| \mathrm{d}x = \|f\|_\infty \int_I |g| \mathrm{d}x = \|f\|_\infty \|g\|_1\,.$$

□