Fredholm Operators

We define Fredholm operators and head for the Atkinson's Theorem.

Definition 1. Let E be a Banach space and $A \in L(E)$. A is called a *Fredholm operator*, if

(i) $\dim \ker A < \infty$,

(ii) $A(E) = \operatorname{ran} A$ is closed in E and

(iii) $\dim(E/\operatorname{ran} A) = \operatorname{codim} \operatorname{ran} A < \infty$.

To prove Atkinson's Theorem we need the following Lemma.

Lemma 2. *Let $A \in L(E)$. Then we have that $\dim \ker A < \infty$ and $\operatorname{ran} A$ is closed if and only if each sequence (x_n) satisfying $\|x_n\| \leq 1$ for all $n \in \mathbb{N}$ and $Ax_n \to 0$, $n \to \infty$, has a convergent subsequence.*

Proof. "⇒". Assume that $\dim \ker A < \infty$ and $\operatorname{ran} A$ is closed. Then

$$E = \ker A \dotplus W$$

with a complementary subspace W (for $\ker A$). Let $(x_n) \subset E$ be a sequence with $\|x_n\| \leq 1$ for all $n \in \mathbb{N}$ and $Ax_n \to 0$. Then $x_n = u_n + w_n$ with $u_n \in \ker A$ and $w_n \in W$. Let $A_0 \colon W \to \operatorname{ran} A$, $A_0 x := Ax$. The operator A_0 is bijective and thus boundedly invertible (i.e. it has a bounded inverse) by the Open Mapping Theorem. We have $A_0 w_n = A_0 x_n = Ax_n \to 0$ and thus $w_n = A_0^{-1} A_0 w_n \to 0$. Further,

$$\|u_n\| = \|x_n - w_n\| \leq 1 + \|w_n\| \leq c$$

for some $c > 0$. Hence there exists a subsequence (u_{n_k}) and $u \in \ker A$ with $u_{n_k} \to u$ and thus $x_{n_k} \to u$.

"⇐". Conversely, assume that each sequence (x_n) satisfying $\|x_n\| \leq 1$ for all $n \in \mathbb{N}$ and $Ax_n \to 0$, $n \to \infty$, has a convergent subsequence. Let $(x_n) \subset \ker A$ with $\|x_n\| \leq 1$ for all $n \in \mathbb{N}$. Then by $Ax_n = 0 \to 0$, (x_n) contains a convergent subsequence, i.e. the closed unit ball in $\ker A$ is compact and thus $\dim \ker A < \infty$. Therefore, $E = \ker A \dotplus W$ as above. Consider $A_0 := A|_W$, again as above. Then $\operatorname{ran} A = \operatorname{ran} A_0$. Assume there does not exist some $c > 0$ such that $\|A_0 w\| \geq c\|w\|$ for all $w \in W$. Then there is also no $c > 0$ such that $\|A_0 w\| \geq c$ for all $w \in W$ with $\|w\| = 1$. Hence for all $n \in \mathbb{N}$ there exists $w_n \in W$ with $\|w_n\| = 1$ and $\|A_0 w_n\| < \frac{1}{n}$. By assumption, $w_{n_k} \to w \in W$ and therefore $A_0 w_{n_k} \to A_0 w$ and $A_0 w = 0$. Injectivity of A_0 yields $w = 0$. But $1 = \|w_{n_k}\| \to \|w\| = 0$. ↯ Thus $\operatorname{ran} A_0$ is closed [1]. □

Theorem 3 (Atkinson's Theorem)**.** *Let E be a Banach space and $A \subset L(E)$. Then the following are equivalent:*

(i) A is a Fredholm operator.

(ii) There exists $B \in L(E)$ such that $\operatorname{Id} - BA$ and $\operatorname{Id} - AB$ are finite-dimensional (i.e. have a finite-dimensional range).

[1] If E is a Banach space, F a normed space, $T \in L(E, F)$ and if there exists some $c > 0$ such that $\|Ax\| \geq c\|x\|$ for all $x \in E$, then $\operatorname{ran} T$ is closed. (Exercise)

(iii) *There exists* $B \in L(E)$ *such that* $\operatorname{Id} - BA$ *and* $\operatorname{Id} - AB$ *are compact.*

(iv) *There exist* $B, C \in L(E)$ *such that* $\operatorname{Id} - BA$ *and* $\operatorname{Id} - AC$ *are compact.*

Proof. (i)⇒(ii). Let A be a Fredholm operator. Then

$$E = \ker A \dotplus W \text{ and } E = \operatorname{ran} A \dotplus V$$

for complementary spaces V, W. Let P resp. Q be the (continuous) projection onto W resp. V. Define $A_0 \colon W \to \operatorname{ran} A$, $A_0 w := Aw$, $w \in W$. Then A_0 is boundedly invertible. Define $B := A_0^{-1}(\operatorname{Id} - Q)$. Let $x \in E$. Then there exist $u \in \ker A$ and $w \in W$ such that $x = u + w$. Now,

$$BAx = A_0^{-1}(\operatorname{Id} - Q)Ax = A_0^{-1}Ax = A_0^{-1}A_0 w = w = Px\,.$$

By $BA = P = \operatorname{Id} - (\operatorname{Id} - P)$ we have that $\operatorname{Id} - BA = \operatorname{Id} - P$ is finite-dimensional. Moreover,

$$AB = AA_0^{-1}(\operatorname{Id} - Q) = \operatorname{Id}|_{\operatorname{ran} A}(\operatorname{Id} - Q) = \operatorname{Id} - Q$$

and thus $\operatorname{Id} - AB = Q$ is also finite-dimensional.

(ii)⇒(iii). Finite-dimensional operators are compact.

(iii)⇒(iv). Choose $C = B$.

(iv)⇒(i). Let $(x_n) \subset E$, $\|x_n\| \leq 1$ for all $n \in \mathbb{N}$, be a sequence with $Ax_n \to 0$. Then $BAx_n = 0$. But $BA = \operatorname{Id} - K$ for some compact operator K. Thus BA is a Fredholm Operator and (x_n) has a convergent subsequence. Now, $\dim \ker A < \infty$ and $\operatorname{ran} A$ is closed. Further, we have that $\operatorname{Id} - C^* A^* = (\operatorname{Id} - AC)^*$ is a compact operator (in $L(E^*)$). We now see that $\dim \ker A^* < \infty$. By

$$\ker A^* = (\operatorname{ran} A)^\perp \cong (E/\operatorname{ran} A)^*$$

also $\dim(E/\operatorname{ran} A)^* < \infty$ and thus $\dim(E/\operatorname{ran} A) < \infty$. □