Functional Analysis 1
Most Important Theorems

Name	**Theorem**	**concept of proof**
Baire's Theorem	(X,d) complete, D_n open and dense. <u>Then:</u> $\bigcap_{n\in\mathbb{N}} D_n$ dense	For $x \in X$ inductively define a sequence with limit in all D_n and in $U_r(x)$: $K_{r_{n+1}}(x_{n+1}) \subseteq D_n \cap U_{r_n}(x_n)$ and $r_n \leq \frac{1}{n}$. $x_1 = x$, $r_1 = \min\{1, r\}$. Can choose next element, since $D_n \cap U_{r_n}(x_n)$ non-empty and open. (x_n) Cauchy sequence.
Corollary from Baire	A_n closed, $X = \bigcup_{n\in\mathbb{N}} A_n$. <u>Then:</u> $\exists n \in \mathbb{N} : \mathring{A}_n \neq \emptyset$	$\mathring{A}_n = \emptyset$, then $X \setminus A_n$ open and dense
Hahn-Banach Theorem	E normed space, F subspace, $f \in F^*$. <u>Then:</u> $\exists \ell \in E^* : \ell\|_F = f,\ \\|\ell\\| = \\|f\\|$	$\rho(x) = \\|f\\|\\|x\\|$ seminorm, $\|f\| \leq \rho$. Other Theorem (using Zorn's Lemma) implies: $\exists$ linear functional ℓ with $\ell\|_F = f$, $\|\ell\| \leq \rho$. This implies $\\|\ell\\| = \\|f\\|$
Corollary from Hahn-Banach	E normed space, F subspace, $x \in E \setminus F$ with $\operatorname{dist}(x,F) > 0$. <u>Then:</u> $\exists \ell \in E^* : \ell\|_F = 0$, $\\|\ell\\| = 1$, $\ell(x) = \operatorname{dist}(x,F)$. *In particular:* $F = \{0\}$	Define $g(y + \lambda x) = \lambda \operatorname{dist}(x,F)$ on $F \dotplus \operatorname{span}\{x\}$. g linear, $g\|_F = 0$, $\\|g\\| = 1$, Hahn-Banach
Open Mapping Theorem	E, F Banach, $T \in L(E,F)$ surjective. <u>Then:</u> T open	$U \subseteq E$ open, $x \in U$. Some previous Lemmas (proved with Baire's Theorem) imply $K_1(0_F) \subseteq T(K_{2r}(0_E))$ for some $r > 0$. For some $t > 0$ with $K_t(x) \subseteq U$ we get $\frac{t}{2r} K_1(0_F) \subseteq T(U) - T(x)$. $K_{\frac{t}{2r}}(T(x)) \subseteq T(U) \rightsquigarrow T(U)$ open
Closed Graph Theorem	E, F Banach, $T\colon E \to F$ closed and linear. <u>Then:</u> T bounded	$E \times F$ Banach, G_T closed $\rightsquigarrow G_T$ Banach. $S\colon (x, Tx) \mapsto x$ is bijective, linear, bounded ($\\|(x,Tx)\\| = \max\{\\|x\\|, \\|Tx\\|\}$), Corollary from Open Mapping Theorem implies S^{-1} bounded, $\\|Tx\\| \leq \\|S^{-1}(x)\\|$
Uniform Boundedness Principle	E Banach, F normed space, $\mathcal{T} \subseteq L(E,F)$ pointwise bounded. <u>Then:</u> $\mathcal{T}$ bounded	$E_n = \{x \in E : \\|Tx\\| \leq n\ \forall T \in \mathcal{T}\}$. Corollary from Baire $\rightsquigarrow \mathring{E}_{n_0} \neq \emptyset$. $K_r(x) \subseteq E_{n_0}$, $x \in E_{n_0}$. $\\|y\\| \leq r \rightsquigarrow \\|Ty\\| = \\|T(y+x) - Tx\\| \leq 2n_0$. With multiplying by $\frac{r}{\\|y\\|}$ for each $y \in E$: $\\|Ty\\| \leq \frac{2n_0}{r}\\|y\\|$
Banach-Steinhaus Theorem	E Banach, F normed space, $T_n \in L(E,F)$ pointwise convergent to a linear operator. <u>Then:</u> (T_n) bounded And: E normed space, F Banach, $(T_n)_n \subseteq L(E,F)$ bounded, $E_0 \subseteq E$ dense subspace, such that $(T_n x)$, $x \in E_0$, converges. <u>Then:</u> (T_n) pointwise convergent to some $T \in L(E,F)$.	First: $(\\|T_n x\\|)$ bounded $\rightsquigarrow$ Uniform boundedness principle. Second: $\exists T \in L(E,F) : T\|_{E_0} = T_0 := \lim_{n\to\infty} T_n$ (pointwise). For $x \in E$ choose $y \in E_0$ with $\\|x - y\\| < \varepsilon$. $\rightsquigarrow \\|T_n x - Tx\\| \leq \\|T_n x - T_n y\\| + \\|T_n y - T_0 y\\| + \\|T_0 y - Tx\\| \leq \varepsilon(\sup\\|T_n\\| + 1 + \\|T\\|)$
Riesz Representation Theorem	$\mathcal{H}$ Hilbert space, $y \in \mathcal{H}$, $\ell \in \mathcal{H}^*$. <u>Then:</u> $\langle \cdot, y\rangle =: f_y \in \mathcal{H}^*$, $\\|f_y\\| = \\|y\\|$ and $\exists! y \in \mathcal{H} : \ell = f_y$ and $y \mapsto f_y$ conjugate linear	Cauchy-Schwarz and $f_y(y) = \\|y\\|^2$ give $\\|f_y\\| = \\|y\\|$, linearity is obvious. For $\ell \in \mathcal{H}^*$ write $z' = x + z$, where $\ell(z') = 1$, $x \in \ker \ell$, $z \in (\ker \ell)^\perp$. Set $y = \frac{z}{\langle z, z\rangle}$.

unnamed (8.13)	$\mathcal{H}$ Hilbert space, $\{x_i : i \in I\}$ ONS. <u>THEN:</u> (i) $\{x_i : i \in I\}$ complete $\Leftrightarrow$ (ii) $\overline{\operatorname{span}\{x_i : i \in I\}}$ dense $\Leftrightarrow$ (iii) $x = \sum_i \langle x, x_i \rangle x_i$ $\Leftrightarrow$ (iv) $\langle x, y \rangle = \sum_i \langle x, x_i \rangle \langle x_i, y \rangle$ $\forall x, y \in \mathcal{H}$	(i)$\Rightarrow$(ii) A Remark states $\{x_i : i \in I\}^\perp = \{0\}$ as the ONS is complete, $\mathcal{H} = \overline{\operatorname{span}\{x_i\}} \oplus \overline{\operatorname{span}\{x_i\}}^\perp$. (ii)$\Rightarrow$(iii) One can show that (ii) $\Rightarrow \oplus_i \mathcal{H}_i \cong \mathcal{H}$. Therefore $x = \sum_i \lambda_i x_i$ which implies $\langle x, x_i \rangle = \lambda_i$. (iii)$\Rightarrow$(iv) Insert (iii) in $\langle x, y \rangle$, pull out scalars. (iv)$\Rightarrow$(i) Assume there exists $x \neq 0$, $\|x\| = 1$, $x \perp \{x_i\}$, then $\|x\|^2 = \langle x, x \rangle = 0$.
unnamed (9.12)	E Banach space, $K \in \mathcal{K}(E)$. <u>THEN:</u> $T := \operatorname{id} - K$ Fredholm.	$\dim \ker T < \infty$ since $K\|_{\ker T} = \operatorname{id}\|_{\ker T}$ compact. $T(E)$ closed: $T\|_G$ bijective, continuous, G closed complementary space to $\ker T$, ... $\rightsquigarrow T\|_G(G) = T(E)$ closed. $\dim(E/T(E)) < \infty$ since $(E/T(E))$ isometrically isomorphic to $T(E)^* = \ker T^*$, $T^* = \operatorname{id} - K^*$ and K^* compact.
Spectral Theorem (compact, self-adjoint)	$\mathcal{H}$ Hilbert space, $K \neq 0$ on $\mathcal{H}$ compact, self-adjoint. <u>THEN:</u> $\exists (\lambda_n), (x_n)$ finite or infinite, $(\|\lambda_n\|)$ non-increasing, $\lambda_n \to 0$ (if infinite), $K x_n = \lambda x_n$, (x_n) orthonormal system, $\dim E(\lambda)$ number of appearences of λ in (λ_n) and (most important) $Kx = \sum_n \lambda_n \langle x, x_n \rangle x_n$	Choose λ_1 with $\|\lambda_1\| = \|K\|$, set $\mathcal{H}_1 = \operatorname{span}\{x_1\}$. Consider $K \colon \mathcal{H}_1^\perp \to \mathcal{H}_1^\perp$, continue ($\mathcal{H}_2 = \operatorname{span}\{x_1, x_2\}$). Cases: Process stops or not; check conditions.