Dunford and Pettis Integral

While it is well known how to integrate functions from a measure space to $\mathbb{R}$ or $\mathbb{R}^n$, we do not yet have an integral for a function taking values in a normed space. The key idea is to reduce this problem to a scalar problem like we did in reducing the integration of $\mathbb{R}^n$ valued functions to the integration of scalar valued functions. Since mapping a vector in $\mathbb{R}^n$ to one of its components is just a continuous linear functional, we would like to apply elements of the dual space to a function taking values in a normed space, do our integration theory and somehow rebuild the desired integral. But first, we have to give means to measurability and integrability before defining an integral. In the following, let (Ω, Σ, μ) a measure space, X a (real) normed space and $f\colon \Omega \to X$ some map.

Definition 1. The function f is called *weakly measurable* if, for any $x^* \in X^*$, the real valued function $x^* f\colon \Omega \to X \to \mathbb{R}$ is measurable. Similarly, f is called *Dunford integrable* if $x^* f$ is integrable for any $x^* \in X^*$.

Remark 2.

- The term "weakly" will become more clear in Chapter 6 when we introduce weak convergence and weak topologies – it can be translated as "reduced to the scalar case by applying functionals".

- Nelson Dunford is best known for his comprehensive book *Linear Operators* he wrote with Jacob T. Schwartz. Its three volumes together contain more than 2500 pages.

- You may have noticed that we would obtain a notion of measurability by endowing X with its Borel σ-algebra. This would not turn out to be helpful.

For a Dunford integrable f the integrals $\int x^* f \, \mathrm{d}\mu$ are well-defined and we would like to reconstruct some element $\int f \, \mathrm{d}\mu$ of X satisfying $x^* \int f \, \mathrm{d}\mu = \int x^* f \, \mathrm{d}\mu$ for any $x^* \in X^*$. In general, this will not be possible. Instead, this integral $\int f \, \mathrm{d}\mu$ will live in the bidual X^{**} and satisfy $\left(\int f \, \mathrm{d}\mu\right) x^* = \int x^* f \, \mathrm{d}\mu$ for any $x^* \in X^*$.

Lemma 3. *If f is Dunford integrable, the mapping*

$$X^* \to \mathbb{R}, \ x^* \mapsto \int x^* f \, \mathrm{d}\mu$$

is linear and continuous, thus an element of the bidual and denoted by $\int f \, \mathrm{d}\mu \in X^{**}$ – the Dunford integral.

Proof. Linearity is obvious, so we check boundedness. For this, let $T\colon X^* \to L^1(\mu)$, $x^* \mapsto x^* f$ such that $\left|\int x^* f \, \mathrm{d}\mu\right| \leq \int |x^* f| \, \mathrm{d}\mu = \|Tx^*\|$. Now we want to show continuity of T, i.e. the existence of $c > 0$ such that

$$\left|\int x^* f \, \mathrm{d}\mu\right| \leq \|Tx^*\| \leq c\|x^*\|$$

which will imply the continuity of our functional. Since $L^1(\mu)$ and X^* are Banach spaces, the closed graph theorem will tell us that it suffices to show closedness of T. If $x_n^* \to x^*$ in X^* and $Tx_n^* \to g$ in $L^1(\mu)$, we note that $Tx_n^* = x_n^* f \to x^* f = Tx^*$ pointwise, so Tx_n^* converges to g in L^1 and pointwise to Tx^*, which implies $Tx^* = g$. $\square$

One may find it inconvenient that the integral of an X valued function appears to live in X^{**}. Indeed, it is possible that $\int f \, \mathrm{d}\mu \in X^{**}$ is not in X, i.e. not contained in the image of the canonical embedding of X in X^{**}.

Example 4. As a measure space we choose $[0,1]$ with μ as the Lebesgue measure. Also, let

$$f \colon [0,1] \to c_0 \,, \ t \mapsto \left(\mathbb{1}_{(0,1]}(t), 2\,\mathbb{1}_{(0,\frac{1}{2}]}(t), 3\,\mathbb{1}_{(0,\frac{1}{3}]}(t), \dots \right) .$$

For $\alpha = (\alpha_n) \in \ell_1 \cong c_0^*$, we note that

$$\alpha f = \sum_n \alpha_n n \, \mathbb{1}_{(0,\frac{1}{n}]}$$

is measurable and also integrable with

$$\int \alpha f \, \mathrm{d}\mu = \sum_n \alpha_n n \left(\frac{1}{n} - 0 \right) = \sum_n \alpha_n \,.$$

Finally, the functional $(\alpha_n) \mapsto \sum_n \alpha_n$ is given by $(1,1,\dots,) \in \ell_\infty$, i.e.

$$\int f \, \mathrm{d}\mu \cong (1,1,1,\dots) \in \ell_\infty \setminus c_0 \,.$$

Thus it is natural to consider X as embedded into X^{**}. One may also write $X \subset X^{**}$ which makes it more convenient to think of $\int f \, \mathrm{d}\mu$ just living in some larger space instead of some space of functionals on a dual space. In the same manner, by a slight abuse of notation we write $\int f \, \mathrm{d}\mu \in X$ if $\int f \, \mathrm{d}\mu$ can be identified via the canonical embedding with an element of X. In the setting of the above example, we could also integrate the function g being constantly some element $x = (x_n) \in c_0$. Then the integral $\int g \, \mathrm{d}\mu$ corresponds the sequence (x_n) as an element of ℓ_∞. Hence, we may write $\int g \, \mathrm{d}\mu = x \in c_0$.

If the integral of f over any measurable subset of Ω is in the image of the canonical embedding (which we identify with X), we shall call f Pettis integrable.

Definition 5. If

$$\int_E f \, \mathrm{d}\mu := \int \mathbb{1}_E f \, \mathrm{d}\mu \in X$$

for any $E \in \Sigma$, f is called *Pettis integrable* and $\int f \, \mathrm{d}\mu$ is its *Pettis integral*.

Here, again, $\int \mathbb{1}_E f \, \mathrm{d}\mu \in X$ means that this element of the bidual actually is of the form $\hat{x}$, where $x \in X$ and the hat denotes the canonical embedding. (then we identify $\int \mathbb{1}_E f \, \mathrm{d}\mu = x$.)

Example 6. We have already seen an example for a Dunford integrable but not Pettis integrable function $f \colon [0,1] \to c_0$. More caution is needed in the following. For this f, let

$$g \colon [0,2] \to c_0 \,, \ t \mapsto \begin{cases} f(t) & t \le 1 \,, \\ -f(t-1) & t > 1 \,. \end{cases}$$

This is again weakly measurable and Dunford integrable when endowing $[0,2]$ with its Lebesgue measure μ, where

$$\int_{[0,2]} g \, \mathrm{d}\mu = \int_{[0,1]} f \, \mathrm{d}\mu - \int_{[0,1]} f \, \mathrm{d}\mu = 0 \,,$$

thus $\int g \, \mathrm{d}\mu \in X$. However, g is not Pettis integrable, since $\int_{[0,1]} g \, \mathrm{d}\mu = \int_{[0,1]} f \, \mathrm{d}\mu \notin c_0$.

Finally we use the Hahn-Banach separation theorem in this theory.

Theorem 7. *Let f be Dunford integrable with $\int f \, \mathrm{d}\mu \in X$ and let $\mu(\Omega) < \infty$. Then*

$$\frac{1}{\mu(\Omega)} \int f \, \mathrm{d}\mu \in \overline{\mathrm{conv}}(f(\Omega)) \, .$$

Intuitively, the mean value of f is in the closed convex hull of its values.

Proof. If this were not the case, the Hahn-Banach separation theorem would yield a functional $x^* \in X^*$ with

$$x^* \left(\frac{1}{\mu(\Omega)} \int f \, \mathrm{d}\mu \right) < 0 \leq x^* f(\omega)$$

for any $\omega \in \Omega$. Integrating over Ω gives

$$\mu(\Omega) x^* \left(\frac{1}{\mu(\Omega)} \int f \, \mathrm{d}\mu \right) = x^* \int f \, \mathrm{d}\mu < 0 \leq \int x^* f \, \mathrm{d}\mu \, .$$

But now, by the definition of the Dunford integral, we may interchange x^* and integration, yielding

$$\int x^* f \, \mathrm{d}\mu < \int x^* f \, \mathrm{d}\mu \, ,$$

which is a contradiction. $\qquad\square$

Remark 8. As a final remark we mention the existence of a notion of strong measurability (also called Bochner measurability) which essentially means that a function can be approximated by simple functions. Suprisingly, a strongly measurable and Dunford integrable function is automatically Pettis integrable if X contains no copy of c_0, i.e. has no subspace isomorphic to c_0.