Theory of Closed Operators

We will now consider closed operators and prove some Theorems concerning them. First of all, we have to define a closed operator.

Definition 1. Let E and F be normed spaces and $D \subset E$ a subspace. A linear operator $T\colon D \to F$ is called *closed*, if for each sequence $(x_n) \subset D$ we have that $x_n \to x$, $Tx_n \to y$ imply $x \in D$ and $Tx = y$.

This means, a closed operator is an operator $T\colon D \to F$ which graph is closed, i.e.

$$\begin{pmatrix} x_n \\ Tx_n \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix} \text{ implies } \begin{pmatrix} x \\ y \end{pmatrix} \in \operatorname{graph} T\,.$$

Remark 2. If a linear operator is only defined on a subspace D, we call D the *domain* of T and denote it by $\operatorname{dom} T$ or $\mathcal{D}(T)$.

Now we derive some properties from the closedness of an operator.

Theorem 3. *Let E, F be Banach spaces and let $T\colon E \supset \operatorname{dom} T \to F$ be a closed operator. Then*

(i) $E_T := (\operatorname{dom} T, \|\cdot\|_T)$ with $\|x\|_T := \|x\| + \|Tx\|$, $x \in \operatorname{dom} T$, is a Banach space.

(ii) T, considered as an operator from E_T to F is bounded, i.e. $T \in L(E_T, F)$.

Proof. (i). Let $(x_n) \subset E_T$ be a Cauchy sequence in E_T. We have

$$\|x_n - x_m\|_T = \|x_n - x_m\| + \|Tx_n - Tx_m\|\,.$$

thus (x_n) is a Cauchy sequence in E and (Tx_n) is a Cauchy sequence in F. Hence there exist $x \in E$ and $y \in F$ such that $x_n \to x$, $Tx_n \to y$. Since T is closed, we have $x \in \operatorname{dom} T$ and $Tx = y$ and therefore

$$\|x_n - x\|_T = \|x_n - x\| + \|Tx_n - Tx\| \to 0,\ n \to \infty\,.$$

(ii). Follows from

$$\|Tx\| \le \|x\| + \|Tx\| = \|x\|_T,\ x \in \operatorname{dom} T\,.$$

□

Theorem 4. *Let E and F be Banach spaces and let $T\colon E \supset \operatorname{dom} T \to F$ be closed and bijective. Then $T^{-1}\colon F \to E$ is bounded, i.e. $T^{-1} \in L(F, E)$.*

Proof. The operator $\tilde{T}\colon E_T \to F$ is bounded and bijective, thus $\tilde{T}^{-1} \in L(F, E_T)$, i.e.

$$\|T^{-1}y\|_T = \|T^{-1}y\| + \|y\| \le c\|y\|$$

for some $c > 0$. This implies

$$\|T^{-1}y\| \le c\|y\|\,.$$

□

Finally, we give an example for a closed operator. For this, we recall the definition of an absolutely continuous function:

Definition. A function $f\colon [0,1] \to \mathbb{C}$ is called *absolutely continuous*, if there exists $g \in L^1(0,1)$ with

$$f(x) = f(0) + \int_0^x g(t)\mathrm{d}t\,.$$

Now let $E = F = L^2(0,1)$ and

$$\operatorname{dom} T := \{f \in L^2(0,1) : f \text{ is absolutely continuous and there exists } f' \in L^2(0,1)\}\,.$$

Define

$$T\colon E \supset \operatorname{dom} T \to F,\ Tf := f'\,.$$

We claim that T is closed.

Proof. Let $(f_n) \subset \operatorname{dom} T$ such that $f_n \to f$ and $Tf_n = f_n' \to g$ in $L^2(0,1)$. We have $f_n(x) = f_n(0) + \int_0^x f_n'(t)\mathrm{d}t$ and by Hölder's inequality

$$\left|\int_0^x f_n(t)\mathrm{d}t - \int_0^x g(t)\mathrm{d}t\right| \le \int_0^1 |f_n'(t) - g(t)|\mathrm{d}t \le \left(\int_0^1 |f_n'(t) - g(t)|^2\mathrm{d}t\right)^{\frac{1}{2}} = \|f_n' - g\|_2 \to 0\,.$$

We observe that for all $x \in (0,1)$ we have

$$f_n(0) - f_m(0) = f_n(x) - f_m(x) + \int_0^x (f_m'(t) - f_n'(t))\mathrm{d}t\,.$$

By the triangle inequality in $L^2(0,1)$ we now have

$$\begin{aligned}
|f_n(0) - f_m(0)| &= \left(\int_0^1 |f_n(0) - f_m(0)|\mathrm{d}x\right)^{\frac{1}{2}} \\
&= \left(\int_0^1 \left|f_n(x) - f_m(x) + \int_0^x |f_m'(t) - f_n'(t)|\mathrm{d}t\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le \left(\int_0^1 |f_n(x) - f_m(x)|^2\mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_0^1 \left|\int_0^x |f_m'(t) - f_n'(t)|\mathrm{d}t\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} \to 0
\end{aligned}$$

for $n \to \infty$. Thus, $(f_n(0))$ is a Cauchy sequence. Set $\alpha := \lim_{n\to\infty} f_n(0)$. Define

$$h(x) := \alpha + \int_0^x g(t)\mathrm{d}t\,.$$

Then h is absolutely continuous and $h' = g \in L^2(0,1)$, i.e. $h \in \operatorname{dom} T$ and $Th = g$. There holds

$$f_n(x) - h(x) = f_n(0) - \alpha + \int_0^x (f_n'(t) - g(t))\mathrm{d}t \to 0,\ n \to \infty$$

and thus

$$\|f_n - h\|_2 \le \|f_n - h\|_\infty \to 0\,.$$

Hence $h = f$ almost everywhere. □